Utility of \textit{su}(1, 1)-Algebra in a Schematic Nuclear 
\textit{su}(2)-Model

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Abstract

The \textit{su}(2)-algebraic model interacting with an environment is investigated from a viewpoint of treating the dissipative system. By using the time-dependent variational approach with a coherent state and with the help of the canonicity condition, the time-evolution of this quantum many-body system is described in terms of the canonical equations of motion in the classical mechanics. Then, it is shown that the \textit{su}(1, 1)-algebra plays an essential role to deal with this model. An exact solution with appropriate initial conditions is obtained by means of Jacobi’s elliptic function. The implication to the dissipative process is discussed.
§1. Introduction

One of the recent interests in nuclear theory is to investigate so-called thermal effect in nucleus and the dissipative process of the collective motion based on the quantum many-body theory or the quantum field theory. To deal with the thermal effects in the quantum field theory, it is well known that there are some methods such as the imaginary-time formalism, the real-time formalism in thermo field dynamics and so on. Four of the present authors (Y.T., J.P., A.K. & M.Y.) have proposed a method to describe the thermal effects and dissipation in many-body systems.\(^1\),\(^2\) As for the description of the thermal effects in quantum many-particle systems, one of the key points in our formalism is to use the mixed-mode coherent state, which we have defined in the series of our papers,\(^1\)\(^-\)\(^3\) as a trial state in the time-dependent variational approach. Furthermore, we learned that the \(su(1,1)\)-algebraic structure which the Hamiltonian has in the system under consideration plays an essential role when the non-equilibrium time-evolution and/or the dissipative process are realized in the quantum many-particle systems. Usually, so-called phase space doubling is carried out in order that the Hamiltonian has the \(su(1,1)\)-algebraic structure.\(^4\) However, in general, the degree of freedom introduced by the phase space doubling cannot be interpreted as the environment or heat bath. We have given a formalism to describe the thermal effects and dissipation in a system in which the relevant and the irrelevant degrees of freedom interact each other and the irrelevant degree of freedom can be regarded as the environment.\(^2\) Further, we have shown that the boson mapping theory\(^5\) presents the powerful technique to find the \(su(1,1)\)-algebraic behavior in the system governed by the \(su(2)\)-algebra. The recent review of the series of our work is seen in Ref.\(^6\)

In this paper, we apply the above-mentioned formulation to the nuclear \(su(2)\)-model which interacts with the environment represented by a harmonic oscillator. Main purpose in this paper is to show the utility of the \(su(1,1)\)-algebra. As was mentioned previously, with the help of the \(su(1,1)\)-algebraic structure, we will find the possibility of the description of the dissipative process. We are restricted ourselves in this paper to the zero temperature system. Namely, we will adopt a coherent state as a trial state instead of the mixed-mode coherent state in the time-dependent variational principle. Further, the exact solution for the time-dependent variational equation of motion will be obtained with the help of the \(su(1,1)\)-algebra. It is shown that the solution is given in terms of the elliptic function. This solution is similar to that encountered in the investigation of nuclear collective motions in the classical \(su(2)\)-models.\(^7\) However, the physical situation is different.

This paper is organized as follows. In the next section, we introduce the nuclear \(su(2)\)-model interacting with a harmonic oscillator. The \(su(1,1)\)-algebraic structure is found by
means of the Schwinger boson representation for the $su(2)$-algebra. The coherent state, which presents the classical counterpart of the original quantum many-body system, is introduced in §3. By the use of the canonicity conditions, the state is parameterized in terms of the canonical variables. Thus, it is shown that the equations of motion derived by the time-dependent variational principle are nothing but the canonical equations of motion. In §4, the equation of motion is solved in a certain case, and it is shown that the solution is expressed in terms of Jacobi’s elliptic function. The implication to the dissipative process is discussed in §5. The last section is devoted to a concluding remarks.

§2. Nuclear $su(2)$-model and the construction of $su(1, 1)$-algebra

Let us start with the following Hamiltonian:

$$
\hat{H} = \hbar \omega \left( a^* a + \frac{1}{2} \right) + 2\epsilon \hat{S}_0 - G \hat{S}_+ \hat{S}_- - \frac{\chi}{2} (\hat{S}_+^2 + \hat{S}_-^2) + \gamma \sqrt{\hbar} (a \hat{S}_+ + \hat{S}_- a^*) .
$$

(2.1)

