The convergence of the derivative expansion of the exact renormalisation group is investigated via the computation of the $\beta$ function of massless scalar $\lambda \phi^4$ theory. The derivative expansion of the Polchinski flow equation converges at one loop for certain fast falling smooth cutoffs. Convergence of the derivative expansion of the Legendre flow equation is trivial at one loop, but also can occur at two loops and in particular converges for an exponential cutoff.

1 Introduction

The derivative expansion of the effective action within the context of the exact renormalisation group has been shown to provide an accurate non-perturbative approximation method for scalar quantum field theory. While this statement rests on empirical fact, it is a challenging task to prove the applicability of the derivative expansion non-perturbatively and in all generality since it is not a controlled expansion in a small parameter. Rather, it results in a numerical series since the approximation lies with neglecting higher powers of $p/\Lambda$ (where $\Lambda$ is the effective cutoff and $p$ some typical momentum) and the flow equations require the contributing typical momentum to be of order $\Lambda$. Thus it is a non-trivial question to ask whether such an expansion converges and, if so, whether it converges to the correct answer. Here we address this question within perturbation theory.

2 Wilson/Polchinski Flow Equation

We define modified propagators $\Delta_{UV} = C_{UV}(q^2/\Lambda^2)$ and $\Delta_{IR} = C_{IR}(q^2/\Lambda^2)$, where $C_{UV}(C_{IR})$ is an as yet unspecified function acting as an UV (IR) cutoff with the properties $C_{UV}(0) = 1$ and $C_{UV} \to 0$ (sufficiently fast) as $q \to \infty$, with $C_{IR} \equiv 1 - C_{UV}$. We write Polchinski’s version of Wilson’s flow equation for the individual vertices as

$$\frac{\partial}{\partial \Lambda} S(p_1, \ldots, p_n; \Lambda) = \sum_{\{I_1, I_2\}} S(-P_1, I_2; \Lambda) \frac{d}{d\Lambda} \Delta_{UV}(P_1) S(P_1, I_2; \Lambda) - \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{d}{d\Lambda} \Delta_{UV}(q) S(q, -q, p_1, \ldots, p_n; \Lambda),$$

where $I_1$ and $I_2$ are disjoint subsets of external momenta such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{p_1, \ldots, p_n\}$. The sum over $\{I_1, I_2\}$ utilises the Bose symmetry so pairs are counted only once i.e. $\{I_1, I_2\} = \{I_2, I_1\}$. The momentum $P_1$ is defined to be $P_1 = \sum_{p_i \in I_1} p_i$. 

1
With the renormalisation condition
\[ S(0,0,0,0;\Lambda) = \lambda \]
imposed and the \( \beta \) function defined as
\[ \beta(\Lambda) \equiv \Lambda \frac{\partial}{\partial \Lambda} \lambda, \]
the only contribution to the \( \beta \) function at one loop takes the form of figure 1. Within
the flow equation, this contribution arises from the tree-level six-point function
with two of its legs tied together.

The \( \beta \) function at one loop is found to be
\[ \beta = 3\lambda^2 \Lambda \int \frac{d^4q}{(2\pi)^4} \frac{d}{d\Lambda} \Delta_{UV}(q^2/\Lambda^2) \left[ \int_{\Lambda}^{\infty} d\Lambda_1 \frac{d}{d\Lambda_1} \Delta_{UV}(q^2/\Lambda_1^2) \right] \]
\[ = \frac{6\lambda^2}{(4\pi)^2} \sum_{n=1}^{\infty} \frac{C_U(n)(0)}{n!} \int_0^\infty dx \ x^n \ C_{UV}(x), \]  

where we have taken the opportunity to perform a derivative expansion. The deriva-
tive expansion corresponds to an expansion in the momentum dependent part of
the six-point function which relates to the term in square brackets in the first line
of (2). This expression is evidently dependent on the exact form of the cutoff
function. Since the sharp cutoff should not be considered within the context of
the Wilson/Polchinski flow equation, we shall restrict ourselves to smooth profiles.

