Consistent superconformal boundary states

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Abstract

We propose a supersymmetric generalization of Cardy’s equation for consistent $N = 1$ superconformal boundary states. We solve this equation for the superconformal minimal models $SM(p/p + 2)$ with $p$ odd, and thereby provide a classification of the possible superconformal boundary conditions. In addition to the Neveu-Schwarz ($NS$) and Ramond ($R$) boundary states, there are $\tilde{NS}$ states. The $NS$ and $\tilde{NS}$ boundary states are related by a $Z_2$ “spin-reversal” transformation. We treat the tricritical Ising model as an example, and in an appendix we discuss the (non-superconformal) case of the Ising model.
1 Introduction

Two fundamental developments of two-dimensional conformal field theory (CFT) [1, 2] have been the incorporation of supersymmetry [3] and the extension to manifolds with boundary [4]. The concept of conformal boundary state [5] is of central importance in the formulation of boundary CFT. Hence, in string theory [6, 7], boundary states also figure prominently [8]. (For further references and recent reviews of the boundary state formalism for describing D-branes, see e.g. [9].) The non-supersymmetric (Virasoro algebra) case is well understood [5]: at “tree” level, conformal invariance implies the constraint

\[(L_n - \bar{L}_{-n}) |\alpha\rangle = 0 \quad (1.1)\]

on the boundary state \(|\alpha\rangle\). This equation has a vector space of solutions which is spanned by the so-called Ishibashi states [10]. For the conformal minimal models, there is an Ishibashi state \(|j\rangle\rangle\) corresponding to each chiral primary field \(\Phi_j(z)\) (or Virasoro highest weight representation with highest weight \(j\)),

\[|j\rangle\rangle = \sum_N |j; N\rangle \otimes U|j; N\rangle, \quad (1.2)\]

where \(U\) is an antiunitary operator satisfying \(U^\dagger \bar{L}_n U = \bar{L}_n\), and \(|j; N\rangle\) is an orthonormal basis of the representation.

There is a further consistency constraint

\[\text{tr} e^{-\mathcal{R}H_{\alpha\beta}^{\text{open}}} = \langle \alpha | e^{-\mathcal{L}H_{\alpha\beta}^{\text{closed}}} | \beta \rangle, \quad (1.3)\]

which arises for the model on a flat cylinder of length \(L\) and circumference \(R\), as represented in Figure 1. Here \(H_{\alpha\beta}^{\text{open}} = \frac{\pi}{L}(L_0 - \frac{c}{24})\) is the Hamiltonian in the “open” channel, with spatial boundary conditions denoted by \(\alpha\) and \(\beta\); and \(H_{\alpha\beta}^{\text{closed}} = \frac{2\pi}{R}(L_0 + \bar{L}_0 - \frac{c}{12})\) is the Hamiltonian

\[R\]
\[\alpha\]
\[\beta\]
\[L\]

Figure 1: Cylinder of length \(L\) and circumference \(R\).
in the “closed” channel. In the string literature, a similar constraint (with integrations with respect to the corresponding moduli) is known as “world-sheet duality” or “open/closed string duality”. The LHS of Eq. (1.3) can be expressed as

$$\text{tr} e^{-R H^{\text{open}}_{\alpha\beta}} = \sum_i N^{i}_{\alpha\beta} \chi_i(q),$$  \hspace{1cm} (1.4)

where the Virasoro characters $\chi_i(q)$ are defined as

$$\chi_i(q) = \text{tr}_i q^{L_0 - \frac{c}{24}},$$ \hspace{1cm} (1.5)

and $q = e^{-\pi R/L}$. Under the modular transformation $S$, the characters transform according to

$$\chi_i(q) = \sum_j S_{ij} \chi_j(\tilde{q}),$$ \hspace{1cm} (1.6)

where $\tilde{q} = e^{-4\pi L/R}$. Thus,

$$\text{tr} e^{-R H^{\text{open}}_{\alpha\beta}} = \sum_{i,j} N^{i}_{\alpha\beta} S_{ij} \chi_j(\tilde{q}).$$  \hspace{1cm} (1.7)

Expressing the RHS of Eq. (1.3) in the Ishibashi basis, one obtains

$$\langle \alpha | e^{-L H^{\text{closed}}_{\beta}} | \beta \rangle = \sum_j \langle \alpha | j \rangle \langle \langle j | \beta \rangle \chi_j(\tilde{q}),$$ \hspace{1cm} (1.8)

assuming that each representation $j$ appears once in the spectrum of $H^{\text{closed}}$. Comparing Eqs. (1.7) and (1.8), one arrives at the Cardy equation

$$\sum_i N^{i}_{\alpha\beta} S_{ij} = \langle \alpha | j \rangle \langle \langle j | \beta \rangle \chi_j(\tilde{q}).$$ \hspace{1cm} (1.9)

Cardy solved this equation for the consistent boundary states

$$|k\rangle = \sum_j \frac{S_{kj}}{\sqrt{S_{0j}}} |j\rangle \rangle.$$

Moreover, with the help of the Verlinde formula [11], Cardy identified $N^{i}_{kl}$ as the fusion rule coefficients for $\Phi_k \times \Phi_l \rightarrow \Phi_i$. The important result (1.10) provides a classification of the possible conformal boundary conditions for the minimal models, and gives explicit values for the corresponding $g$-factors [12],

$$g_k = \langle \langle 0 | k \rangle \rangle = \frac{S_{k0}}{\sqrt{S_{00}}},$$ \hspace{1cm} (1.11)
Renormalization-group (RG) flows between the various conformal boundary conditions have been investigated in integrable boundary field theories. (See e.g. [13] - [16], and references therein.)

The aim of this paper is to generalize the above considerations to the case of $N = 1$ superconformal field theory [3], which encompasses many important models, including superstrings. Some progress on this problem has been made by Apikyan and Sahakyan in [17]. We have been motivated in part by our effort to better-understand RG boundary flows in supersymmetric integrable boundary field theories [18, 19].

It is evident that Cardy’s results cannot be naively carried over to the supersymmetric case. Indeed, (1.11) would imply that the $g$-factor of any Ramond boundary state is zero, since modular $S$ matrix elements between Ramond ($R$) and Neveu-Schwarz ($NS$) representations generally vanish (see Eq. (2.12) below).

Although for the Virasoro algebra case the consistent boundary states are in one-to-one correspondence with the irreducible representations, this is no longer true for the superconformal algebra case. Indeed, we find that in the latter case there are more such boundary states. This can be traced to the fact that under $S$ modular transformation, $R$ characters do not transform into $NS$ characters, but rather, into new characters denoted by $\tilde{NS}$. The $NS$ and $\tilde{NS}$ Cardy states are related by a $Z_2$ “spin-reversal” transformation, as are the “fixed +” and “fixed −” boundary states of the Ising model.

The outline of this article is as follows. In Section 2, we briefly review some necessary results about the $N = 1$ superconformal algebra, its representations, and the modular transformation properties of its characters. In Section 3, we formulate a supersymmetric generalization of Cardy’s equation, and we find its solutions. We also identify certain coefficients which appear in the super Cardy equations with the fusion rule coefficients of the chiral primary superconformal fields. In Section 4 we work out in detail the case of the tricritical Ising model (TIM). This example also serves as a check on our general formalism, since the TIM is also a member of the conformal minimal series. In Section 5, we briefly discuss some implications of our results, and mention several possible further generalizations. In an appendix we present an extended discussion of the case of the Ising model (IM). Although the IM does not have superconformal invariance, it does have $NS$ and $R$ sectors, and it provides valuable insight into how to treat the sectors of superconformal models.
2 Superconformal representation theory

In this Section, we first briefly review the superconformal algebra and its representations [3]. We then recall how the characters [20] transform under $S$ modular transformations [21, 22].

