Enhancement of superhorizon scale inflationary curvature perturbations

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We show that there exists a simple mechanism which can enhance the amplitude of curvature perturbations on superhorizon scales during inflation, relative to their amplitude at horizon crossing. The enhancement may occur even in a single-field inflaton model, and occurs if the quantity $a\dot{\phi}/H$ becomes sufficiently small, as compared to its value at horizon crossing, for some time interval during inflation. We give a criterion for this enhancement in general single-field inflation models.

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\section{I. INTRODUCTION}

The standard, single-field, slow-roll inflation model predicts that the curvature perturbation on comoving hypersurfaces, $\mathcal{R}_c$, remains constant from soon after the scale crosses the Hubble horizon, giving the formula \cite{1,2}

$$\mathcal{R}_c \approx \mathcal{R}_c(t_k) \approx \left(\frac{H^2}{2\pi\dot{\phi}}\right)_{k=aH}$$  \hspace{0.5cm} (1)

where $H$ is the Hubble parameter, $\dot{\phi}$ is the time derivative of the inflaton field $\phi$, and $t_k$ is a time shortly after horizon crossing. However, one may consider a model in which slow-roll is violated during inflation. Recently, Leach & Liddle \cite{3} studied the behavior of the curvature perturbation in a model in which inflation is temporarily suspended, finding a large amplification of the curvature perturbation relative to its value at horizon crossing for a range of scales extending significantly beyond the Hubble horizon.

In this short paper, we consider single-field inflation models and analyze the general behavior of the curvature perturbation on superhorizon scales. We show analytically when and how this large enhancement occurs. We find that a necessary condition is that the quantity $z \equiv a\dot{\phi}/H$ becomes smaller than its value at the time of horizon crossing. We then present an integral which involves the above quantity and which gives a criterion for enhancement.

\section{II. ENHANCEMENT OF THE CURVATURE PERTURBATION}

We assume a background metric of the form

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^idx^j$$

$$= a^2(\eta)\left(-d\eta^2 + \delta_{ij}dx^idx^j\right).$$  \hspace{0.5cm} (2)

On this background the growing mode solution of the curvature perturbation on comoving hypersurfaces is known to stay constant in time on super-horizon scales in the absence of any entropy perturbation \cite{1,2,4-6}. This follows from the equation for $\mathcal{R}_c$

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' + k^2\mathcal{R}_c = 0,$$  \hspace{0.5cm} (3)

where the prime denotes the conformal time derivative, $d/d\eta$, and $z = a\dot{\phi}/H$. One readily sees that on superhorizon scales, when the last term can be neglected, there exists a solution with $\mathcal{R}_c$ constant, which corresponds to the growing adiabatic mode.

However, this does not necessarily mean that $\mathcal{R}_c$ must stay constant in time after its scale crosses the Hubble horizon. In fact, if the contribution of the other independent mode (i.e. the decaying mode) to $\mathcal{R}_c$ is large at horizon crossing, $\mathcal{R}_c$ will not become constant until the decaying mode dies out. The important point here is that the decaying mode is, by definition, the mode that decays asymptotically in the future, but it does not necessarily start to decay right after horizon crossing. In what follows, we show that there indeed exists a situation in which the mode dies out for a while after the horizon crossing before it starts to decay. In such a case, the contribution of the two modes to the curvature perturbation is found to almost cancel at horizon crossing. This gives a small initial amplitude of $\mathcal{R}_c$, but results in a large final amplitude for $\mathcal{R}_c$ after the decaying mode becomes negligible.

