Bell’s theorem for general N-qubit states

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We obtain a single general Bell inequality that summarizes all possible Bell type constrains on the correlation function for the N-particle system. This is for the case where measurements on each particle can be chosen between two arbitrary dichotomic observables. We also obtain a simple condition for an arbitrary N-qubit mixed state to violate that general Bell inequality.

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A. Introduction

Bell type inequalities [1–5] are bounds on certain combinations of statistical correlations for measurements on multi-particle systems which can be understood within a realistic picture based on local properties of each individual particle. In such a picture particles carry properties prior to and independent of observation that are performed on other particle(s) at space-like separation. Quantum mechanics predicts violation of the inequalities even if the individual particles are separated. This is known as Bell’s theorem in quantum mechanics. However, the problem which states violate Bell type inequalities has been solved only for pure states [6] and for the mixed states only in the most simple case of two-qubit systems [7].

Here we present a generalized single Bell inequality that is a necessary and sufficient condition for local realistic description of a correlation function for the N-particle system, for the case where the measurements on each particle are chosen between two dichotomic observables. From this inequality we obtain, for example, Mermin-Klyshko [4,5] inequality as a particular case. We also formulate the condition for an arbitrary N-qubit mixed state to violate the general Bell inequality. The presentation will be given for N = 3. The general proof for arbitrary N follows exactly the same line of thought.

B. Necessary and sufficient condition for violation of local realism

In a hidden variable theory one assumes that for every individual particle the result of the measurement of an observable is predetermined by hidden variables. In a local hidden variable it is furthermore assumed that this predetermined result does not depend on which other observables are measured simultaneously on space-like separated systems, but only on the local experimental setting and the hidden variables. This implies the existence of a function $I_j(\vec{n}(j), \lambda)$ taking values -1 and 1 which describes the result of a measurement with the setting $\vec{n}(j)$ on a system $j$ characterized by the hidden variable $\lambda$. Then, by the construction the three qubit correlation function is (see e.g. [1], [3])

$$E_{LHV}(\vec{n}(1), \vec{n}(2), \vec{n}(3)) = \int d\lambda \rho(\lambda) I_1(\vec{n}(1), \lambda) I_2(\vec{n}(2), \lambda) I_3(\vec{n}(3), \lambda),$$

(1)

where $\rho(\lambda)$ is the probabilistic distribution over the set of values of local hidden variables.

Analogous to the standard cases of the Bell [1] and GHZ theorems [3] we allow each observer of the particle $j$ to choose between two local settings $\vec{n}(x)^1$ and $\vec{n}(x)^2$. Now, the correlation function for the chosen settings is given by

$$E_{LHV}(\vec{n}(1)^k_1, \vec{n}(2)^k_2, \vec{n}(3)^k_3) = \int d\lambda \rho(\lambda) I_1(\vec{n}(1)^k_1, \lambda) I_2(\vec{n}(2)^k_2, \lambda) I_3(\vec{n}(3)^k_3, \lambda)$$

(2)

with $k_1, k_2, k_3 = 1, 2$. It is important to stress that one must consider arbitrary local hidden variable correlation functions. The only constraint being their structure given by (2).

One can treat the full set of the local hidden variable predictions for correlations as a three index tensor $E_{LHV}$, with the indices $k_1, k_2, k_3 = 1, 2$. One immediately notices that (2) implies that $E_{LHV}$ is built out of the tensorial...
products of two dimensional real vectors $v^1_x = (I_x(i(x)^1, \lambda), J_x(i(x)^2, \lambda))$ with $x = 1, 2, 3$ each representing the two possible results of a given observer for the given value of the hidden variable:

$$\hat{E}_{LHV} = \int d\lambda p(\lambda)v_1^* \otimes v_2^* \otimes v_3^*. \tag{3}$$

The actual values of the components of the two dimensional vectors $v^1_x$ can be equal to only either $(1, 1)$, or $(1, -1)$, or $(-1, 1)$, or finally $(-1, -1)$. Let us denote these four possible vectors by $v(x)^j$ with $j = 1, 2, 3, 4$, respectively. Thus, the local hidden variable correlation function (tensor) can be simplified to a discrete sum over hidden probabilities $p_{k, l, m}$ of the tensorial products of all possible measurement results

$$\hat{E}_{LHV} = \sum_{k, l, m=1}^{4} p_{k, l, m} v(1)^k \otimes v(2)^l \otimes v(3)^m. \tag{4}$$