The $N=1$ superconformal algebra is defined by the (anti) commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n,0},$$

$$[L_m, G_r] = \left(\frac{1}{2} - r\right)G_{m+r},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r+s,0},$$  (2.1)

where $r, s \in \mathbb{Z}$ for the Ramond ($R$) sector and $r, s \in \mathbb{Z} + \frac{1}{2}$ for the Neveu-Schwarz ($NS$) sector. The two modings of $G_r$ are consistent with the $Z_2$ symmetry ($L_n \rightarrow L_n, G_r \rightarrow -G_r$) of the algebra. Highest weight irreducible representations are generated from highest weight states $|\Delta\rangle$ satisfying

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_n|\Delta\rangle = G_r|\Delta\rangle = 0, \quad n > 0, \quad r > 0.$$  (2.2)

For simplicity, we restrict to the superconformal minimal models that are unitary $SM(p/p+2)$, for which the central charge $c$ has the values

$$c_p = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right), \quad p = 3, 4, \ldots,$$  (2.3)

and the highest weights $\Delta$ are given by

$$\Delta_{(n,m)} = \frac{(n(p+2) - mp)^2 - 4}{8p(p+2)} + \frac{1}{32}(1 - (-1)^{n+m}),$$  (2.4)

where $1 \leq n \leq p - 1$ and $1 \leq m \leq p + 1$. The $NS$ representations have $n - m$ even, and the $R$ representations have $n - m$ odd. Following [21] we denote by $\Delta_{NS}$ and $\Delta_{R}$ the following independent sets of $NS$ and $R$ weights, respectively:

$$\Delta_{NS} = \left\{\Delta_{(n,m)} | 1 \leq m \leq n \leq p - 1, \quad n - m \quad \text{even}\right\},$$

$$\Delta_{R} = \left\{\Delta_{(n,m)} | 1 \leq m \leq n - 1 \quad \text{for} \quad 1 < n \leq (p - 1)/2; \right.$$  

$$1 \leq m \leq n + 1 \quad \text{for} \quad (p + 1)/2 \leq n \leq p - 1, \quad n - m \quad \text{odd}\right\}.  \quad (2.5)$$

\[1\]We generally follow the conventions of Matsuo and Yahikozawa [21], with the main exception that our characters (2.6) have an extra factor $q^{-\frac{n}{2\tau}}$. 

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In the $R$ sector, there is a zero mode $G_0$ which commutes with $L_0$. Hence, the highest weight states are generally two-fold degenerate, $|\Delta\rangle$ and $G_0|\Delta\rangle$. These states have opposite Fermion parity, since $G_0$ anticommutes with the Fermion parity operator $(-1)^F$. Due to the relation $G_0^2 = L_0 - \frac{c}{24}$, if $\Delta = \frac{c}{24}$, then $G_0|\Delta\rangle$ is a null state and decouples, in which case there is a unique highest weight state.

For $p$ even, there exists a $R$ representation $(n, m) = \left(\frac{p}{2}, \frac{p+2}{2}\right)$ which has weight $\Delta = \left(\frac{p}{2}, \frac{p+2}{2}\right) = \frac{c}{24}$, and so the corresponding highest weight state is unpaired. For $p$ odd, all the highest weight states in the $R$ sector are paired.

We define the characters \[ \chi_{i}^{NS}(q) = \text{tr}_{i} q^{L_0 - \frac{c}{24}}, \quad \chi_{i}^{\tilde{NS}}(q) = \text{tr}_{i} (-1)^F q^{L_0 - \frac{c}{24}}, \quad i \in \Delta_{NS}, \]
\[ \chi_{i}^{R}(q) = \text{tr}_{i} q^{L_0 - \frac{c}{24}}, \quad \chi_{i}^{\tilde{R}} = \text{tr}_{i} (-1)^F q^{L_0 - \frac{c}{24}}, \quad i \in \Delta_{R}. \] (2.6)

From the above remarks, it follows that for $p$ odd, $\chi_{i}^{\tilde{R}} = 0$ for all representations $i$; and for $p$ even, \[ \chi_{i}^{\tilde{R}} = \pm \delta_{i, \Delta_{\left(\frac{p}{2}, \frac{p+2}{2}\right)}}. \] (2.7)

The characters transform under the $S$ modular transformation according to \[ \chi_{i}^{NS}(q) = \sum_{j \in \Delta_{NS}} S_{ij}^{[NS,NS]} \chi_{j}^{NS}(\tilde{q}), \]
\[ \chi_{i}^{\tilde{NS}}(q) = \sum_{j \in \Delta_{R}} S_{ij}^{[\tilde{NS},R]} \sqrt{2} \chi_{j}^{R}(\tilde{q}), \]
\[ \sqrt{2} \chi_{i}^{R}(q) = \sum_{j \in \Delta_{NS}} S_{ij}^{[R,\tilde{NS}]} \chi_{j}^{\tilde{NS}}(\tilde{q}), \] (2.8)

where $\tilde{q} = e^{-4\pi L/R}$. As already mentioned in the Introduction, the characters $\chi_{i}^{\tilde{NS}}$ appear when the characters $\chi_{i}^{R}$ undergo a modular transformation. For the superconformal minimal models $SM(p/p+2)$ with $p$ odd, the modular $S$ matrices are given by

\[ S_{(n,m),(n',m')}^{[NS,NS]} = \frac{4}{\sqrt{p(p+2)}} \sin \frac{\pi nn'}{p} \sin \frac{\pi mm'}{p+2}, \] (2.9)
\[ S_{(n,m),(n',m')}^{[\tilde{NS},R]} = \frac{4}{\sqrt{p(p+2)}} (-1)^{(n-m)/2} \sin \frac{\pi nn'}{p} \sin \frac{\pi mm'}{p+2}, \] (2.10)
\[ S_{(n,m),(n',m')}^{[R,\tilde{NS}]} = \frac{4}{\sqrt{p(p+2)}} (-1)^{(n'-m')/2} \sin \frac{\pi nn'}{p} \sin \frac{\pi mm'}{p+2}. \] (2.11)
These matrices can be arranged into the matrix $S$

$$
S = \begin{pmatrix} 
S^{[NS,NS]} & 0 & 0 \\
0 & 0 & S^{[\tilde{\bar{NS}},R]} \\
0 & S^{[R,\bar{NS}]} & 0 
\end{pmatrix},
$$

(2.12)

which is real, symmetric, and satisfies $S^2 = I$. We do not quote the corresponding expressions for the case $p$ even, which are somewhat more complicated due to the special representation $(\frac{p}{2}, \frac{p+2}{2})$.

### 3 Consistent boundary states

The full operator algebra of the $NS$ and $R$ superconformal primary fields is nonlocal. We consider here the so-called spin model [3] which has a local operator algebra. It is obtained by projecting on even Fermion parity $(-1)^F = 1$ in the $NS$ sector, and either even or odd Fermion parity $(-1)^F = \pm 1$ in the $R$ sector.\(^2\) For definiteness, we treat only the case with even Fermion parity also in the $R$ sector. Also, for simplicity, we restrict to models all of whose representations satisfy $\Delta \neq \frac{c}{24}$; that is, we consider only the superconformal minimal models $SM(p/p+2)$ with $p$ odd. Moreover, we again assume that the bulk theory is diagonal, with each representation appearing once.

Our goal is to construct for such spin models the complete set of consistent superconformal boundary states $|\alpha\rangle$, by solving the various constraints which they must obey. The restriction to even Fermion parity implies the constraint

$$
(-1)^F |\alpha\rangle = |\alpha\rangle,
$$

(3.1)

where here $F$ is the total Fermion number of right and left movers. Superconformal invariance implies the constraints [10, 17]

$$
(L_n - \bar{L}_{-n}) |\alpha\rangle = 0,
$$

$$
(G_r + i\gamma \bar{G}_{-r}) |\alpha\rangle = 0,
$$

(3.2)

where $\gamma$ is either $+1$ or $-1$. Finally, we impose the further constraint

$$
\text{tr}_{NS} \frac{1}{2}(1 + (-1)^F)e^{-R_{\alpha\beta}^{\text{open}}} + \text{tr}_{R} \frac{1}{2}(1 + (-1)^F)e^{-R_{\alpha\beta}^{\text{open}}} = \langle \alpha | e^{-L_{H}\text{closed}} | \beta \rangle,
$$

(3.3)

for a spin model on the cylinder in Figure 1. The projectors $\frac{1}{2}(1 + (-1)^F)$ project onto states of even Fermion parity in the open channel. The Hamiltonians in the open and closed

\(^2\)An analysis of consistent boundary states for the so-called fermionic model, which is obtained by keeping only the $NS$ sector, is given in [23].
channels are (as in the non-supersymmetric case which was reviewed in the Introduction) given by
\[ H_{\alpha\beta}^{open} = \frac{1}{2}(L_0 - \frac{c}{24}) \] and \[ H_{\alpha\beta}^{closed} = \frac{2}{\pi} (L_0 + \bar{L}_0 - \frac{c}{12}) \], respectively. This constraint is similar to the Ising-model constraint (A.32), except without the term involving the projector \( \frac{1}{2}(1 - (-1)^F) \) in the NS sector.