Here, $a^*$ and $a$ are boson creation and annihilation operators, respectively, and the set $(\hat{S}_0, \hat{S}_\pm)$ composes the $su(2)$ algebra:

$$
[a, a^*] = 1 , \quad [a, \hat{S}_{0,\pm}] = 0 , \quad [\hat{S}_+, \hat{S}_-] = 2\hbar \hat{S}_0 , \quad [\hat{S}_0, \hat{S}_\pm] = \pm \hbar \hat{S}_\pm .
$$

(2.2)

In the case $\omega = \chi = \gamma = 0$, this model Hamiltonian is known as that of the pairing model in which the interaction is active for particle-particle pair with coupled angular momentum being 0. In the case $\omega = G = \gamma = 0$, this corresponds to the Lipkin model which consists of two energy-levels with the same degeneracy. The level spacing is $2\hbar \epsilon$ and the interaction is active for particle-hole pair with coupled angular momentum being 0. These are the schematic nuclear models governed by the $su(2)$-algebra. We call these models the nuclear $su(2)$-models. In the case $G = \chi = 0$, this model is nothing but the Jaynes-Cummings model if the representation of the $su(2)$-algebra is adopted as the spin 1/2 one. The Jaynes-Cummings model is well known in the field of quantum optics. In general, we will regard in this paper the above model given in Eq.(2.1) as a schematic nuclear model including both the relevant and the irrelevant degree of freedom which are represented by the $su(2)$-generators and $a$-boson, respectively. Namely, this model will be regarded as a typical model that the many-body system represented by the nuclear $su(2)$-model interacts with the environment represented by $a$-boson. Other possibility to interpret this model is as follows: This model in Eq.(2.1) can be interpreted as a schematic nuclear model including both the collective and
the intrinsic degree of freedom which are represented by $a$-boson and the $su(2)$-generators, respectively.

In this paper, we use the Schwinger boson representation for the $su(2)$-generators. The new boson operators $(b, b^*)$ and $(c, c^*)$ are introduced and the $su(2)$-generators are expressed in terms of these two kinds of boson operators:

$$
\hat{S}_0 = \frac{\hbar}{2}(c^*c - b^*b), \quad \hat{S}_+ = \hbar c^*b, \quad \hat{S}_- = \hbar b^*c. \quad (2.3)
$$

The Casimir operator $\hat{\Gamma}_{su(2)}$ is written as

$$
\hat{\Gamma}_{su(2)} = \hat{S}_0^2 + \frac{1}{2}(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+) = \hat{S}(\hat{S} + \hbar), \quad (2.4)
$$

$$
\hat{\hat{S}} = \frac{\hbar}{2}(c^*c + b^*b).
$$

Further, the following operators can be also defined in terms of the original boson operators $(a,a^*)$ and the Schwinger boson $(b,b^*)$ as

$$
\hat{T}_0 = \frac{\hbar}{2}(bb^* + a^*a), \quad \hat{T}_+ = \hbar b^*a^*, \quad \hat{T}_- = \hbar ab. \quad (2.5)
$$

These operators compose the $su(1,1)$-algebra, and the Casimir operator $\hat{\Gamma}_{su(1,1)}$ is also defined as follows:

$$
[\hat{\hat{T}}, \hat{\hat{T}_0}] = -2\hbar\hat{T}_0, \quad [\hat{\hat{\Gamma}}, \hat{\hat{T}_0}] = \pm\hbar\hat{T}_\pm, \quad (2.6)
$$

$$
\hat{\hat{\Gamma}}_{su(1,1)} = \hat{T}_0^2 - \frac{1}{2}(\hat{T}_+\hat{T}_- + \hat{T}_-\hat{T}_+) = \hat{T}(\hat{T} - \hbar), \quad (2.7)
$$

$$
\hat{T} = \frac{\hbar}{2}(bb^* - a^*a).
$$

It should be noted that the two algebras are not independent in this model. Actually, we can calculate the commutation relations: $[\hat{S}, \hat{T}_0] = [\hat{S}, \hat{T}] = [\hat{T}, \hat{S}_0] = 0$. However, we obtain $[\hat{S}, \hat{T}_\pm] \neq 0$ and $[\hat{T}, \hat{S}_\pm] \neq 0$.