Let $u(\eta)$ be a solution of Eq. (3) for any given $k$. For much of the following discussion it is not necessary to specify the nature of the solution $u$, but for clarity let us identify it straightaway as the late-time asymptotic solution at $\eta_\ast$ (taking $\eta_\ast$ for instance as the end of inflation). For any other solution, $v(\eta)$, independent of $u(\eta)$, it is easy to show from Eq. (3) that the Wronskian $W = v'u - u'v$ obeys

$$W' = -2\frac{z'}{z}W$$  \hspace{0.5cm} (4)

and hence $W \propto 1/z^2$. Therefore we have
Hence the decaying mode, \(v\), which vanishes as \(\eta \to \eta_*\), may be expressed in terms of the growing mode, \(u\), as

\[
v(\eta) \propto u(\eta) \int_{\eta_*}^{\eta} \frac{d\eta'}{\eta^2(\eta') (u' (\eta')^2)} .
\]

Without loss of generality, we may assume \(v = u\) at some initial epoch, which we take to be shortly after horizon crossing, \(\eta = \eta_k\) \((< \eta_*)\). Then \(v\) is expressed as

\[
v(\eta) = u(\eta) \frac{D(\eta)}{D(\eta_k)} ,
\]

where

\[
D(\eta) = 3H_k \int_{\eta}^{\eta_k} d\eta' \frac{z^2(\eta_k)u^2(\eta_k)}{z^2(\eta')u^2(\eta')} .
\]

and, for convenience, the conformal Hubble parameter \(H_k = (a'/a)_k\) at \(\eta = \eta_k\) is inserted to make \(D\) dimensionless. In terms of \(u\) and \(v\), the general solution of \(R_c\) may be expressed as

\[
R_c(\eta) = \alpha u(\eta) + \beta v(\eta) ,
\]

where \(\alpha\) and \(\beta\) are constants and we assume \(\alpha + \beta = 1\) without loss of generality. Thus, if the amplitude of \(R_c\) at horizon crossing differs significantly from that of the growing mode, \(\alpha u(\eta_k)\), it can only be because \(|\beta| \gg 1\).

Using Eq. (6) and noting \(\alpha + \beta = 1\), \(R_c\) and \(R'_c\) at the initial epoch \(\eta = \eta_k\) are given by

\[
R_c(\eta_k) = u(\eta_k) ,
\]

\[
R'_c(\eta_k) = u'(\eta_k) - \frac{3(1-\alpha)H_k u(\eta_k)}{D_k} ,
\]

where \(D_k = D(\eta_k)\). Then \(\alpha\) can be expressed in terms of the initial conditions as

\[
\alpha = 1 + D_k \frac{1}{3H_k} \left[ \frac{R'_c}{R_c} - \frac{u'}{u} \right]_{\eta=\eta_k} .
\]

If we assume \(R_c(\eta_k)\) to be a complex amplitude determined by an initial vacuum state for quantum fluctuations, then \([R'_c/R_c - u'/u]\) at the time of horizon crossing will be at most of order \(H_k\). This implies that \(|\alpha|\), and hence \(|\beta|\), can become large only if \(D_k \gg 1\).

### III. LONG-WAVELENGTH APPROXIMATION

Equation (3) can be written in terms of the canonical field perturbation, \(Q = zR\), as

\[
Q'' + \left( k^2 - \frac{z''}{z} \right) Q = 0 .
\]

From this we see that the general solution for \(k^2 \ll |z''/z|\) is given approximately by

\[
R_c \approx A + B \int_{\eta_k}^{\eta} \frac{d\eta'}{z^2(\eta')} .
\]

where \(A\) and \(B\) are constants.

The requirement that \(v \to 0\) as \(\eta \to \eta_*\) uniquely identifies the decaying mode as proportional to \(\int_{\eta_k}^{\eta} d\eta'/z^2(\eta')\) in Eq. (12), but one is always free to include arbitrary contributions from the decaying mode in the growing mode. Nonetheless, it is convenient to identify the constant \(A\) in Eq. (12) as an approximate solution for the growing mode, \(u\), on large scales. Then for \(k^2 \ll |z''/z|\) Eq. (10) for \(\alpha\) may be approximated as

\[
\alpha \approx 1 + D_k \frac{1}{3H_k} \left[ \frac{R'_c}{R_c} \right]_{\eta=\eta_k} .
\]

where for definiteness we will take \((k/H_k)^2 = 0.1\). \(D_k\) is given, from the integral in Eq. (7), by

\[
D_k \approx 3H_k \int_{\eta_k}^{\eta} d\eta' \frac{z^2(\eta_k)}{z^2(\eta')} ,
\]

which may become large if there is a period at which \(z(\eta) \ll z(\eta_k)\).