We first consider the open channel. We define the coefficients \( n^i_{\alpha\beta}, \text{etc.} \) by
\[
tr_{NS} e^{-RH_{\alpha\beta}^{open}} = \sum_{i \in \Delta_{NS}} n^i_{\alpha\beta} \chi^i_{NS}(q),
\]
\[
tr_{NS} (-1)^F e^{-RH_{\alpha\beta}^{open}} = \sum_{i \in \Delta_{NS}} \tilde{n}^i_{\alpha\beta} \chi^{NS}(\tilde{q}),
\]
\[
tr_{R} e^{-RH_{\alpha\beta}^{open}} = \sum_{i \in \Delta_{R}} m^i_{\alpha\beta} \chi^i_{R}(q),
\]
\[
tr_{R} (-1)^F e^{-RH_{\alpha\beta}^{open}} = \sum_{i \in \Delta_{R}} \tilde{m}^i_{\alpha\beta} \chi^{R}(\tilde{q}) = 0,
\] (3.4)

where \( q = e^{-\pi R/L} \), and the various characters are defined in (2.6). In the last line, we have made use of our restriction to \( p \) odd, together with the result (2.7). It follows that
\[
\text{LHS of Eq. (3.3)} = \frac{1}{2} \sum_{i \in \Delta_{NS}} \left( n^i_{\alpha\beta} \chi^i_{NS}(q) + \tilde{n}^i_{\alpha\beta} \chi^{NS}(\tilde{q}) \right) + \frac{1}{2} \sum_{i \in \Delta_{R}} m^i_{\alpha\beta} \chi^i_{R}(q)
\]
\[
= \frac{1}{2} \sum_{i \in \Delta_{NS}} \left( \sum_{j \in \Delta_{NS}} n^i_{\alpha\beta} S^{[NS,NS]}_{ij} \chi^j_{NS}(q) + \sum_{j \in \Delta_{R}} \tilde{n}^i_{\alpha\beta} S^{[\tilde{NS},R]}_{ij} \sqrt{2} \chi^j_{R}(\tilde{q}) \right)
\]
\[
+ \frac{1}{2} \sum_{i \in \Delta_{R}} \sum_{j \in \Delta_{NS}} m^i_{\alpha\beta} S^{[R,\tilde{NS}]}_{ij} \frac{1}{\sqrt{2}} \chi^j_{\tilde{NS}}(\tilde{q}),
\] (3.5)

where \( \tilde{q} = e^{-4\pi L/R} \). In passing to the second equality, we have made use of the modular transformation properties (2.8) of the characters.

Turning now to the closed channel, we recall [10, 17] that corresponding to each irreducible representation \( j \) of the superconformal algebra, one can construct a pair of Ishibashi states \( |j_{\pm}\rangle\rangle \) satisfying
\[
(L_n - \bar{L}_{-n}) |j_{\pm}\rangle\rangle = 0,
\]
\[
(G_r \pm i\bar{G}_{-r}) |j_{\pm}\rangle\rangle = 0.
\] (3.6)

From the explicit expressions for the Ishibashi states, it is easy to see that the states in the NS sector have even Fermion parity
\[
(-1)^F |j_{\pm}^{NS}\rangle\rangle = |j_{\pm}^{NS}\rangle\rangle,
\] (3.7)
where (as in Eq. (3.1)) $F$ is the total Fermion number of right and left movers. For the $R$ sector, the computation of Fermion parity is more subtle due to the presence of zero modes [17]. We assume that, in analogy with the Ising model result (A.25),

$$(-1)^F |j^{R|}_{\pm}\rangle = \pm |j^{R|}_{\pm}\rangle . \tag{3.8}$$

We propose that the set of Ishibashi states $\{|j^{NS|}_{\pm}\rangle,|j^{R|}_{\pm}\rangle\}$ constitutes a basis for the boundary states. That is,

$$|\alpha\rangle = \sum_{j \in \Delta_{NS}} (|j^{NS|}_{+}\rangle \langle j^{NS|}_{+}|\alpha\rangle + |j^{NS|}_{-}\rangle \langle j^{NS|}_{-}|\alpha\rangle) + \sum_{j \in \Delta_{R}} |j^{R|}_{+}\rangle \langle j^{R|}_{+}|\alpha\rangle . \tag{3.9}$$

Indeed, Eqs. (3.7) and (3.8) imply that the constraint (3.1) is already satisfied. For a given value of $\gamma$, the constraints (3.2) can be satisfied by keeping in the expansion (3.9) only the terms involving $|j^{\gamma|}_{\pm}\rangle$, i.e. setting $\langle j^{\gamma|}_{-}|\alpha\rangle = 0$. Moreover, the number of basis vectors (twice the number of $NS$ representations plus the number of $R$ representations) is the same as the dimension of the vector space on which the full modular $S$ matrix (2.12) acts, which we expect is the number of consistent boundary states. The expansion (3.9) is also motivated by the corresponding result (A.34) for the Ising model.

In this basis, we have

$$\text{RHS of Eq. (3.3)} = \sum_{j \in \Delta_{NS}} \left( \langle \alpha|j^{NS|}_{+}\rangle \langle j^{NS|}_{+}|e^{-LH_{closed}}|j^{NS|}_{+}\rangle \langle j^{NS|}_{+}|\beta\rangle + \langle \alpha|j^{NS|}_{+}\rangle \langle j^{NS|}_{+}|e^{-LH_{closed}}|j^{NS|}_{+}\rangle \langle j^{NS|}_{-}|\beta\rangle \right)$$

$$+ \sum_{j \in \Delta_{R}} \langle \alpha|j^{R|}_{+}\rangle \langle j^{R|}_{+}|e^{-LH_{closed}}|j^{R|}_{+}\rangle \langle j^{R|}_{+}|\beta\rangle$$

$$= \sum_{j \in \Delta_{NS}} \left[ \left( \langle \alpha|j^{NS|}_{+}\rangle \langle j^{NS|}_{+}|\beta\rangle + \langle \alpha|j^{NS|}_{-}\rangle \langle j^{NS|}_{-}|\beta\rangle \right) \chi^{NS}_{j}(\tilde{q}) \right. \right.$$  

$$+ \left. \left( \langle \alpha|j^{NS|}_{+}\rangle \langle j^{NS|}_{+}|\beta\rangle + \langle \alpha|j^{NS|}_{-}\rangle \langle j^{NS|}_{-}|\beta\rangle \right) \chi^{NS}_{j}(\tilde{q}) \right]$$

$$+ \sum_{j \in \Delta_{R}} \langle \alpha|j^{R|}_{+}\rangle \langle j^{R|}_{+}|\beta\rangle \chi^{R}_{j}(\tilde{q}) . \tag{3.10}$$

In passing to the second equality, we have used the relations

$$\langle j^{NS|}_{\pm}|e^{-LH_{closed}}|j^{NS|}_{\pm}\rangle = \chi^{NS}_{j}(\tilde{q}) ,$$
which are analogous to the results (A.26) for the Ising model.