A further simplification of the tensor is possible since $(-1, -1) = (1, 1)$ and $(-1, 1) = (1, -1)$, or in other words $v(x)^{k+2} = -v(x)^k$. The set of tensorial products $v(1)^k \otimes v(2)^l \otimes v(3)^m$ with $k, l, m = 1, 2$ forms a complete orthogonal (product) basis in the (real) Hilbert space of tensors $R^2 \otimes R^2 \otimes R^2$. One can thus rewrite the expansion (4) so that it becomes an expansion in terms of the mentioned basis

$$\hat{E}_{LHV} = \sum_{k, l, m=1, 2} c_{k, l, m} v(1)^k \otimes v(2)^l \otimes v(3)^m. \tag{5}$$

The relation between the coefficients in Eq. (5) and the probabilities of Eq. (4) is given by

$$c_{k, l, m} = p_{k, l, m} - p_{k+2, l, m} - p_{k, l+2, m} - \cdots + p_{k+2, l+2, m} + \cdots - p_{k+2, l+2, m+2}. \tag{6}$$

The expansion coefficients are of course unique, and since $\sum_{k, l, m=1}^{4} p_{k, l, m, n} = 1$ they satisfy the following obvious inequality

$$\sum_{k, l, m=1, 2} |c_{k, l, m}| \leq 1. \tag{7}$$

Please note that

$$c_{k, l, m} = 2^{-3}\langle \hat{E}_{LHV}, v(1)^k \otimes v(2)^l \otimes v(3)^m \rangle, \tag{8}$$

where $\langle ... \rangle$ denotes a scalar product in $R^2 \otimes R^2 \otimes R^2$. The factor $2^{-3}$ is due to the fact that the square of the norm of vectors $v$ is 2. Therefore the necessary condition for existence of a local realistic description for any three particle correlation function is given by

$$\sum_{k, l, m=1, 2} |\langle \hat{E}, v(1)^k \otimes v(2)^l \otimes v(3)^m \rangle| \leq 2^3, \tag{9}$$

where $\hat{E}$ is a tensor built out of elements $E(i(1)^k, i(2)^k, i(3)^k)$.

Let us move to the proof of the sufficiency of the condition (9). We shall obtain this by directly constructing a local hidden variable model with the aid of the positive real numbers which are summed up in (9). Simply if (9) holds then

$$\hat{E} = \sum_{k, l, m=1}^{4} p^{'}_{k, l, m} (\pm v(1)^k \otimes v(2)^l \otimes v(3)^m) + \frac{1}{4^3}(1 - p^{'}) \sum_{k, l, m=1}^{4} v(1)^k \otimes v(2)^l \otimes v(3)^m, \tag{10}$$

where $p^{'}_{k, l, m} = |c_{k, l, m}|$, and $p^{'} = \sum_{k, l, m=1, 2} |c_{k, l, m}|$. The sign in $\pm v(1)^k$ is that of $c_{k, l, m}$. Note that the latter implies that either $v(1)^k$ or $v^{k+2}(1)$ enters the given tensorial product. The contribution of the last term in Eq. (10) is exactly zero. Therefore the Eq. (10) is a hidden variable model for $\hat{E}$. Note that, if required, the model can be refined to give correct statistics of counts at local observation stations, for example if one wants each of the terms to give for each local settings with equal probability $\pm 1$ one can replace them by
\[
\frac{1}{7}F_{k,l,m} \left[ \mathbf{v}(1)^k \otimes \mathbf{v}(2)^l \otimes \mathbf{v}(3)^m + (-\mathbf{v}(1)^k) \otimes (-\mathbf{v}(2)^l) \otimes \mathbf{v}(3)^m
\right.
+ (-\mathbf{v}(1)^k) \otimes \mathbf{v}(2)^l \otimes (-\mathbf{v}(3)^m) + \mathbf{v}(1)^k \otimes (-\mathbf{v}(2)^l) \otimes (-\mathbf{v}(3)^m) \right].
\]

In the case of \(N\)-qubit correlations one builds a tensor \(\hat{E}\) out of \(E(\bar{n}(1)^{k_1}, \bar{n}(2)^{k_2}, ..., \bar{n}(N)^{k_N})\), and the necessary and sufficient condition for existence of a local realistic model for any \(N\) qubit correlation function \(E\) is given by

\[
\sum_{k_1,k_2,\ldots,k_N=1,2} \left| \left\langle \hat{E}, \mathbf{v}(1)^{k_1} \otimes \mathbf{v}(2)^{k_2} \ldots \otimes \mathbf{v}(N)^{k_N} \right\rangle \right| \leq 2^N.
\]