The Hamiltonian (2.1) is then expressed by using the $su(2)$- and the $su(1,1)$-generators together with their Casimir operators. The Hamiltonian is recast into

$$
\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2, \quad (2.8)
$$

$$
\hat{H}_0 = \hbar \omega \left(a^*a + \frac{1}{2}\right) + 2\epsilon\hat{S}_0 - G\hat{S}_+\hat{S}_-
$$

$$
= \omega \left[\hat{T}_0 - \hat{T} + \frac{\hbar}{2}\right] + 2\epsilon(\hat{S} - \hat{T}_0 - \hat{T} + \hbar) - G(\hat{T}_0 + \hat{T})(2\hat{S} - \hat{T}_0 - \hat{T} + \hbar), \quad (2.9)
$$

$$
\hat{H}_1 = \gamma\sqrt{\hbar}(a\hat{S}_+ + \hat{S}_-a^*) = \gamma\sqrt{\hbar}(\hat{T}_-c^* + c\hat{T}_+), \quad (2.10)
$$

$$
\hat{H}_2 = -\frac{\chi}{2}(\hat{S}_+^2 + \hat{S}_-^2). \quad (2.11)
$$
In any case, we have the commutation relation, $[\hat{H}, \hat{S}] = 0$. Thus, $\hat{S}$ is a constant of motion. Besides $\hat{S}$, we have conserved variables in some cases: (i) If $\chi = 0$, that is, the case of no particle-hole interaction, then $\hat{H}$ does not contain $\hat{S}_\pm$. Thus, we can derive $[\hat{H}, \hat{T}] = 0$, so that $\hat{T}$ is a constant of motion in addition to $\hat{S}$. (ii) If $\gamma = 0$, that is, there is no interaction between the relevant and the irrelevant motion, then $\hat{H}$ does not contain any linear term with respect to $a$ and $a^*$. Thus, we obtain $[\hat{H}, \hat{T}_0 - \hat{T}] = 0$ because $\hat{T}_0 - \hat{T} = h a^* a$. Namely, we have another constant of motion $\hat{T}_0 - \hat{T}$ in addition to $\hat{S}$. (iii) If $\chi = \gamma = 0$, then $[\hat{H}, \hat{T}] = [\hat{H}, \hat{T}_0] = 0$. Thus the variables $\hat{T}$ and $\hat{T}_0$ are conserved independently.

In any case, the operators $\hat{S}$, $\hat{T}$, $\hat{T}_0$ and $\hat{S}_0$ commute each other. However, $\hat{S}_0$ is expressed in terms of the other three operators as $\hat{S}_0 = \hat{S} - \hat{T}_0 - \hat{T} + h$. Thus, it is enough to introduce the simultaneous eigenstate, $|s, t, t_0\rangle$, for $\hat{S}, \hat{T}$ and $\hat{T}_0$ with the eigenvalues $hs$, $ht$ and $ht_0$, respectively. Except for the normalization constant, the eigenstate is given as

$$|s, t, t_0\rangle = (a^*)^{t_0 - t}(b^*)^{t_0 + t - 1}(c^*)^{2s - t + t_0 + 1}|0\rangle. \quad (2.12)$$

Here, $a|0\rangle = b|0\rangle = c|0\rangle = 0$. The eigenvalue equations are written as

$$\hat{S}|s, t, t_0\rangle = hs|s, t, t_0\rangle, \quad \hat{T}|s, t, t_0\rangle = ht|s, t, t_0\rangle, \quad \hat{T}_0|s, t, t_0\rangle = ht_0|s, t, t_0\rangle,$$

$$\hat{S}_0|s, t, t_0\rangle = h(s - t - t_0 + 1)|s, t, t_0\rangle. \quad (2.13)$$

As for the $su(1, 1)$-algebra, the case $t \geq 1/2$ will only be treated from now on. In this case, the eigenstate $|s, t, t_0\rangle$ is expressed by the use of $\hat{S}_+$ and $\hat{T}_+$ as

$$|s, t, t_0\rangle = \mathcal{N}(\hat{T}_+)^{t_0 - t}(\hat{S}_+)^{2s - t + t_0 + 1}(b^*)^{2s - t + t_0 + t}|0\rangle. \quad (2.14)$$

Then, we can show the following relations:

$$\hat{S}_-(b^*)^{2s - t_0 + t}|0\rangle = \hat{T}_-(b^*)^{2s - t_0 + t}|0\rangle = 0. \quad (2.15)$$

Therefore, $\hat{S}_+$ and $\hat{T}_+$ play a role of the raising operators from the state $(b^*)^{2s - t_0 + t}|0\rangle$.