Comparing Eqs. (3.5) and (3.10), we arrive at the “super” Cardy equations (cf. (1.9))

\[
\frac{1}{2} \sum_{i \in \Delta_{NS}} n^i_{\alpha \beta} \mathcal{S}^{[NS,NS]}_{ij} = \langle \alpha | j^+_{NS} | \rangle \langle j^+_{NS} | \beta \rangle + \langle \alpha | j^-_{NS} | \rangle \langle j^-_{NS} | \beta \rangle,
\]

\[
\frac{1}{\sqrt{2}} \sum_{i \in \Delta_{NS}} \tilde{n}^i_{\alpha \beta} \mathcal{S}^{[NS,R]}_{ij} = \langle \alpha | j^+_{R} | \rangle \langle j^+_{R} | \beta \rangle,
\]

\[
\frac{1}{2\sqrt{2}} \sum_{i \in \Delta_{R}} m^i_{\alpha \beta} \mathcal{S}^{[R,NS]}_{ij} = \langle \alpha | j^+_{NS} | \rangle \langle j^-_{NS} | \beta \rangle + \langle \alpha | j^-_{NS} | \rangle \langle j^+_{NS} | \beta \rangle.
\] (3.12)

We now proceed to solve these equations, together with the constraints (3.2), for the consistent superconformal boundary states. Defining the state \(|0^{NS}\rangle\) as the solution with \(n^i_{0^{NS}0^{NS}} = \tilde{n}^i_{0^{NS}0^{NS}} = \delta^i_0\), \(m^i_{0^{NS}0^{NS}} = 0\), we obtain

\[
|0^{NS}\rangle = \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{NS}} \sqrt{\mathcal{S}^{[NS,NS]}_{0j}} | j^+_{NS} \rangle + \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{R}} \sqrt{\mathcal{S}^{[NS,R]}_{0j}} | j^+_{R} \rangle.
\] (3.13)

We then define the states \(|k^{NS}\rangle\) and \(|\tilde{k}^{NS}\rangle\) with \(k \in \Delta_{NS}\) by

\[
n^i_{0^{NS}k^{NS}} = \tilde{n}^i_{0^{NS}k^{NS}} = \delta^i_k, \quad m^i_{0^{NS}k^{NS}} = 0,
\]

\[
n^i_{0^{NS}k^{NS}} = -\tilde{n}^i_{0^{NS}k^{NS}} = \delta^i_k, \quad m^i_{0^{NS}k^{NS}} = 0,
\] (3.14)

respectively, and we obtain

\[
|k^{NS}\rangle = \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{NS}} \sqrt{\mathcal{S}^{[NS,NS]}_{kj}} | j^+_{NS} \rangle + \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{R}} \sqrt{\mathcal{S}^{[NS,R]}_{kj}} | j^+_{R} \rangle,
\] (3.15)

\[
|\tilde{k}^{NS}\rangle = \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{NS}} \sqrt{\mathcal{S}^{[NS,NS]}_{kj}} | j^+_{NS} \rangle - \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{R}} \sqrt{\mathcal{S}^{[NS,R]}_{kj}} | j^+_{R} \rangle.
\] (3.16)

Finally, we define the states \(|k^{R}\rangle\) with \(k \in \Delta_{R}\) by

\[
n^i_{0^{NS}k^{R}} = \tilde{n}^i_{0^{NS}k^{R}} = 0, \quad m^i_{0^{NS}k^{R}} = 2\delta^i_k,
\] (3.17)
and we obtain

$$|k^R\rangle = \sum_{j \in \Delta_{NS}} \frac{S_{kj}^{[R,\bar{NS}]} S_{ij}^{[NS,NS]} |j^\text{NS}\rangle}{\sqrt{S_{0j}^{[NS,NS]}}}.$$  (3.18)

We shall refer to the states (3.15), (3.16) and (3.18) as the $NS$, $\bar{NS}$ and $R$ Cardy states, respectively. These states manifestly satisfy the constraints (3.2), with the $R$ states and the $NS$, $\bar{NS}$ states having opposite signs of $\gamma$. The $NS$ and $\bar{NS}$ states differ by the $Z_2$ “spin-reversal” transformation $|j^{NS}\rangle \to |j^{\bar{NS}}\rangle$, just like the “fixed +” and “fixed −” boundary states of the Ising model (A.36).

The Eqs. (3.12) and their solutions (3.15), (3.16), (3.18) are the main results of this paper. These solutions provide a classification of the possible superconformal boundary conditions for the superconformal minimal models $SM(p/p + 2)$ with $p$ odd.

The $g$-factor [12] of a boundary state $|\alpha\rangle$ is given by

$$g_\alpha = \left(\langle 0^\text{NS}_+ | + \langle 0^\text{NS}_- |\right) |\alpha\rangle. \quad (3.19)$$

Hence, the $g$-factors of the Cardy states are

$$g_{k^{NS}} = g_{k^{\bar{NS}}} = \frac{1}{\sqrt{2}} \frac{S_{k0}^{[NS,NS]}}{\sqrt{S_{00}^{[NS,NS]}}}, \quad (3.20)$$

$$g_{k^R} = \frac{S_{k0}^{[R,\bar{NS}]} S_{ij}^{[\bar{NS},R]} (S_{0j}^{[\bar{NS},R]})^{-1}}{\sqrt{S_{00}^{[NS,NS]}}}, \quad (3.21)$$

We see from (3.20) that, for a $NS$ state, the naive use of the modular $S$ matrix (2.9) in the original Cardy result (1.11) would give a $g$-factor which is a factor $\sqrt{2}$ too big. Moreover, the $g$-factor (3.21) of a $R$ state does not generally vanish.

As in the non-supersymmetric case, the various coefficients $n_{\alpha\beta}^i$, etc. in Eq. (3.4) can now be expressed in terms of modular $S$ matrices and be related to fusion rule coefficients. Indeed, by substituting the expression (3.15) for two $NS$ Cardy states $|k^{NS}\rangle$ and $|1^{NS}\rangle$ back into the super Cardy formula (3.12), we obtain

$$n_{k^{NS}1^{NS}}^i = \sum_{j \in \Delta_{NS}} \frac{S_{kj}^{[NS,NS]} S_{ij}^{[NS,NS]} (S_{0j}^{[NS,NS]})^{-1}}{S_{0j}^{[NS,NS]}},$$

$$\tilde{n}_{k^{NS}1^{NS}}^i = \sum_{j \in \Delta_R} \frac{S_{kj}^{[\bar{NS},R]} S_{ij}^{[\bar{NS},R]} (S_{0j}^{[\bar{NS},R]})^{-1}}{S_{0j}^{[NS,NS]}}.$$  

---

3In [17] a different set of equations is proposed, which gives the $NS$ states (3.15), but not the $\bar{NS}$ and $R$ states (3.16), (3.18).
\[ m^i_{k \text{NS1NS}} = 0. \] (3.22)

From the work [24] (see also [25]) on a generalized Verlinde formula, we can identify \( n^i_{k \text{NS1NS}} \) as the fusion rule coefficient for \( \Phi^\text{NS}_k \times \Phi^\text{NS}_l \rightarrow \Phi^\text{NS}_i \). Similarly, for two \( R \) Cardy states (3.18), we obtain

\[
n^i_{k \text{R1R}} = 0, \quad \widetilde{n}^i_{k \text{R1R}} = 2 \sum_{j \in \Delta \text{NS}} S_{kj}^{[R, \overline{\text{NS}}]} S_{lj}^{[R, \overline{\text{NS}}]} (S_{[\text{NS,NS}]}^{-1})_{ji}, \]

and we identify \( n^i_{k \text{R1R}} \) as the fusion rule coefficient for \( \Phi^\text{R}_k \times \Phi^\text{R}_l \rightarrow \Phi^\text{NS}_i \). Finally, for one \( \text{NS} \) state and one \( R \) state, we obtain

\[
m^i_{k \text{NS1R}} = 2 \sum_{j \in \Delta \text{NS}} S_{kj}^{[\text{NS,NS}]} S_{lj}^{[R, \overline{\text{NS}}]} (S_{[\text{R,\overline{NS}}]}^{-1})_{ji}, \quad n^i_{k \text{NS1R}} = 0, \]

and we identify \( m^i_{k \text{NS1R}} \) as the fusion rule coefficient for \( \Phi^\text{NS}_k \times \Phi^\text{R}_l \rightarrow \Phi^\text{R}_i \). The results for the coefficients involving \( \overline{\text{NS}} \) states (3.16) are very similar to those for the corresponding \( \text{NS} \) states.

### 4 Tricritical Ising model

As an example of the general formalism presented in the previous section, we now work out in detail the first nontrivial case: namely, the superconformal minimal model \( S\mathcal{M}(3/5) \) (\( p = 3 \)), which has been identified [3] as the tricritical Ising model (TIM). This model is equivalent to the conformal minimal model \( \mathcal{M}(4/5) \), for which the Cardy states are already known [5, 14]. Hence, this example also serves as a valuable check on our general formalism.