The condition (9) can be rewritten in the following way (recall that the components of the \(\mathbf{v}(x)^k\) vectors, for arbitrary qubit number \(x\), are \(v(x)^1_k = v(x)^2_k = 1\) and \(v(x)^3_k = -v(x)^2_k = 1\))

\[
\sum_{k,l,m=1,2} \left| \sum_{k_1,k_2,k_3=1} E_{k_1,k_2,k_3} v^k_{k_1} v^l_{k_2} v^m_{k_3} \right| \leq 2^4,
\]

where \(E_{k_1,k_2,k_3} = E_{LHV}(\bar{n}(1)^{k_1}, \bar{n}(2)^{k_2}, \bar{n}(3)^{k_3})\), and we have dropped the number of particle, \(x\), from the symbols denoting the coordinates of the \(v\) vectors. The inequality implies that for any sign function \(s(k, l, m) = \pm 1\) one has

\[
\left| \sum_{k,l,m=1,2} s(k,l,m) \sum_{k_1,k_2,k_3=1} E_{k_1,k_2,k_3} v^k_{k_1} v^l_{k_2} v^m_{k_3} \right| \leq 2^3.
\]

Therefore the Eq. (14) forms a full set of Bell-Mermin type inequalities where each inequality is defined by a specific sign function. For example, it is easy to show that for a choice of \(s(k, l, m)\) in the form of \(s(1,1,1) = s(2,2,2) = -1\) and \(s(1,2,2) = 1\) for all other \(k, l, m\), one obtains

\[
|E_{122} + E_{212} + E_{222} - E_{221}| \leq 2.
\]

This is one of the standard Bell-Mermin inequalities, which for an appropriate choice of local settings leads to a GHZ-type contradiction (what in terms of the inequality means that the value of the left hand side reaches 4, which is the maximal possible value for any, not only quantum, correlation function).

### C. \(N\)-qubit states that violate local realism

In this section we present the necessary and sufficient condition for arbitrary (pure as well as mixed) states to violate the condition (9). Again the presentation will be given for \(N = 3\), and the generalization to arbitrary \(N\) is obvious.

We describe the general three qubit state in the form

\[
\hat{\rho} = 2^{-3} \sum_{i,j,k=0}^3 T_{ijk} (\hat{\sigma}_i^1 \otimes \hat{\sigma}_j^2 \otimes \hat{\sigma}_k^3).
\]

where \(\hat{\sigma}_k^x\) stands for \(\hat{\sigma}_0^x = I\) and the three Pauli matrices for the given qubit, and \(T_{ijk}\) is a set of real coefficients and \(T_{000} = 1\).

Then the three qubit correlation function for a Bell-GHZ type experiment is given by

\[
E_{QM}(\bar{n}(1), \bar{n}(2), \bar{n}(3)) = \text{Tr} \left[ \hat{\rho} (\hat{n}(1) \cdot \hat{\sigma}_1^1 \otimes \hat{n}(2) \cdot \hat{\sigma}_2^2 \otimes \hat{n}(3) \cdot \hat{\sigma}_3^3) \right],
\]

and it reads

\[
E_{QM}(\bar{n}(1), \bar{n}(2), \bar{n}(3)) = \sum_{i,j,k=1}^3 T_{ijk} n(1)_i n(2)_j n(3)_k.
\]
I.e. the three particle correlations are fully defined once one knows the components of $T_{ijk}$, $i,j,k = 1,2,3$, of the tensor $\hat{T}$ which belongs to the linear space $R^3 \otimes R^3 \otimes R^3$. The equation (18) can be written down in a more compact way as

$$E_{QM}(\vec{n}(1), \vec{n}(2), \vec{n}(3)) = \hat{T} \cdot \vec{n}(1) \otimes \vec{n}(2) \otimes \vec{n}(3).$$  \hspace{1cm} (19)$$

where "$\cdot$" denotes now the scalar product in the space $R^3 \otimes R^3 \otimes R^3$.