§3. Construction of coherent state and classical counterpart

In the previous section, we have introduced the eigenstate with respect to the operators $\hat{S}, \hat{T}, \hat{T}_0$ and $\hat{S}_0$. Then, $\hat{S}_+$ and $\hat{T}_+$ are regarded as the raising operators. Thus, it is allowed to construct the following coherent state:

$$|c\rangle = N_c \exp\left(VU^{-1}/h \cdot \hat{T}_+\right) \cdot \exp\left(UYW^{-1}/h \cdot \hat{S}_+\right) \cdot \exp\left(\sqrt{2/h} \cdot WU^{-1}b^*\right)|0\rangle. \quad (3.1)$$
Here, $V$, $W$ and $Y$ are complex variables, $U = \sqrt{1 + |V|^2}$ and $N_c$ is a normalization factor. The state $|c\rangle$ is rewritten as

$$|c\rangle = N_c \exp \left( \sqrt{\frac{2}{\hbar}} Y c^* + VU^{-1} b^* a^* + \sqrt{\frac{2}{\hbar}} WU^{-1} b^* \right) |0\rangle$$

$$= |c_c\rangle \otimes |c_{ab}\rangle ,$$

$$|c_c\rangle = e^{-\frac{i}{\hbar} |Y|^2} \cdot \exp \left( \sqrt{\frac{2}{\hbar}} Y c^* \right) |0\rangle ,$$

$$|c_{ab}\rangle = U^{-1} e^{-\frac{i}{\hbar} |W|^2} \cdot \exp \left( VU^{-1} b^* a^* + \sqrt{\frac{2}{\hbar}} WU^{-1} b^* \right) |0\rangle .$$

By introducing the following new operators,

$$a' = Ua - Vb^* , \quad b' = Ub - V a^* - \sqrt{2/\hbar} W , \quad c' = c - \sqrt{2/\hbar} Y , \quad (3.3)$$

it is shown that the state $|c\rangle$ is vacuum for $a'$, $b'$ and $c'$:

$$a'|c\rangle = b'|c\rangle = c'|c\rangle = 0 . \quad (3.4)$$

Inversely, $a$, $b$ and $c$ are expressed as

$$a = U a' + V b'^* + \sqrt{2/\hbar} W^* V , \quad b = V a'^* + U b' + \sqrt{2/\hbar} W U , \quad c = c' + \sqrt{2/\hbar} Y . \quad (3.5)$$

By the use of the above relations, we can easily calculate the expectation values of the $su(2)$- and the $su(1,1)$-generators and their Casimir operators. For example,

$$T = \langle c | \hat{T} | c \rangle = |W|^2 + \frac{\hbar}{2} ,$$

$$S = \langle c | \hat{S} | c \rangle = \left( |W|^2 + \frac{\hbar}{2} \right) |V|^2 + |W|^2 + |Y|^2 ,$$

$$T_0 = \langle c | \hat{T}_0 | c \rangle = \left( |W|^2 + \frac{\hbar}{2} \right) \left( 1 + 2|V|^2 \right) . \quad (3.6)$$

It is here noted that the dynamical variables are $(V, V^*, W, W^*, Y, Y^*)$. As is similar to the time-dependent Hartree-Fock theory with canonical form, it is convenient to parameterize the coherent state in terms of canonical variables. For this purpose, we impose the following canonicity conditions:

$$\langle c | i \hbar \partial_{\phi} | c \rangle = T_0 - \frac{\hbar}{2} , \quad \langle c | i \hbar \partial_{T_0} | c \rangle = 0 ,$$

$$\langle c | i \hbar \partial_{\phi} | c \rangle = T - \frac{\hbar}{2} , \quad \langle c | i \hbar \partial_{T} | c \rangle = 0 ,$$

$$\langle c | i \hbar \partial_{\psi} | c \rangle = S , \quad \langle c | i \hbar \partial_{S} | c \rangle = 0 . \quad (3.7)$$
The sets of variables \((T_0, \phi_0), (T, \phi)\) and \((S, \psi)\) are those of the canonical variables. The above canonicity conditions can be easily solved by the use of the following relation:

\[
\langle c | i \hbar \partial_z | c \rangle = i \left[ (|W|^2 + \hbar/2) \cdot (V^* \partial_z V - V \partial_z V^*) 
+ (W^* \partial_z W - W \partial_z W^*) + (Y^* \partial_z Y - Y \partial_z Y^*) \right].
\] (3.8)

As a result, the original variables are expressed by the canonical variables as

\[
V = \sqrt{\frac{T_0 - T}{2T}} \cdot e^{-i\phi_0} e^{-i\psi/2},
\]
\[
W = \sqrt{\frac{T - \hbar}{2}} \cdot e^{-i\phi_0/2} e^{-i\phi/2} e^{-i\psi/2},
\]
\[
Y = \sqrt{\frac{1}{2} (2S - T_0 - T + \hbar)} \cdot e^{-i\psi/2}.
\] (3.9)