The Kac table for \( S\mathcal{M}(3/5) \), which is obtained using Eq. (2.4), is given in Table 1. The

<table>
<thead>
<tr>
<th>( k )</th>
<th>( l )</th>
<th>( m )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{7}{16} )</td>
<td>( \frac{1}{10} )</td>
<td>( \frac{3}{80} )</td>
<td>( \frac{1}{10} )</td>
</tr>
</tbody>
</table>

Table 1: Kac table for \( S\mathcal{M}(3/5) \)
modular $S$ matrix is (2.9) - (2.12)

\[
S = \begin{pmatrix}
2a & 2b & 0 & 0 & 0 & 0 \\
2b & -2a & 0 & 0 & 0 & \sqrt{2c} \sqrt{2d} \\
0 & 0 & 0 & \sqrt{2c} & \sqrt{2d} & 0 \\
0 & 0 & \sqrt{2c} & -\sqrt{2d} & 0 & 0 \\
0 & 0 & \sqrt{2d} & \sqrt{2c} & 0 & 0
\end{pmatrix},
\]

(4.1)

where the rows and columns are labeled by the highest weights $(0_{NS}, \frac{1}{10}^{NS}, \bar{0}^{\bar{NS}}, \frac{1}{10}^{\bar{NS}}, \frac{7}{16} R, \frac{3}{80} R)$, and where

\[
a = \frac{1}{2} \sqrt{\frac{1}{10} (5 - \sqrt{5})}, \quad b = \frac{1}{2} \sqrt{\frac{1}{10} (5 + \sqrt{5})},
\]

\[
c = \frac{1}{2} \sqrt{\frac{1}{5} (5 - \sqrt{5})}, \quad d = \frac{1}{2} \sqrt{\frac{1}{5} (5 + \sqrt{5})}.
\]

(4.2)

According to our results (3.15), (3.16), (3.18), there are 6 Cardy states, given by

\[
|0^{NS}\rangle = \sqrt{a} |0^{NS}_+\rangle + \sqrt{b} |\frac{1}{10}^{NS}_+\rangle + \sqrt{c} |\frac{7}{16}^{R}_+\rangle + \sqrt{d} |\frac{3}{80}^{R}_+\rangle,
\]

\[
|0^{\bar{NS}}\rangle = \sqrt{a} |0^{NS}_+\rangle + \sqrt{b} |\frac{1}{10}^{NS}_+\rangle - \sqrt{c} |\frac{7}{16}^{R}_+\rangle - \sqrt{d} |\frac{3}{80}^{R}_+\rangle,
\]

\[
|\frac{1}{10}^{NS}\rangle = b |0^{NS}_+\rangle - a |\frac{1}{10}^{NS}_+\rangle - \sqrt{c} |\frac{7}{16}^{R}_+\rangle + \sqrt{d} |\frac{3}{80}^{R}_+\rangle,
\]

\[
|\frac{1}{10}^{\bar{NS}}\rangle = b \sqrt{a} |0^{NS}_+\rangle - a \sqrt{b} |\frac{1}{10}^{NS}_+\rangle + \sqrt{c} |\frac{7}{16}^{R}_+\rangle - \sqrt{d} |\frac{3}{80}^{R}_+\rangle,
\]

\[
|\frac{7}{16}^{R}\rangle = \frac{c}{\sqrt{a}} |0^{NS}_+\rangle - \frac{d}{\sqrt{b}} |\frac{1}{10}^{NS}_+\rangle,
\]

\[
|\frac{3}{80}^{R}\rangle = \frac{d}{\sqrt{a}} |0^{NS}_-\rangle + \frac{c}{\sqrt{b}} |\frac{1}{10}^{NS}_-\rangle.
\]

(4.3)

Using (3.19), we obtain the $g$ factors

\[
g_{0^{NS}} = g_{0^{\bar{NS}}} = \sqrt{a}, \quad g_{\frac{1}{10}^{NS}} = g_{\frac{1}{10}^{\bar{NS}}} = \frac{b}{\sqrt{a}},
\]

\[
g_{\frac{7}{16}^{R}} = \frac{c}{\sqrt{a}}, \quad g_{\frac{3}{80}^{R}} = \frac{d}{\sqrt{a}}.
\]

(4.4)

Let us compare these results with those [5, 14] obtained from the $\mathcal{M}(4/5)$ description.

The $\mathcal{M}(4/5)$ Kac table is given in Table 2.
Table 2: Kac table for $M(4/5)$

The modular $S$ matrix is

$$S = \begin{pmatrix}
  a & b & b & a & c & d \\
  b & -a & -a & b & -d & c \\
  b & -a & -a & b & d & -c \\
  a & b & b & a & -c & -d \\
  c & -d & d & -c & 0 & 0 \\
  d & c & -c & -d & 0 & 0 \\
\end{pmatrix},$$

(4.5)

where the rows and columns are labeled by the highest weights $(0, \frac{1}{10}, \frac{3}{5}, \frac{3}{2}, \frac{7}{16}, \frac{3}{80})$, and $a$-$d$ are given by (4.2). As follows from (1.10), the Cardy states are given by

$$|0\rangle = \sqrt{a}|0\rangle + \sqrt{b}|\frac{1}{10}\rangle + \sqrt{a}\sqrt{\frac{3}{5}}|\frac{3}{5}\rangle + \sqrt{c}|\frac{7}{16}\rangle + \sqrt{d}|\frac{3}{80}\rangle,$$

$$|\frac{3}{2}\rangle = \sqrt{a}|0\rangle + \sqrt{b}|\frac{1}{10}\rangle + \sqrt{b}\sqrt{\frac{3}{5}}|\frac{3}{5}\rangle + \sqrt{a}|\frac{3}{2}\rangle - \sqrt{c}|\frac{7}{16}\rangle - \sqrt{d}|\frac{3}{80}\rangle,$$

$$|\frac{1}{10}\rangle = \frac{b}{\sqrt{a}}|0\rangle - a\frac{1}{\sqrt{b}}|\frac{1}{10}\rangle - a\frac{3}{\sqrt{b}}|\frac{3}{5}\rangle + b\frac{3}{\sqrt{a}}|\frac{3}{2}\rangle - d\frac{7}{\sqrt{c}}|\frac{7}{16}\rangle + c\frac{3}{\sqrt{d}}|\frac{3}{80}\rangle,$$

$$|\frac{3}{5}\rangle = \frac{b}{\sqrt{a}}|0\rangle - a\frac{1}{\sqrt{b}}|\frac{1}{10}\rangle - a\frac{3}{\sqrt{b}}|\frac{3}{5}\rangle + b\frac{3}{\sqrt{a}}|\frac{3}{2}\rangle + d\frac{7}{\sqrt{c}}|\frac{7}{16}\rangle - c\frac{3}{\sqrt{d}}|\frac{3}{80}\rangle,$$

$$|\frac{7}{16}\rangle = \frac{c}{\sqrt{a}}|0\rangle - d\frac{1}{\sqrt{b}}|\frac{1}{10}\rangle + d\frac{3}{\sqrt{b}}|\frac{3}{5}\rangle - c\frac{3}{\sqrt{a}}|\frac{3}{2}\rangle,$$

$$|\frac{3}{80}\rangle = \frac{d}{\sqrt{a}}|0\rangle + c\frac{1}{\sqrt{b}}|\frac{1}{10}\rangle - c\frac{3}{\sqrt{b}}|\frac{3}{5}\rangle - d\frac{3}{\sqrt{a}}|\frac{3}{2}\rangle.$$  

(4.6)

We observe that the two modular $S$ matrices (4.1) and (4.5) are related by a unitary transformation, due to the relation of the corresponding characters [21]

$$\chi_0^{NS}(q) = \chi_0(q) + \chi_{\frac{3}{5}}(q), \quad \chi_0^{\bar{NS}}(q) = \chi_0(q) - \chi_{\frac{3}{5}}(q),$$

$$\chi_{\frac{1}{10}}^{NS}(q) = \chi_{\frac{1}{10}}(q) + \chi_{\frac{7}{16}}(q), \quad \chi_{\frac{1}{10}}^{\bar{NS}}(q) = \chi_{\frac{1}{10}}(q) - \chi_{\frac{7}{16}}(q),$$