With an exponential cutoff of the form
\[ C_{UV}(q^2/\Lambda^2) = e^{-q^2/\Lambda^2}, \]
problems arise:
\[ \beta = \frac{6\lambda^2}{(4\pi)^2} \sum_{n=1}^{\infty} (-1)^{n+1}, \]

Clearly this fails to converge to the correct value of \( \beta = \frac{3\lambda^2}{(4\pi)^2} \), and indeed exponen-
tials of any power (i.e. \( C_{UV}(x) = e^{-x^n} \)) suffer from the same affliction. However
there are UV cutoffs that can provide the desired convergence. If we consider the
much faster falling \( C_{UV}(x) = \exp(1-e^x) \), we obtain the converging series
\[ \beta = \frac{3\lambda^2}{(4\pi)^2} \{1.193 + 0 - 0.194 - 0.060 + 0.032 + \cdots \}. \]

3 Legendre flow equation at one loop

Irrespective of the exact form of the cutoff, the only contribution to the flow of the
one-loop four-point 1PI is
\[ \frac{\partial}{\partial \Lambda} \Gamma(p_1, p_2, p_3, p_4; \Lambda) = \int \frac{d^4q}{(2\pi)^4} K(q) \]
\[ \times \sum_{\{I_1, I_2\}} \Gamma(q, -q - p_1, I_1; \Lambda) \Delta_{IR}(|q + p_1|) \Gamma(q - p_2, -q, I_2; \Lambda), \]

where the notation is the same as used in (1). Imposing the renormalisation con-
dition \( \Gamma(0,0,0,0;\Lambda) = \lambda \) and substituting the classical vertex \( \lambda \) for the four-point
functions on the right hand side of (5), we find that
\[
\beta = 3\lambda^2 \Lambda \int \frac{d^4q}{(2\pi)^4} \left( \frac{d}{d\Lambda} \Delta_{UV}(q^2/\Lambda^2) \right) \Delta_{IR}(q^2/\Lambda^2) = \frac{3\lambda^2}{(4\pi)^2}. \tag{6}
\]

Note that the result is independent of cutoff function as expected, and exact irrespective of the derivative expansion because the classical four-point vertex carries no external momentum dependence.

4 Sharp cutoff at two loops

Various forms of the sharp cutoff version of the Legendre flow equation expanded in vertices can be found in the literature.\textsuperscript{7,6,3} By the process of iteration, the usual Feynman diagrams can be constructed but with restrictions on the allowed values of momentum for internal propagators.

To ensure that only renormalised quantities are used, we split the four point function into its momentum free \([\lambda(\Lambda)]\) and momentum dependent parts:\textsuperscript{6}
\[
\Gamma(p_1, p_2, p_3, p_4; \Lambda) = \lambda(\Lambda) + \gamma(p_1, p_2, p_3, p_4; \Lambda), \tag{7}
\]
with \(\gamma(0, 0, 0, 0; \Lambda) = 0\). Momentum expanding the one-loop result, we find
\[
\gamma(p_1, p_2, p_3, p_4; \Lambda)
\]
\[
= -\lambda^2 \int_\Lambda^{\infty} d\Lambda_1 \int \frac{d^4q}{(2\pi)^4} \frac{\delta(q - \Lambda_1)}{q^2} \sum_{i=2}^4 \left\{ \frac{\theta(|q + P_i| - \Lambda_1)}{(q + P_i)^2} - \frac{\theta(q - \Lambda_1)}{q^2} \right\} \tag{8}
\]
\[
= +\frac{\lambda^2}{4\pi^3} \sum_{i=2}^4 \left\{ \frac{1}{6} \frac{P_i}{\Lambda} + \frac{1}{720} \left( \frac{P_i}{\Lambda} \right)^3 + \frac{3}{44800} \left( \frac{P_i}{\Lambda} \right)^5 + \cdots \right\}, \tag{9}
\]
It is \(\gamma(p_1, p_2, p_3, p_4; \Lambda)\) that is iterated through the flow equation to obtain the running of the coupling to second loop order.

![Feynman diagrams](image)

Figure 2. Feynman diagrams contributing to the four-point function at two loops.

The three possible topologies of two-loop four-point 1PI diagrams are displayed in figure 2. Topology (a) does not provide a contribution to the \(\beta\) function; in terms of renormalised quantities (a) is already incorporated in the one-loop running \(\lambda(\Lambda)\).