In slow-roll inflation, the time variation of \(\dot\phi\) is small and \(z\) increases rapidly, approximately proportional to the scale factor \(a\). Hence the integral \(D_k\) cannot become large. Soon after horizon crossing \(R'_c/R_c \ll H\), so that \(\alpha \approx 1\) and the standard result \(R_c(\eta) \approx R_c(\eta_k)\) holds. However, if the slow-roll condition is violated, \(\dot\phi\) may become very small and \(z\) may decrease substantially to give a large value of \(D_k\). (The case where \(z\) actually crosses zero is treated separately in an Appendix.) Then at late times, we have

\[
R_c(\eta_k) = \alpha u(\eta_k) \approx \alpha u(\eta_k) = \alpha R_c(\eta_k) .
\]

Thus the final amplitude will be enhanced by a factor \(|\alpha|\), which can be large if \(D_k \gg 1\).

### IV. STAROBSINSKY’S MODEL

As an example we consider the model discussed by Starobinsky [7], where the potential has a sudden change in its slope at \(\phi = \phi_0\) such that

\[
V(\phi) = \begin{cases} 
V_0 + A_+ (\phi - \phi_0) & \text{for } \phi > \phi_0 \\
V_0 + A_- (\phi - \phi_0) & \text{for } \phi < \phi_0 
\end{cases} .
\]

If the change in the slope is sufficiently abrupt [7] then the slow-roll can be violated and for \(A_+ > A_- > 0\) the field enters a friction-dominated transient (or “fast-roll”) solution with \(\dot{\phi} \approx -3H\dot{\phi}\) [3] until the slow-roll conditions are once again satisfied.

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which implies \( R \) decreases rapidly until the slow-roll condition is violated. Also plotted is the slow-roll amplitude \( R^\phi \) given by Eq. (23).

\[
3H_0 \phi = \begin{cases} 
-A_+ & \text{for } \phi > \phi_0 \\
-A_- - (A_+ - A_-)e^{-3H_0 \Delta t} & \text{for } \phi < \phi_0 
\end{cases}
\]

For \( \phi < \phi_0 \) we have

\[
z \simeq -a_0 \frac{A_- e^{H_0 \Delta t} + (A_+ - A_-)e^{-2H_0 \Delta t}}{3H_0^2}.
\]

This decreases rapidly to a minimum value \( z_{\min} \approx (A_-/A_+)^{2/3}z_0 \) for \( A_+ > A_- \), which can cause a significant change in \( R_e \) on super-horizon scales.

For a mode that leaves the horizon in the slow-roll regime \( z \) grows proportional to \( a \) while \( \phi > \phi_0 \), so that the integrand of \( D(\eta) \) remains small. Hence \( D(\eta) \approx D_1 \), which implies \( R_c(\eta) \approx R_c(\eta_k) \) until \( \eta = \eta_0 \). Even after the slow-roll condition is violated \( R_c(\eta) \) still remains constant until \( z \) becomes smaller than \( z_k \) and the integrand of \( D(\eta) \) becomes small again. Then \( D(\eta) \) may decrease rapidly until \( R_c \) approaches the asymptotic value for \( \eta \to \eta_\ast \), given by Eq. (15). Substituting the above solution for \( z \) in Eq. (18) into Eq. (14) we obtain

\[
D_k \approx \begin{cases} 
1 + \frac{A_+}{A_-} & \text{for } k > (k/\mathcal{H}_0)\mathcal{H}_0 \\
1 + \frac{A_+}{A_-} \left( \frac{\mathcal{H}_0}{k} \right)^3 & \text{for } k < (k/\mathcal{H}_0)\mathcal{H}_0 
\end{cases}
\]

which shows that for \( A_+/A_- \gg 1 \) the curvature perturbation is significantly affected \( (D_k \gg 1) \) by the discontinuity at \( \phi \sim \phi_0 \) even on super-horizon scales from

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**FIG. 1.** The power spectrum for the Starobinsky model [7] with \( A_+/A_- = 10^4 \). Plotted are the exact asymptotic value of the curvature perturbation \( R^\phi(\eta_\ast) \), the horizon-crossing value \( R^\phi(\eta_0) \), and the enhanced horizon-crossing amplitude \( \alpha^2 R^\phi(\eta_k) \) using the long-wavelength approximation. The range of scales between the dotted lines corresponds to modes leaving the horizon during the transient epoch, defined as the region where \( z'/z < 0 \). Also plotted is the slow-roll amplitude \( R^\phi \) given by Eq. (23).