Our notation is related to Chim’s [14] $C = \sqrt{\frac{\sin(\pi/5)}{\sqrt{5}}}$, $\eta = \sqrt{\frac{\sin(2\pi/5)}{\sin(\pi/5)}}$ by

\begin{align*}
a &= C^2, & b &= C^2\eta^2, & c &= C^2\sqrt{2}, & d &= C^2\eta^2\sqrt{2}.
\end{align*}
Moreover, the Cardy states (4.3) and (4.6) can be seen to coincide, upon identifying the Ishibashi states

\[
\begin{align*}
\left|0_{\pm}^{NS}\right\rangle &= \left|0\right\rangle \pm \frac{3}{2} \left|\right\rangle, \\
\left|\frac{1}{10}^{NS}_{\pm}\right\rangle &= \left|\frac{1}{10}\right\rangle \pm \frac{3}{5} \left|\right\rangle, \\
\left|\frac{7}{16}^{R}_{\pm}\right\rangle &= \left|\frac{7}{16}\right\rangle, \\
\left|\frac{3}{80}^{R}_{\pm}\right\rangle &= \left|\frac{3}{80}\right\rangle.
\end{align*}
\]  

(4.8)

Evidently, whether we use the \(\mathcal{M}(4/5)\) or \(\mathcal{SM}(3/5)\) description, the Hamiltonian is the same, and so are the Cardy states and corresponding \(g\) factors. The two descriptions correspond to two equivalent bases.

5 Discussion

We have proposed (3.12) a supersymmetric generalization of Cardy’s equation, and we have solved it for the consistent superconformal boundary states (3.15), (3.16), (3.18), thereby classifying the possible superconformal boundary conditions. In particular, there are \(\tilde{NS}\) boundary states in addition to the \(NS\) and \(R\) states.

Having a better understanding of boundary conditions in boundary superconformal field theories, one is in a better position to investigate integrable perturbations of these theories, and treat problems such as RG boundary flows.

For simplicity, we have restricted here to the unitary superconformal minimal models \(\mathcal{SM}(p/p + 2)\) with \(p\) odd. It should be possible to extend our analysis to the models with \(p\) even, and in fact, to general (nonunitary) models \(\mathcal{SM}(p/q)\). Also, a similar analysis should be possible for \(N = 2\) superconformal models, which are important for superstring compactifications with spacetime supersymmetry [26, 27].

Acknowledgments

I thank C. Ahn for his collaboration at an early stage of this work, C. Efthimiou for bringing references [17, 23] to my attention, and S. Apikyan for helpful correspondence. This work was supported in part by the National Science Foundation under Grant PHY-9870101.
A  Ising model

Although the critical Ising model (i.e., the conformal minimal model $\mathcal{M}(3/4)$) does not have superconformal symmetry, it does have $NS$ and $R$ sectors. Here we work out explicitly how these sectors “transform” between the open and closed channels of the cylinder. Because the Ising model is a free-field theory, the computations are particularly simple. Nevertheless, this exercise is useful, since it gives insight into how to treat the sectors of a superconformal model. Although the Ising model on a cylinder has already been studied extensively [5, 28, 29, 30], this particular aspect does not seem to have been emphasized before.

The critical two-dimensional Ising model (IM) is described by a free Majorana spinor field, whose two components we denote by $\psi(x, y)$ and $\bar{\psi}(x, y)$. We consider this model on the cylinder shown in Figure 1, with $x \in [0, L]$ the coordinate along the axis, and $y \in [0, R]$ the coordinate along the circumference.

A.1  Open channel

In the open channel, we regard $x$ as the space coordinate and $y$ as the time coordinate. The time coordinate thus has period $R$, corresponding to the temperature $T = 1/R$. The conformally-invariant spatial boundary conditions (BC) are [28]

\begin{align}
\psi(0, y) + a i \bar{\psi}(0, y) &= 0 \\
\psi(L, y) - b i \bar{\psi}(L, y) &= 0 ,
\end{align}

(A.1)

where $a, b = +1$ corresponds to “fixed” BC, and $a, b = -1$ corresponds to “free” BC.

Our first task is to find appropriate mode expansions for the fields $\psi$ and $\bar{\psi}$. To this end, we recall that the overall relative sign between these fields is a matter of convention. We can therefore redefine $\bar{\psi}(x, y)$ such that

\begin{equation}
\psi(0, y) = i \bar{\psi}(0, y) ,
\end{equation}

(A.2)

which implies

\begin{equation}
\psi(L, y) = -i \frac{b}{a} \bar{\psi}(L, y) .
\end{equation}

(A.3)

Proceeding as in the case of the superstring [6], we extend the definition of $x$ to $[-L, L]$ and define the new field

\begin{align*}
\Psi(x, y) &= \begin{cases} 
\psi(x, y) & \text{if } x \in [0, L] \\
\bar{\psi}(-x, y) & \text{if } x \in [-L, 0] 
\end{cases} 
\end{align*}

(A.4)
This definition is consistent by virtue of Eq. (A.2). It follows that $\Psi(x, y)$ obeys the (quasi)periodicity condition

$$\Psi(L, y) = -\frac{b}{a} \Psi(-L, y).$$  \hspace{1cm} (A.5)

Thus, $\Psi$ is periodic ($R$) if $a = -b$, and $\Psi$ is antiperiodic ($NS$) if $a = b$. Note that a given set $(a, b)$ of boundary conditions is compatible with only one ($R$ or $NS$) sector. The field $\Psi$ has the standard mode expansion

$$\Psi(x, y) = \frac{1}{\sqrt{2L}} \sum_k b_k e^{-i\frac{\pi}{L} k(x+iy)}, \quad \{b_k, b_l\} = \delta_{k+l,0},$$ \hspace{1cm} (A.6)

with $k \in \mathbb{Z}$ for $R$, and $k \in \mathbb{Z} + \frac{1}{2}$ for $NS$. It follows that the sought-after mode expansions for $\psi$ and $\bar{\psi}$ are

$$\psi(x, y) = \frac{1}{\sqrt{2L}} \sum_k b_k e^{-i\frac{\pi}{L} k(x+iy)},$$

$$\bar{\psi}(x, y) = -\frac{i}{\sqrt{2L}} \sum_k b_k e^{-i\frac{\pi}{L} k(x+iy)}.$$  \hspace{1cm} (A.7)

There is only one independent set of modes $\{b_k\}$ in the open channel.

The Hamiltonian $H_{open}$ is

$$H_{open} = \frac{\pi}{L} \left( e_0 + \sum_{k>0} kb_{-k} b_k \right) = \frac{\pi}{L} \left( L_0 - \frac{c}{24} \right),$$ \hspace{1cm} (A.8)

with

$$e_0^{NS} = -\frac{1}{48}, \quad e_0^R = \frac{1}{24},$$ \hspace{1cm} (A.9)

and $c = \frac{1}{2}$. Standard computations give the partition functions

$$\text{tr}_{NS} e^{-R H_{open}} = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left( 1 + q^{\frac{1}{2}+n} \right),$$

$$\text{tr}_{NS} (-1)^F e^{-R H_{open}} = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left( 1 - q^{\frac{1}{2}+n} \right),$$

$$\text{tr}_{R} e^{-R H_{open}} = 2q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n),$$

$$\text{tr}_{R} (-1)^F e^{-R H_{open}} = 0,$$ \hspace{1cm} (A.10)
where \( q = e^{-\pi R/L} \) and \( F \) is the Fermion number operator. For the case of the IM, the Virasoro algebra has three irreducible representations with highest weights 0, 1/2, 1/16; the corresponding characters (1.5) are given by (see, e.g., [2])

\[
\chi_0(q) = \frac{1}{2} q^{-\frac{1}{4}} \left( \prod_{n=0}^{\infty} \left( 1 + q^{\frac{1}{2}+n} \right) + \prod_{n=0}^{\infty} \left( 1 - q^{\frac{1}{2}+n} \right) \right),
\]

\[
\chi_{\frac{1}{2}}(q) = \frac{1}{2} q^{-\frac{1}{4}} \left( \prod_{n=0}^{\infty} \left( 1 + q^{\frac{1}{2}+n} \right) - \prod_{n=0}^{\infty} \left( 1 - q^{\frac{1}{2}+n} \right) \right),
\]

\[
\chi_{\frac{1}{16}}(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n),
\] (A.11)