There are two contributions of the form of (c), one arising from the one-loop self energy being inserted into the one-loop four-point function and the other from the one-loop six-point 1PI diagram with two legs at the same vertex joined together. It can be shown that for all types of cutoff function, these two contributions cancel one another. Hence the only contributions we need consider are those of topology...
(b). The first comes from the iteration of the renormalised one-loop four-point function and its contribution to the \( \beta \) function is\(^7\)

\[
- 6 \lambda \Lambda \int \frac{d^4 q}{(2\pi)^4} \frac{\delta(q - \Lambda)}{q^2} \frac{\theta(q - \Lambda)}{q^2} \gamma(q, -q, 0; \Lambda) \\
= \frac{\lambda^3}{(4\pi)^4} \frac{1}{\pi} \left( 8 + \frac{1}{15} + \frac{9}{2800} + \cdots \right). \tag{10}
\]

The next two parts arise from the one-loop six-point 1PI diagram with two legs from different vertices joined up. The first contributes

\[
- 6 \lambda^3 \int \frac{d^4 q}{(2\pi)^4} \frac{\delta(q - \Lambda)}{q^2} \int_{\Lambda}^{\infty} d\Lambda_1 \int \frac{d^4 p}{(2\pi)^4} \frac{\delta(p - \Lambda_1)}{p^2} \frac{\theta^2(|p + q| - \Lambda_1)}{|p + q|^4} \\
= -12 \frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda \pi} \left( \frac{\pi}{2} - \frac{10}{9} - \frac{10}{4} - \frac{63}{100} + \frac{\pi}{6} + \cdots \right), \tag{11}
\]

while the other is

\[
- 12 \lambda^3 \int \frac{d^4 q}{(2\pi)^4} \frac{\delta(q - \Lambda)}{q^2} \int_{\Lambda}^{\infty} d\Lambda_1 \int \frac{d^4 p}{(2\pi)^4} \frac{\delta(p - \Lambda_1)}{p^2} \frac{\theta(|p + q| - \Lambda_1) \theta(p - \Lambda_1)}{|p + q|^2} \\
= -12 \frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda \pi} \left( \frac{\pi}{2} - \frac{1}{9} - \frac{1}{300} - \frac{3}{15680} + \cdots \right). \tag{12}
\]

While both of these series converge, the first only does so very slowly and indeed for a \( O(\partial^2) \) operator (or higher) this diagram would diverge.

The final contribution to the \( \beta \) function at two-loop order takes account of the wavefunction renormalisation due to figure 3 which appears through \( \Sigma(k; \Lambda) \big|_{O(k^2)} = [Z(\Lambda) - 1]k^2 \).

From this we find

\[
k^2 \frac{\partial}{\partial \Lambda} Z(\Lambda) \\
= \lambda^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\delta(q - \Lambda)}{q^2} \int_{\Lambda}^{\infty} d\Lambda_1 \int \frac{d^4 p}{(2\pi)^4} \frac{\delta(p - \Lambda_1)}{p^2} \frac{\theta(|p + q + k| - \Lambda_1)}{|p + q + k|^2} \bigg|_{O(k^2)} \\
= - \frac{\lambda^2 k^2}{(4\pi)^4} \frac{1}{\Lambda \pi} \left( \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \cdots \right). \tag{13}
\]

Altogether, these contributions converge towards the correct two-loop answer for the properly normalised \( \beta \) function

\[
\beta(\Lambda) = \lambda \frac{\partial}{\partial \Lambda} \left( \frac{\lambda(\Lambda)}{Z^2} \right) = 3 \frac{\lambda^2}{(4\pi)^2} - \frac{17}{3} \frac{\lambda^3}{(4\pi)^4}. \tag{14}
\]
5 Smooth cutoffs

When a smooth cutoff is utilised, the Legendre flow equation takes on a slightly different form to its sharp counterpart.\(^7,^3\)