From Eq. (13) one knows that $E_{QM}(\vec{n}(1), \vec{n}(2), \vec{n}(3))$ can be described by a local realistic model if and only if

$$\sum_{k,l,m=1,2} |\sum_{k_1,k_2,k_3=1}^2 v_{k_1}^k v_{k_2}^l v_{k_3}^m \hat{T} \cdot \vec{n}(1)^{k_1} \otimes \vec{n}(2)^{k_2} \otimes \vec{n}(3)^{k_3}| \leq 2^3,$$

for any choice of the settings $\vec{n}(1)^{k_1}$, $\vec{n}(2)^{k_2}$ and $\vec{n}(3)^{k_3}$ where $k_1, k_2, k_3 = 1, 2$. This can be transformed into

$$2^{-3} \sum_{k,l,m=1,2} |\hat{T} \cdot (\vec{n}(1)^1 + (-1)^k \vec{n}(1)^2) \otimes (\vec{n}(2)^1 + (-1)^l \vec{n}(2)^2) \otimes (\vec{n}(3)^1 + (-1)^m \vec{n}(3)^2)| \leq 1.$$

(21)

Now since there always exist two mutually orthogonal unit vectors $\vec{a}(x)^1$ and $\vec{a}(x)^2$ such that

$$\vec{n}(x)^1 + (-1)^k \vec{n}(x)^2 = 2\alpha(x)_k \vec{a}(x)^k \hspace{0.5cm} k = 1, 2$$

with $\alpha(x)_1 = \cos \theta(x)$ and $\alpha(x)_2 = \sin \theta(x)$, one gets

$$\sum_{k,l,m=1,2} |\alpha(1)_k \alpha(2) l \alpha(3)_m \hat{T} \cdot \vec{a}(1)^{k} \otimes \vec{a}(2)^{l} \otimes \vec{a}(3)^{m}| \leq 1.$$  \hspace{1cm} (23)

Note that $\hat{T} \cdot \vec{a}(1)^k \otimes \vec{a}(2)^l \otimes \vec{a}(3)^m$ is a component of the tensor $\hat{T}$ after a transformation of the local coordinate systems of each of the particles into such ones where the two first basis vectors are $\vec{a}(x)^1$ and $\vec{a}(x)^2$. We shall denote such transformed components again by $T_{klm}$.

Therefore the necessary and sufficient conditions for a three-qubit correlation to be described within a local realistic model is that in any plane of observations for each particle (which are defined by the two observation directions) one must have

$$\sum_{k,l,m=1,2} |\alpha(1)_k \alpha(2) l \alpha(3)_m T_{klm}| \leq 1.$$  \hspace{1cm} (24)

for arbitrary $\alpha(1)_k$, $\alpha(2)_l$, $\alpha(3)_m$.

Note that using the Schmidt inequality one gets

$$\sum_{k,l,m=1,2} |\alpha(1)_k \alpha(2)_l \alpha(3)_m T_{klm}| \leq \left(\sum_{k,l,m=1,2} T_{klm}^2\right)^{\frac{1}{2}}.$$  \hspace{1cm} (25)

Therefore, if

$$\sum_{k,l,m=1,2} T_{klm}^2 \leq 1$$  \hspace{1cm} (26)

for any set of local coordinate systems the three particle correlation functions of the form of (18) can be understood within the local realism. Of course, the violation of this condition is a necessary condition for violation of the local realism. Also it is worth mentioning, that this condition applies to any three particle correlation function, which is defined by a tensor in the way of the right hand side of the equation (19).

Since the $N$ qubit correlation function has the following structure

$$E_{QM}(\vec{n}(1), \vec{n}(2), ..., \vec{n}(N)) = \hat{T} \cdot \vec{n}(1) \otimes \vec{n}(2) \otimes ... \otimes \vec{n}(N),$$

(27)
where $\hat{T}$ stands for an $N$ index tensor, with components $T_{k_1,k_2...k_N}$, where $k_i = 1, 2, 3,$, the necessary and sufficient condition for a description of the correlation function within local realism in the general case reads

$$\sum_{k_1,k_2,...,k_N=1,2} |\alpha(1)_{k_1}\alpha(2)_{k_2}...\alpha(N)_{k_N} T_{k_1,k_2...k_N}| \leq 1.$$  

(28)

for any possible choice of local coordinate systems for individual particles. Again if

$$\sum_{k_1,k_2,...,k_N=1,2} T_{k_1,k_2...k_N}^2 \leq 1$$  

(29)

for any set of local coordinate systems, the $N$-qubit correlation function can be described by a local realistic model.

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