The partition functions therefore have the following expressions in terms of characters

\[
\text{tr}_{NS} e^{-RH_{\text{open}}} = \chi_0(q) + \chi_{\frac{1}{2}}(q),
\]

\[
\text{tr}_{NS}(-1)^F e^{-RH_{\text{open}}} = \chi_0(q) - \chi_{\frac{1}{2}}(q),
\]

\[
\text{tr}_R e^{-RH_{\text{open}}} = 2 \chi_{\frac{1}{16}}(q).
\] (A.12)

The modular transformation law (1.6) for the characters, together with the explicit modular \( S \) matrix for the case of the IM (see, e.g. [5]), imply

\[
\chi_0(q) + \chi_{\frac{1}{2}}(q) = \chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q}),
\]

\[
\chi_0(q) - \chi_{\frac{1}{2}}(q) = \sqrt{2} \chi_{\frac{1}{16}}(\tilde{q}),
\]

\[
\chi_{\frac{1}{16}}(q) = \frac{1}{\sqrt{2}} \left( \chi_0(\tilde{q}) - \chi_{\frac{1}{2}}(\tilde{q}) \right),
\] (A.13)

where \( \tilde{q} = e^{-4\pi L/R} \). We conclude that the partition functions are given by

\[
\text{tr}_{NS} e^{-RH_{\text{open}}} = \chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q}),
\]

\[
\text{tr}_{NS}(-1)^F e^{-RH_{\text{open}}} = \sqrt{2} \chi_{\frac{1}{16}}(\tilde{q}),
\]

\[
\text{tr}_R e^{-RH_{\text{open}}} = \sqrt{2} \left( \chi_0(\tilde{q}) - \chi_{\frac{1}{2}}(\tilde{q}) \right),
\]

\[
\text{tr}_R(-1)^F e^{-RH_{\text{open}}} = 0.
\] (A.14)

### A.2 Closed channel

In the closed channel, we regard \( x \) as the time coordinate and \( y \) as the space coordinate. Since \( y \) is periodic, the fields \( \psi \) and \( \bar{\psi} \) can be either periodic (\( R \)) or anti-periodic (\( NS \)).
These fields have the standard mode expansions
\[ \psi(x,y) = \frac{1}{\sqrt{R}} \sum_k a_k e^{-\frac{2\pi}{R} k(x-iy)} , \quad \{a_k,a_l\} = \delta_{k+l,0} , \]
\[ \bar{\psi}(x,y) = \frac{1}{\sqrt{R}} \sum_k \bar{a}_k e^{-\frac{2\pi}{R} k(-y+ix)} , \quad \{\bar{a}_k,\bar{a}_l\} = \delta_{k+l,0} , \quad \{a_k,\bar{a}_l\} = 0 , \tag{A.15} \]
with \( k \in \mathbb{Z} \) for \( R \), and \( k \in \mathbb{Z} + \frac{1}{2} \) for NS. There are two independent sets of modes in the closed channel.

The Hamiltonian \( H_{\text{closed}} \) is
\[ H_{\text{closed}} = \frac{2\pi}{R} \left( 2e_0 + \sum_{k>0} k (a_{-k} a_k + \bar{a}_{-k} \bar{a}_k) \right) = \frac{2\pi}{R} \left( L_0 + L_0 - \frac{c}{12} \right) , \tag{A.16} \]
where again \( k \in \mathbb{Z} \) for \( R \), \( k \in \mathbb{Z} + \frac{1}{2} \) for NS, and \( e_0 \) is given in (A.9).

The boundary conditions (A.2), (A.3) now correspond to initial and final conditions on states. Expressing these conditions in terms of modes, we are led to define (up to normalization) the boundary kets \( |B_{\gamma}\rangle \) and the corresponding bras \( \langle B_{\gamma}| \) as the solutions of the constraints [8]
\[ (a_k - i\gamma \bar{a}_{-k}) |B_{\gamma}\rangle = 0 , \quad \langle B_{\gamma}| (a_{-k} + i\gamma \bar{a}_k) = 0 , \tag{A.17} \]
where \( \gamma = \pm1 \). The solutions in the NS sector are given by
\[ |B_{\gamma}^{NS}\rangle = e^{i\gamma \sum_{k=1}^{\infty} a_{-k} \bar{a}_k} |0\rangle , \quad \langle B_{\gamma}^{NS}| = \langle 0| e^{-i\gamma \sum_{k=1}^{\infty} \bar{a}_k a_k} , \tag{A.18} \]
where the NS vacuum state \( |0\rangle \) satisfies \( a_{-k} |0\rangle = 0 \), \( \bar{a}_{-k} |0\rangle = 0 \) for \( k > 0 \). These states have even Fermion parity
\[ (-1)^F |B_{\gamma}^{NS}\rangle = |B_{\gamma}^{NS}\rangle , \tag{A.19} \]
where \( F \) is now the total Fermion number operator in the NS sector,
\[ F = \sum_{k=1}^{\infty} (a_{-k} a_k + \bar{a}_{-k} \bar{a}_k) . \tag{A.20} \]

The solutions of (A.17) in the \( R \) sector are given by
\[ |B_{\gamma}^{R}\rangle = e^{i\gamma \sum_{k=1}^{\infty} a_{-k} \bar{a}_k} |\gamma\rangle , \quad \langle B_{\gamma}^{R}| = \langle \gamma| e^{-i\gamma \sum_{k=1}^{\infty} \bar{a}_k a_k} , \tag{A.21} \]
where the degenerate \( R \) vacuum states \( |\pm\rangle \) satisfy \( a_{-k} |\pm\rangle = 0 \), \( \bar{a}_{-k} |\pm\rangle = 0 \) for \( k > 0 \), as well as
\[ (a_0 - i\gamma \bar{a}_0) |\gamma\rangle = 0 . \tag{A.22} \]
An explicit representation for the zero modes is (see, e.g., [2, 30])

\[ a_0|\pm\rangle = \frac{1}{\sqrt{2}} e^{\pm i \frac{\pi}{4}} |\mp\rangle, \]

\[ \bar{a}_0|\pm\rangle = \frac{1}{\sqrt{2}} e^{\mp i \frac{\pi}{4}} |\mp\rangle, \] (A.23)

using which one can readily verify (A.22). Moreover, \((2ia_0\bar{a}_0)|\pm\rangle = \pm|\pm\rangle\). The total Fermion parity operator \((-1)^F\) in the \(R\) sector is given by

\[ (-1)^F = (2ia_0\bar{a}_0) e^{i\pi \sum_{k=1}^{\infty} (a_{-k}\bar{a}_k + \bar{a}_{-k}a_k)}, \] (A.24)

(we choose the sign so that \(|+\rangle\) has \((-1)^F = 1\)), and thus, the boundary states satisfy

\[ (-1)^F |B^R_\pm\rangle = \pm |B^R_\pm\rangle. \] (A.25)

Using standard techniques, we find

\[ \langle B^NS_\pm | e^{-L H_{\text{closed}}} | B^NS_\pm \rangle = \tilde{q}^{\frac{1}{16}} \prod_{n=0}^{\infty} \left( 1 + \tilde{q}^{\frac{1}{2}+n} \right) = \chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q}), \]

\[ \langle B^NS_\pm | e^{-L H_{\text{closed}}} | B^NS_\pm \rangle = \tilde{q}^{\frac{1}{16}} \prod_{n=0}^{\infty} \left( 1 - \tilde{q}^{\frac{1}{2}+n} \right) = \chi_0(\tilde{q}) - \chi_{\frac{1}{2}}(\tilde{q}), \]

\[ \langle B^R_\pm | e^{-L H_{\text{closed}}} | B^R_\pm \rangle = \tilde{q}^{\frac{1}{16}} \prod_{n=1}^{\infty} (1 + \tilde{q}^n) = \chi_{\frac{1}{16}}(\tilde{q}), \]

\[ \langle B^R_\pm | e^{-L H_{\text{closed}}} | B^R_\pm \rangle = 0. \] (A.26)