If we consider a power law cutoff \(C_{UV}(q^2/\Lambda^2) = \frac{1}{1+(q/\Lambda)^{2m}}\) (with \(\kappa\) a non-negative integer) the derivative expansion fails at two loops. For instance with the iteration of the one-loop four-point function, we have the following contribution to the two-loop \(\beta\) function:

\[
\sim \frac{\lambda^3}{\Lambda^{2\kappa+3}} \int d^4q \frac{q^{4\kappa}}{[1+(q/\Lambda)^{2\kappa+2}]^3} \int_\Lambda^{\infty} \frac{d\Lambda_1}{\Lambda_1^{2\kappa+3}} \int d^4p \times 
\]

\[
\times \left[ 1 - \frac{1}{1+((q+p)/\Lambda_1)^{2\kappa+2}} \right] \frac{1}{|q+p|^2}. \tag{15}
\]

With a derivative expansion the inner integral is expanded in terms of \(q\), but when the power \(q^{2m}\) is such that \(m \geq \kappa + 1\), the outer integral fails to converge. Hence the coefficients of the derivative expansion are themselves infinite.

However, the situation is much better when an exponential cutoff of the form \(C_{UV}(q^2/\Lambda^2) = e^{-q^2/\Lambda^2}\) is used. The renormalised one-loop four-point function is\(^9\)

\[
\gamma(p_1, p_2, p_3, p_4, \Lambda) = -\frac{\lambda^2}{2(4\pi)^2} \sum_{i=2}^{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)! n} \left( \frac{p_i^2}{2\Lambda_1^2} \right)^n. \tag{16}
\]

When this is iterated through the flow equation, its contribution to the \(\beta\) function is\(^9\)

\[
-12 \frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \left( \frac{1}{2^n} - \frac{1}{2^{n+1}} \right), \tag{17}
\]

which can be shown to sum exactly to \(\frac{6\lambda^3}{(4\pi)^4} \frac{1}{\Lambda} [6 \ln 3 + 4 \ln 2 - 5 \ln 5 - 1]\). The analogue of (11) is

\[
12 \frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda} \left( \ln \frac{4}{3} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \ln \frac{3}{4} - \frac{1}{n} \sum_{s=2}^{n} \frac{(-1)^s}{s} \left( \frac{2}{s} \right)^n \right), \tag{18}
\]

which numerically sums to \(\frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda} (-2.45411725)\), and the equivalent of (12) is

\[
-24 \frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda} \left( \ln \frac{4}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \left\{ \left( \frac{2}{3} \right)^n - \left( \frac{1}{2} \right)^n \right\} \right), \tag{19}
\]

which equates to \(12 \frac{\lambda^3}{(4\pi)^4} \frac{1}{\Lambda} [9 \ln 3 - 2 \ln 2 - 5 \ln 5]\). The flow of the wavefunction renormalisation becomes

\[
\frac{\partial}{\partial \Lambda} Z(\Lambda) = \frac{\lambda^2}{(4\pi)^4} \frac{1}{\Lambda} \sum_{n=2}^{\infty} \frac{(-1)^n}{2^n} = \frac{1}{6} \frac{\lambda^2}{(4\pi)^4} \frac{1}{\Lambda}. \tag{20}
\]

Gathering together these equations, we obtain fast convergence to the correct \(\beta\) function (14).

\(^9\)Similar calculations using the exponential cutoff have been performed.\(^8\)
6 Summary

We have seen that convergence of the derivative expansion is an issue sensitive both to the form of cutoff and to the specific flow equation that is used. Nevertheless, the derivative expansion can converge even for massless field theory, at least to the two loops tested. If the Wilson/Polchinski flow equation is employed, convergence for the one-loop $\beta$ function can only be obtained with the use of very fast falling smooth cutoffs. Convergence at one loop is trivial if the Legendre flow equation is utilised. With a sharp cutoff in the Legendre flow equation, the expansion for the two-loop $\beta$ function also converges but clearly diverges for operators dependent on external momentum, whilst for a simple power law cutoff the expansion ceases to make sense. Finally, fast convergence of the $\beta$ function at two loops was obtained using an exponential cutoff. These results thus shed light on the accuracy seen in the non-perturbative calculations and further work along these lines could be used to bound the accuracy and reliability of practical calculations.

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References