Thus, we can obtain the expectation values of various operators with respect to \(|c\rangle\) in terms of the canonical variables:

\[
\langle c | \hat{S}_0 | c \rangle = S - T_0 - T + \hbar,
\]
\[
\langle c | \hat{S}_+ | c \rangle = \sqrt{2(2S - T_0 - T + \hbar)} \sqrt{T - \frac{\hbar}{2}} \sqrt{\frac{T_0 + T}{2T}} \cdot e^{-i(\phi_0 + \phi)/2},
\]
\[
\langle c | \hat{T}_0 | c \rangle = T_0,
\]
\[
\langle c | \hat{T}_+ | c \rangle = (T_0 - T)(T_0 + T) \cdot e^{i\phi_0} e^{i\psi/2},
\]
\[
\langle c | \hat{S}_+ \hat{S}_- | c \rangle = (2S - T_0 - T + \hbar)(T_0 + T),
\]
\[
\langle c | a^* a | c \rangle = \left( T_0 - T \right) / \hbar,
\]
\[
\langle c | \hat{S}_+^2 | c \rangle = \left( 1 - \frac{\hbar}{2T} \right) (T_0 + T)(2S - T_0 - T + \hbar) e^{-i(\phi_0 + \phi)},
\]
\[
\langle c | \hat{S}_-^2 | c \rangle = \left( 1 - \frac{\hbar}{2T} \right) (T_0 + T)(2S - T_0 - T + \hbar) e^{i(\phi_0 + \phi)},
\]
\[
\langle c | \sqrt{\hbar} a^* \hat{S}_- | c \rangle = \sqrt{(T_0 - T)(T_0 + T)(2S - T_0 - T + \hbar)} \cdot e^{i\phi_0},
\]
\[
\langle c | \sqrt{\hbar} a \hat{S}_+ | c \rangle = \sqrt{(T_0 - T)(T_0 + T)(2S - T_0 - T + \hbar)} \cdot e^{-i\phi_0}.
\] (3.10)

Thus, the expectation value of the Hamiltonian with respect to \(|c\rangle\) is given by

\[
H = \langle c | \hat{H} | c \rangle = H_0 + H_1 + H_2
\]
\[
H_0 = \langle c | \hat{H}_0 | c \rangle = \omega(T_0 - T + \hbar/2) + 2\epsilon(S - T_0 - T + \hbar)
- G(2S - T_0 - T + \hbar)(T_0 + T)
\]
\[ H_1 = \langle c|\hat{H}_1|c\rangle = 2\gamma \sqrt{(T_0 - T)(T_0 + T)(2S - T_0 - T + \hbar)} \cos \phi_0, \]
\[ = g(T_0, T; S) \cos \phi_0, \]
\[ H_2 = \langle c|\hat{H}_2|c\rangle = -\chi(1 - h/(2T))(T_0 + T)(2S - T_0 - T + \hbar) \cos(\phi_0 + \phi), \]
\[ = h(T_0, T; S) \cos(\phi_0 + \phi). \quad (3.11) \]

Here, \( \hat{H}_0, \hat{H}_1 \) and \( \hat{H}_2 \) have been defined in Eqs. (2.9) ∼ (2.11).

The equations of motion are derived from the time-dependent variational principle:
\[ \delta \int_{t_1}^{t_2} dt \langle c|ih\partial_t - \hat{H}|c\rangle = 0. \quad (3.12) \]

With the help of the canonicity conditions (3.7), the derived equations of motion have the form of canonical equations of motion:
\[ \dot{T}_0 = -\frac{\partial H}{\partial \phi_0} = g(T_0, T; S) \sin \phi_0 + h(T_0, T; S) \sin(\phi_0 + \phi), \]
\[ \dot{\phi}_0 = \frac{\partial H}{\partial T_0} = \frac{\partial f(T_0, T; S)}{\partial T_0} + \frac{\partial g(T_0, T; S)}{\partial T_0} \cos \phi_0 + \frac{\partial h(T_0, T; S)}{\partial T_0} \cos(\phi_0 + \phi), \quad (3.13) \]
\[ \dot{T} = -\frac{\partial H}{\partial \phi} = h(T_0, T; S) \sin(\phi_0 + \phi), \]
\[ \dot{\phi} = \frac{\partial H}{\partial T} = \frac{\partial f(T_0, T; S)}{\partial T} + \frac{\partial g(T_0, T; S)}{\partial T} \cos \phi_0 + \frac{\partial h(T_0, T; S)}{\partial T} \cos(\phi_0 + \phi), \quad (3.14) \]
\[ \dot{S} = -\frac{\partial H}{\partial \psi} = 0, \]
\[ \dot{\psi} = \frac{\partial H}{\partial S} = \frac{\partial f(T_0, T; S)}{\partial S} + \frac{\partial g(T_0, T; S)}{\partial S} \cos \phi_0 + \frac{\partial h(T_0, T; S)}{\partial S} \cos(\phi_0 + \phi). \quad (3.15) \]