**FIG. 2.** Power-spectrum for the false-vacuum quartic model as in Fig. 1.

\[
k \sim (A_-/A_+)^{1/3} \mathcal{H}_0 \text{ up to } k \sim (A_+/A_-)^{1/3} \mathcal{H}_0.
\]

Super-imposed upon the amplification of the horizon-crossing amplitude, the exact solution in Fig. 1 shows scale-dependent oscillations due to “accidental” cancellations in \( [R'_c/R_c - u'/u] \) at the time of horizon crossing in Eq. (10).

Similar behavior was observed in the model studied by Leach & Liddle [3] for false-vacuum inflation with a quartic self-interaction potential [8], whose power spectrum is shown in Fig. 2. In this model there is no discontinuity in the potential, so the oscillations seen in Starobinsky’s model are washed out.

In both cases comparison of our analytic estimate of the enhancement on super-horizon scales with the numerical results shows that, although there is excellent agreement on shorter wavelengths \( (k \gtrsim \mathcal{H}_0) \), it does much worse for modes which leave the horizon long before the enhancement occurs \( (k \ll \mathcal{H}_0) \). This seems to reflect a failure of our approximation that \( u'/u \equiv 0 \) at \( \eta = \eta_k \) to pick out to sufficient accuracy the constant mode on large scales, because the growing mode and decaying mode solutions are both very nearly constant soon after horizon-crossing. In the Leach & Liddle model there is the added complication that the long-wavelength condition, \( k^2 \ll |z''/z| \), is violated for these modes and even the growing mode does not remain constant.

**V. INVARIANT SPECTRA**

A striking feature of these results is that the modes which leave the horizon during the transient regime share the same underlying spectrum as that produced during the subsequent slow-roll era. This is a manifestation of the ‘duality invariance’ of perturbation spectra produced in apparently different inflationary scenarios [9].

Starting from a particular asymptotic background solution, \( z(\eta) \), one finds a two parameter family of solutions
\[ \ddot{z}(\eta) = C_1 z(\eta) + C_2 \frac{d\eta'}{z^2(\eta')} \tag{20} \]

which leave \( z''/z \) unchanged in the perturbation equation (11) and thus generate the same perturbation spectrum from vacuum fluctuations [9] (up to the overall normalization \( C_1 \)). The variable \( z \) itself obeys the second-order equation

\[ z'' + \left( a^2 \frac{d^2 V}{d\phi^2} - 5\mathcal{H}^2 \right) z = 0. \tag{21} \]

Thus for a weakly interacting field \( (d^2 V/d\phi^2 \approx \text{constant}) \) in a quasi-de Sitter background \( (H \approx \text{constant}) \) the equation can be approximated by the linear equation of motion

\[ z'' + \left( a^2 \frac{d^2 V}{d\phi^2} - 2\mathcal{H}^2 \right) z \approx 0. \tag{22} \]

The general solution \( \tilde{z}(\eta) \) is related to the asymptotic late-time solution \( z(\eta) \) by the expression given in Eq. (20).

This means that the usual slow-roll result \( [\text{taking } \dot{\phi} \approx -(dV/d\phi)/3H] \) for the amplitude of the curvature perturbations in Eq. (1)

\[ \mathcal{R}_c \approx -\left( \frac{3H^3}{2\pi(dV/d\phi)} \right)_{k=\mathcal{H}}, \tag{23} \]

may continue to be a useful approximation even when the actual background scalar field solution at horizon crossing is no longer described by slow-roll, as was noted previously by Seto, Yokoyama and Kodama [10] and seen in our figures.