Recalling the results (A.14) from the open channel, we obtain the sought-after relations

\[ \text{tr}_{\text{NS}} e^{-R H_{\text{open}}} = \langle B^NS_\pm | e^{-L H_{\text{closed}}} | B^NS_\pm \rangle, \]

\[ \text{tr}_{\text{NS}} (-1)^F e^{-R H_{\text{open}}} = \sqrt{2} \langle B^R_\pm | e^{-L H_{\text{closed}}} | B^R_\pm \rangle, \]

\[ \text{tr}_{\text{R}} e^{-R H_{\text{open}}} = \sqrt{2} \langle B^NS_\pm | e^{-L H_{\text{closed}}} | B^NS_\pm \rangle, \]

\[ \text{tr}_{\text{R}} (-1)^F e^{-R H_{\text{open}}} = 0 = \langle B^R_\pm | e^{-L H_{\text{closed}}} | B^R_\pm \rangle, \] (A.27)

which show explicitly how the \(NS\) and \(R\) sectors “transform” between the open and closed channels of the cylinder. Similar results are known in string theory.

\[ ^5\text{Numerical factors appear in these relations because the \(NS\) and \(R\) sectors are not irreducible representations of the Virasoro algebra and also the states \(|B^NS_\pm\rangle\) are not properly normalized. See Eqs. (A.28) and (A.30) below.} \]
We conclude this subsection by noting that the boundary states (A.18), (A.21) are closely related to the Ishibashi states (1.2). Namely,

\[ |0\rangle = \frac{1}{2} (|B_{+}^{NS}\rangle + |B_{-}^{NS}\rangle), \]
\[ \frac{1}{2}\rangle = \frac{1}{2} (|B_{+}^{NS}\rangle - |B_{-}^{NS}\rangle), \]
\[ \frac{1}{16}\rangle = |B_{\pm}^{R}\rangle. \quad (A.28) \]

Indeed, recalling that the boundary states satisfy (A.17) and that

\[ L_n = \frac{1}{2} \sum_{k} (k + \frac{n}{2}) : a_{-k} a_{n+k} :; \quad \bar{L}_n = \frac{1}{2} \sum_{k} (k + \frac{n}{2}) : \bar{a}_{-k} \bar{a}_{n+k} :, \quad (A.29) \]

one can easily show that the boundary states satisfy the constraint (1.1). Moreover, expanding the exponentials in the expressions (A.18), (A.21) and comparing the leading terms with (1.2), one can infer (A.28). Regarding the Ishibashi states as orthonormal vectors \((i, j) = \delta_{ij}\), it follows from (A.28) that the boundary states have the normalization

\[ (B_{\pm}^{NS}, B_{\pm}^{NS}) = 2, \quad (B_{\pm}^{R}, B_{\pm}^{R}) = 1. \quad (A.30) \]

Strictly speaking, the Ishibashi states and boundary states \(|B_{\pm}\rangle\) are not normalizable. However, one can define an inner product [10, 5] and argue

\[ \frac{(B_{\pm}^{NS}, B_{\pm}^{NS})}{(0, 0)} = \lim_{q \to 1} \frac{\langle B_{\pm}^{NS}|q^{L_{0} + \bar{L}_{0}}|B_{\pm}^{NS}\rangle}{\langle 0|q^{L_{0} + \bar{L}_{0}}|0\rangle} = \lim_{q \to 1} \frac{\chi_{0}(q^{2}) + \chi_{1}(q^{2})}{\chi_{0}(q^{2})} = 2. \quad (A.31) \]

### A.3 Consistent boundary states

Finally, it is also instructive to rederive Cardy’s results for the consistent IM boundary states in our basis \(|B_{\pm}\rangle\). We begin by rewriting the fundamental consistency constraint (1.3) as

\[ \text{tr}_{NS} \frac{1}{2} (1 + (-1)^{F}) e^{-RH_{open}^{NS}} + \text{tr}_{NS} \frac{1}{2} (1 - (-1)^{F}) e^{-RH_{open}^{NS}} + \text{tr}_{R} \frac{1}{2} (1 + (-1)^{F}) e^{-RH_{open}^{R}} = \langle \alpha|e^{-LH_{closed}} |\beta\rangle. \quad (A.32) \]

From the results (A.12), it is evident that

\[ \text{tr}_{NS} \frac{1}{2} (1 + (-1)^{F}) e^{-RH_{open}^{NS}} = N_{0}^{0\alpha\beta} \chi_{0}(q), \]
\[ \text{tr}_{NS} \frac{1}{2} (1 - (-1)^{F}) e^{-RH_{open}^{NS}} = N_{0}^{1\alpha\beta} \chi_{1}(q), \]
\[ \text{tr}_{R} \frac{1}{2} (1 + (-1)^{F}) e^{-RH_{open}^{R}} = N_{1}^{\frac{1}{2}} \chi_{\frac{1}{2}}(q), \quad (A.33) \]
and so the LHS of (A.32) is indeed equal to \( \sum_i N_{i\alpha\beta} \chi_i(q) \). In the RHS of (A.32), we expand the boundary states in the basis \( |B_\pm\rangle \) using

\[
|\alpha\rangle = \frac{1}{2} \left( |B^+_{NS}\rangle \langle B^+_{NS}|\alpha\rangle + |B^-_{NS}\rangle \langle B^-_{NS}|\alpha\rangle \right) + |B^R_{+}\rangle \langle B^R_{+}|\alpha\rangle ,
\]

(A.34)

keeping in mind the normalization (A.30). Then, making use also of the relations (A.27), we arrive at the Cardy equations

\[
N^0_{\alpha\beta} = \frac{1}{4} \langle \alpha | B^+_{NS}\rangle \langle B^+_{NS}| \beta \rangle + \frac{1}{4} \langle \alpha | B^-_{NS}\rangle \langle B^-_{NS}| \beta \rangle + \frac{1}{2} \sqrt{2} \langle \alpha | B^R_{+}\rangle \langle B^R_{+}| \beta \rangle ,
\]

\[
N^{1/2}_{\alpha\beta} = \frac{1}{4} \langle \alpha | B^+_{NS}\rangle \langle B^+_{NS}| \beta \rangle + \frac{1}{4} \langle \alpha | B^-_{NS}\rangle \langle B^-_{NS}| \beta \rangle - \frac{1}{2} \sqrt{2} \langle \alpha | B^R_{+}\rangle \langle B^R_{+}| \beta \rangle ,
\]

\[
N^{1/16}_{\alpha\beta} = \frac{1}{2\sqrt{2}} \langle \alpha | B^+_{NS}\rangle \langle B^-_{NS}| \beta \rangle + \frac{1}{2\sqrt{2}} \langle \alpha | B^-_{NS}\rangle \langle B^+_{NS}| \beta \rangle .
\]

(A.35)

Following Cardy [5], we define the states \( |k\rangle \) by \( N^k_{0\alpha} = \delta^k_\alpha \), and we obtain

\[
|0\rangle = \frac{1}{\sqrt{2}} |B^-_{NS}\rangle + \frac{1}{\sqrt{2}} |B^R_{+}\rangle ,
\]

\[
|\frac{1}{2}\rangle = \frac{1}{\sqrt{2}} |B^-_{NS}\rangle - \frac{1}{\sqrt{2}} |B^R_{+}\rangle ,
\]

\[
|\frac{1}{16}\rangle = |B^+_{NS}\rangle .
\]

(A.36)

These states correspond to the boundary conditions “fixed +”, “fixed −”, and “free”, respectively. The \( g \)-factor [12] of a boundary state \( |\alpha\rangle \) is given by

\[
g_\alpha = \langle \langle 0 | \alpha \rangle = \frac{1}{2} \left( \langle B^+_{NS}\rangle | + \langle B^-_{NS}\rangle | \right) \langle \alpha \rangle .
\]

(A.37)

We therefore obtain (again remembering the normalization (A.30)) the well-known results

\[
g_0 = g_{\frac{1}{2}} = \frac{1}{\sqrt{2}} , \quad g_{\frac{1}{16}} = 1 .
\]

(A.38)

References