Let us consider simple cases: (i) If \( \chi = 0 \), then \( h(T_0, T; S) = 0 \). In this case, from Eq. (3.14), the quantity \( T \) is conserved in addition to \( S \). This fact is, of course, originated from the fact \( [\hat{H}, \hat{T}] = 0 \) mentioned in §2. It is thus necessary to solve the time-evolution of \( T_0 \):
\[ \dot{T}_0 = g(T_0, T; S) \sin \phi_0, \]
\[ \dot{\phi}_0 = \frac{\partial f(T_0, T; S)}{\partial T_0} + \frac{\partial g(T_0, T; S)}{\partial T_0} \cos \phi_0. \quad (3.16) \]

The expectation value of the total Hamiltonian, which is a constant of motion, is expressed as
\[ H = f(T_0, T; S) + g(T_0, T; S) \cos \phi_0 = E_0. \quad (3.17) \]

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From Eqs. (3.17) and (3.16), we obtain
\[
\left( \frac{T_0}{g(T_0, T; S)} \right)^2 + \left( \frac{E_0 - f(T_0, T; S)}{g(T_0, T; S)} \right)^2 = 1 .
\] (3.18)

Namely,
\[
\dot{T}_0^2 = [g(T_0, T; S)]^2 - [E_0 - f(T_0, T; S)]^2 .
\] (3.19)

(ii) If \( \gamma = 0 \), then \( g(T_0, T; S) = 0 \). In this case, from Eqs. (3.13) and (3.14), the quantity \( T_0 - T \) is conserved in addition to \( S \). It is thus necessary to solve the time-evolution of \( T_0 - T \). However it is enough to know the time-dependence for \( T_0 \) because \( \dot{T}_0 = \dot{T} = 0 \):
\[
\dot{T}_0 = h(T_0, T; S) \sin(\phi_0 + \phi) ,
\]
\[
\dot{\phi}_0 = \frac{\partial f(T_0, T; S)}{\partial T_0} + \frac{\partial h(T_0, T; S)}{\partial T_0} \cos(\phi_0 + \phi) .
\] (3.20)

The expectation value of the total Hamiltonian is expressed as
\[
H = f(T_0, T; S) + h(T_0, T; S) \cos(\phi_0 + \phi) = E_0 .
\] (3.21)

From Eqs. (3.20) and (3.21), we obtain
\[
\left( \frac{T_0}{h(T_0, T; S)} \right)^2 + \left( \frac{E_0 - f(T_0, T; S)}{h(T_0, T; S)} \right)^2 = 1 .
\] (3.22)

Namely,
\[
\dot{T}_0^2 = [h(T_0, T; S)]^2 - [E_0 - f(T_0, T; S)]^2 .
\] (3.23)

Our next task is to solve the equation of motion (3.19) or (3.23) which have the same form.

§4. Utility of elliptic function for a solution of equations of motion

In the previous section, it has been shown that in the simple cases the equation of motion for \( T_0 \) which has a form in Eq. (3.19) or (3.23) should be solved. Hereafter, let us consider the case (i), that is, \( \chi = 0 \), in which the particle-hole interaction is not active. In the region \( T_0 \geq T \geq \hbar/2 \) and the adequate model parameters, the right-hand side of Eq. (3.19) is always non-negative. However, we regard the right-hand side of Eq. (3.19) as a function of \( T_0 \) with arbitrary parameter set :
\[
\dot{T}_0^2 = F(T_0) = [g(T_0, T; S)]^2 - [E - f(T_0, T; S)]^2
\]
\[
= -G^2(T_0 - \tau_0)(T_0 - \tau_1)(T_0 - \tau_2)(T_0 - \tau_3) ,
\] (4.1)
where we used the knowledge that the function $F$ is the function of fourth order for $T_0$ and the coefficient of $T_0^4$ is negative: $F(T_0) = -G^2 T_0^4 + \cdots$. It should be noted here that, in case (ii), if the parameters satisfy an inequality $\chi^2(1 - h/(2T))^2 < G^2$, the same situation as the case (i) is realized. From the fact $F(T_0 \to \pm \infty) < 0$, it is concluded that the two real solutions for $F(T_0) = 0$ except for $\dot{T}_0 \equiv 0$ must exist because the physical region $F > 0$ must exist. In addition to this fact, in the second line in Eq.(4.1), we assume that there exist four real solutions for $F(T_0) = 0$ which are denoted as $\tau_0, \tau_1, \tau_2$ and $\tau_3$ with $\tau_0 < \tau_1 < \tau_2 < \tau_3$. We can numerically check the existence of four real solutions for $F(T_0) = 0$. For example, if the parameters are adopted as $\bar{h}G/\epsilon = 1/2, \omega/\epsilon = 1, \gamma\sqrt{\bar{h}/\epsilon} = 0.1$ and $E_0/\epsilon = 0$ and the variables are $S/\bar{h} = 5$ and $T/\bar{h} = 1/2$, the above-mentioned situation is realized. Further, in this parameter set, the physical region is given in $\tau_0 < T_0 < \tau_1$. Thus, hereafter, the physical region is taken as $\tau_0 < T_0 < \tau_1$. Then, $T_0$ corresponding to the physical value is parameterized as follows:

$$T_0 = \tau_0 - (\tau_0 - \tau_1)x^2, \quad -1 \leq x \leq 1. \quad (4.2)$$

Thus, we can derive the equation for $x$ from Eq.(4.1):

$$\dot{x}^2 = \frac{G^2}{4} (\tau_0 - \tau_2)(\tau_0 - \tau_3)(1 - x^2)(1 - k_1 x^2)(1 - k_2 x^2), \quad (4.3)$$

where $k_1$ and $k_2$ are defined as

$$k_1 = \frac{\tau_0 - \tau_1}{\tau_0 - \tau_2}, \quad k_2 = \frac{\tau_0 - \tau_1}{\tau_0 - \tau_3}, \quad 0 \leq k_2 \leq k_1 \leq 1. \quad (4.4)$$

Further, if we set

$$x(t) = \frac{m(t)}{\sqrt{1 - k_2 + k_2 m^2(t)}}, \quad (-1 \leq m \leq 1) \quad (4.5)$$

the variable $m(t)$ satisfies the following differential equation:

$$\dot{m}^2 = Z^2 (1 - m^2)(1 - k^2 m^2), \quad (4.6)$$

$$Z = \frac{K}{\pi} \omega_0, \quad (4.7)$$

where $\omega_0, K$ and $k$ are defined as

$$\omega_0 = \frac{G}{2} \sqrt{(\tau_0 - \tau_2)(\tau_1 - \tau_3)} \frac{\pi}{K},$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} = K(k),$$

$$k = \sqrt{\frac{k_1 - k_2}{1 - k_2}} = \sqrt{\frac{(\tau_0 - \tau_1)(\tau_2 - \tau_3)}{(\tau_0 - \tau_2)(\tau_1 - \tau_3)}}. \quad (0 \leq k \leq 1) \quad (4.8)$$
Here, $K$ and $k$ are the complete elliptic integral of the first kind and the modulus, respectively. If we take the initial value $m(t = 0) = 0$, that is, $T_0(t = 0) = \tau_0$, we thus obtain the following solution for $m(t)$:

$$m(t) = \text{sn}(Zt, k),$$  \hspace{1cm} (4.9)

where $\text{sn}(Zt, k)$ is Jacobi’s elliptic function:

$$Zt = \int_0^{m(t)} \frac{dm}{\sqrt{(1 - m^2)(1 - k^2m^2)}}. \hspace{1cm} (4.10)$$

As a result, we obtain $T_0$ as a function of time $t$:

$$T_0 = \tau_0 - \frac{(\tau_0 - \tau_1)(\tau_0 - \tau_3)\text{sn}^2(Zt, k)}{(\tau_1 - \tau_3) + (\tau_0 - \tau_1)\text{sn}^2(Zt, k)}. \hspace{1cm} (4.11)$$

It should be noted that, if the physical region is given by $\tau_2 < T_0 < \tau_3$, the solution for $T_0$ is also expressed in terms of Jacobi’s elliptic function similar to Eq.(4.11).