### VI. SUMMARY

In summary, we have studied the enhancement of the curvature perturbation on superhorizon scales possible in some models of inflation. We have found that the curvature perturbation can be enhanced on superhorizon scales even in single-field inflation, provided that the slow-roll condition is violated and \( a\dot{\phi}/H \) becomes small compared to its value at horizon crossing. We have presented a quantitative criterion for this enhancement, namely that the integral \( D_k \), given in terms of \( z = a\dot{\phi}/H \) in Eq. (7), becomes larger than unity. In the long-wavelength limit \( (k^2 \ll |z''/z|) \) it is possible to express \( D_k \) in terms of the background quantities, so an analytical formula for the final curvature perturbation amplitude may be derived without assuming slow-roll inflation. In the case of a weakly self-interacting field in de Sitter inflation we recover the usual slow-roll formula for the amplitude of the scalar perturbations even when the background solution is far from slow-roll at horizon crossing.

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**APPENDIX A: IF \( \dot{\phi} \) CROSSES ZERO**

The case when \( \dot{\phi} \) and hence \( z \) changes its sign can be treated as follows. For simplicity, let us assume \( z \) changes the sign only once at \( \eta = \eta_0 \). Then in the vicinity of \( \eta = \eta_0, z \) can be expressed as \( z = z^0_\epsilon (\eta - \eta_0) \) where \( z^0_\epsilon = z' (\eta_0) \). Hence the equation for \( \mathcal{R}_c \) becomes

\[ \left[ \frac{d^2}{d\eta^2} + 2\frac{d}{\eta - \eta_0} + k^2 \right] \mathcal{R}_c = 0. \tag{A1} \]

The two independent solutions can be found as

\[ u \approx C \left( 1 - \frac{1}{6}k^2(\eta - \eta_0)^2 + \cdots \right), \tag{A2} \]

\[ v \approx D \left( \frac{1}{\eta - \eta_0} - \frac{1}{2}k^2(\eta - \eta_0)^2 + \cdots \right). \tag{A3} \]

It is apparent that \( u \) corresponds to the growing mode, and it still remains constant on superhorizon scales in this case.

We require \( v \) to describe the decaying mode. As before, we consider an integral expression of \( v \) in terms of \( z^2 \) and \( u \). Then

\[ v = u \int_{\eta}^{\eta_0} \frac{d\eta'}{z^2 u^2} \approx u \int_{\eta}^{\eta_0} \frac{d\eta'}{z^2 C^2} \tag{A4} \]

for \( \eta > \eta_0 \). This \( v \) behaves in the limit \( \eta \to \eta_0 \) as

\[ v \sim \frac{1}{z_0^2 C^2(\eta - \eta_0)}. \tag{A5} \]

This should be extended to the region \( \eta < \eta_0 \) as the solution (A3), which implies

\[ v = u \lim_{\epsilon \to 0} \left( \int_{\eta}^{\eta_0 - \epsilon} \frac{d\eta'}{z^2 u^2} + \int_{\eta_0 + \epsilon}^{\eta_0} \frac{d\eta'}{z^2 u^2} - \frac{2}{z_0^2 C^2} \right), \tag{A6} \]

for \( \eta < \eta_0 \). Thus introducing the function \( \tilde{D}(\eta) \) by

\[ \tilde{D}(\eta) = \lim_{\epsilon \to 0} \left( \int_{\eta}^{\eta_0 - \epsilon} \frac{d\eta'}{z^2 u^2} + \int_{\eta_0 + \epsilon}^{\eta_0} \frac{d\eta'}{z^2 u^2} - \frac{2}{z_0^2 C^2} \right), \tag{A7} \]

and \( \tilde{D}_k = \tilde{D}(\eta_k) \), where \( u_0 = u(\eta_0) \), the decaying mode \( v \) normalized to \( u \) at \( \eta = \eta_k \) is given by
\[ v(\eta) = u(\eta) \frac{\bar{D}(\eta)}{D_k}. \] 

(A8)

Thus exactly the same argument applies to this case, by replacing the original \( D_k \) by the above \( \bar{D}_k \).

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