In the Fourier series representation for Jacobi’s elliptic function, the function $k^2\text{sn}^2(Zt, k)$ is expressed as

$$k^2\text{sn}^2(Zt, k) = 1 - \frac{E}{K(k)} - \frac{\pi^2}{K(k)^2} \cdot \sum_{n=1}^{\infty} \frac{n}{\sinh(n\pi K(k')/K(k))} \cdot \cos(n\omega_0 t), \hspace{1cm} (4.12)$$

where $k' = \sqrt{1 - k^2}$ and $E$ represents the complete elliptic integral of the second kind:

$$E = \int_0^1 \sqrt{\frac{1 - k^2x^2}{1 - x^2}} \, dx. \hspace{1cm} (4.13)$$

From Eqs.(4.11) and (4.12), it can be seen that the time-dependent variable $T_0$ has a fundamental period $t_0$ given by

$$t_0 = \frac{2\pi}{\omega_0}. \hspace{1cm} (4.14)$$

§5. Discussion

As was mentioned in §1, it is interesting to investigate the dissipative process which is governed by the $su(1,1)$-algebraic structure. In this model-Hamiltonian we are treating in this paper, the $su(1,1)$-algebraic structure is hidden and appears by means of the Schwinger boson representation for $su(2)$-generators. In the previous section, the parameter $\chi$, which is the strength of the particle-hole interaction, is taken as $0$. It has been shown that in this case there exists a fundamental period $t_0$ in the solution expressed by the elliptic function. However, if this period is long compared with the typical time scale we consider, the dissipative situation may be realized. In this section, we will investigate this possibility.
The fundamental period $t_0$ is given in Eq.\(4.14\). Therefore, if the angular frequency $\omega_0$, which has been given in Eq.\(4.8\), is infinitesimal or very small value, the period $t_0$ substantially becomes infinity. Thus, the periodicity disappears. In order to realize this situation, it is necessary that the modulus $k$ is close to 1 because the complete elliptic integral of the first kind, $K$, reveals the behavior of a logarithmic divergence with $k \rightarrow 1$. As a result, $\omega_0$ is close to 0 and $t_0$ becomes infinity. If $\tau_1 \sim \tau_2$, then $k \sim 1$ is realized. In this case, Jacobi’s elliptic function $\text{sn}(Zt, k = 1)$ is expressed in terms of the hyperbolic tangent, namely,

$$\text{sn}(Zt, 1) = \tanh Zt.$$ \hspace{1cm} (5.1)

Thus, the solution $T_0$ is expressed by

$$T_0 = \tau_0 - \frac{\tau_0 - \tau_3}{\frac{\tau_0 - \tau_1}{\tanh^2 Zt} + 1}.$$ \hspace{1cm} (5.2)

This is a monotonically increasing function from $\tau_0$ at $t = 0$ to $\tau_1$ at $t \rightarrow \infty$. Thus, the periodicity is lost.

In this special case, the energy of the degree of freedom represented by $a$-boson increases monotonically, that is,

$$E_a = \langle c | \hbar \omega (a^* a + 1/2) | c \rangle = \omega \left( T_0 - T + \frac{\hbar}{2} \right)$$ \hspace{1cm} (5.3)

increases for time because $T_0$ increases monotonically. Here, it is possible to introduce a “coordinate” $x$ which is defined and expressed in terms of the canonical variables:

$$\dot{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^*),$$

$$x = \langle c | \hat{x} | c \rangle = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{2}{\hbar}} (W^* V + WV^*)$$

$$= -\frac{2}{\sqrt{m\omega}} \sqrt{T - \frac{\hbar}{2}} \sqrt{\frac{T_0 - T}{2T}} \cos \frac{\phi - \phi_0}{2}.$$ \hspace{1cm} (5.4)

It is seen that the amplitude of $x$ increases monotonically. Thus, $x$ reveals the behavior of the amplified oscillation. If the degrees of freedom represented by $a$-boson and by the $su(2)$-generators are regarded as the environment and the relevant degree of freedom, respectively, the process in which the energy of the relevant motion dissipates to the environment is described. This is nothing but the dissipative process. This process may be interpreted as the “cooling” process for the relevant motion, although we do not treat the system at finite temperature. Actually, if the state $|c\rangle$ is replaced into the mixed-mode coherent state, the thermal effect may be treated in this framework.
§6. Concluding remarks

We have investigated the nuclear $su(2)$-model interacting with the harmonic oscillator. By the use of the Schwinger boson representation for the $su(2)$-algebra, the $su(1,1)$-algebraic behavior was come in sight. In this model we have considered in this paper, it is allowed to obtain the exact solution for the classical equation of motion. It has been shown that the solution is expressed in terms of Jacobi’s elliptic function. The solution has a fundamental period. However, if the period is so long, it may be possible to realize the dissipative process. In this paper, we are restricted ourselves to deal with the zero temperature case. If we apply our formalism to the finite temperature system, it is necessary to replace the coherent state into the mixed-mode coherent state. We have already investigated the mixed-mode coherent state in the many-body system composed of three kinds of boson operators.\textsuperscript{10} The investigation of the behavior in the nuclear $su(2)$-model interacting with the environment at finite temperature is left as a future problem.

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