and describe from spatially extensive objects to high redshifts.

In the context of reionization models, it has been suggested that reionization may proceed in a patchy, stochastic manner. This idea is supported by the observed clustering of galaxies. However, the observed clustering of galaxies is also consistent with a smooth, continuous reionization process. The question of whether reionization proceeded in a patchy or smooth manner remains open.

In this paper, we present a new method to infer the details of the reionization process. We use a combination of observational data and theoretical models to constrain the properties of the reionization process. Our results suggest that reionization proceeded in a patchy manner.

Introduction

1. Observational Measurements

In this section, we present observational measurements of the reionization process. We use a combination of spatially extensive objects and high redshifts as tracers of the reionization process. The observations are consistent with a patchy, stochastic reionization process.

2. Theoretical Models

In this section, we present theoretical models of the reionization process. We use a combination of spatially extensive objects and high redshifts as tracers of the reionization process. The models are consistent with a patchy, stochastic reionization process.

Conclusion

In conclusion, we find that reionization proceeded in a patchy manner. This is supported by both observational and theoretical measurements. Further studies are needed to constrain the properties of the reionization process in more detail.
use of both observables, and based on a perturbative
development of general physical hypothesis, this method
allow us to test some very general physical hypothesis of
the gas (hydrostatic equilibrium, global thermodynamic
equilibrium) and also provide naturally some X observa-
tion predictions.

Observations only provide us with $2 - D$ projected
quantities (e.g. mass, gas pressure, ...). This quantities
are related by some physical hypothesis which are ex-
plained in $3 - D$ equalities (e.g. hydrostatic equilibrium,
equation of state). The point is that these $3 - D$ equalities
do not have any tractable equivalent relating projected
$2 - D$ quantities: in particular, projection along the line of
sight does not provide an equation of state or a projected
hydrostatic equilibrium equation. Therefore as soon as
we want to compare this data (WL, SZ, X), we have to
project the relevant physical quantities $\mathcal{P}^2, \mathcal{P}^0, \mathcal{P}_0, ...$.
This can be done only using strong assumptions, either
by using parametric models (e.g. a $\beta$ model (Cavaliere
& Fusco-Femiano 1976)) or by assuming mere geometrical
hypothesis (the former necessarily encompassing the latter)
(Fabian et al. 1981; Yoshikawa and Suto 1999).
We choose the geometric approach in order to use as
general physical grounds as possible and to avoid as many
theoretical biases as possible.

This simplest choice might be naturally motivated first
by looking at some images of observed clusters (Désert
et al. 1998; Grego et al. 1999). Their regularity is striking:
some have almost circular or ellipsoidal appearance as we
expect for fully relaxed system. Then since relaxed clus-
ters are expected to be spheroidal in favored hierarchical
structure formation scenario, it is natural to try to relate
the observed quasi-circularity (quasi-sphericity) to the
$3 - D$ quasi-sphericity (quasi-spheroidality). We perform
this using some linearly perturbed spherical (spheroidal)
symmetries in a self-consistent approach.

We proceed as follows: in section 2 we defined our physical
hypothesis and our notations. The method is precisely
described in section 3. We consider both the spherical as
well as spheroidal cases and obtain a predicted X surface
brightness map from a SZ decrement map and a WL grav-
titational distortion map. In section 4 a demonstration with
simulated clusters is presented before discussing its appli-
cation to genuine data as well as further developments in
section 5.

2. Hypothesis, Sunyaev-Zel’dovich effect and the
Weak lensing

We now briefly describe our notations as well as our physical
hypothesis.

2.1. General hypothesis

Following considerations fully detailed in (Sarazin 1988)
the ICM can be regarded as a hot and dilute plasma
constituted from ions and electrons, whose respective kin-
etic temperatures $T_e$ and $T_i$ will be considered as equal
$T_e = T_i = T$. This is the global thermodynamic hypo-
thesis which is expected to hold up to $r_{virial}$ ( see
(Teyssier et al. 1997; Chièze et al. 1998) for a precise
discussion). Given the low density (from $n_e \sim 10^{-5} \text{cm}^{-3}$
in the core to $\sim 10^{-3} \text{cm}^{-3}$ in the outer part) and high
temperature of this plasma ($\sim 10^8 \text{K}$), it can be treated
as a perfect gas satisfying the equation of state:

$$ p_g = \frac{\rho_g k_B T_g}{\mu_e m_p} = \beta \rho_g T_g $$

(1)

with $\beta \equiv \frac{k_B}{\mu_e m_p}$. Let us neglect then the gas mass
regards to the dark matter mass, and assume stationaritv
(no gravitational potential variation on time scale smaller
than the hydrodynamic time scale, e.g. no recent mergers).
Then the gas assumed to be in hydrostatic equilibrium in:
the dark matter gravitational potential satisfies:

$$ \nabla (\rho_g v_g^2) = 0 $$

$$ \nabla P_g = -\rho_g \nabla \Phi_{DM} $$

(2)

(3)

At this point there is no need to assume isothermality.

2.2. Sunyaev-Zel’dovich effect and weak lensing

Inverse Compton scattering of cosmic background (CMB)
photons by the electrons in the ICM modifies the CMB
spectrum (Zel’dovich and Sunyaev 1969; Sunyaev and
Zel’dovich 1972; Sunyaev and Zel’dovich 1980). The
amplitude of the SZ temperature decrement $\Delta T_{SZ}$ is
directly proportional to the Comptonisation parameter $y$ which
is given by:

$$ y = \frac{\sigma_T}{m_e c^2} \int d l \ n_e k_B T_e = \frac{\sigma_T}{m_e c^2} \int d l \ p_e $$

$$ = \frac{\sigma_T}{m_e c^2} \int d l \frac{p_e k_B T_e}{\mu_e m_p} = \alpha \int d l P_g $$

(4)

(5)

where $\alpha \equiv \frac{\sigma_T}{m_e c^2}$, $k_B$ is the Boltzmann’s constant,
$\sigma_T$ is the Thomson scattering cross section and $d l$ is the
physical line-of-sight distance. $m_e, n_e, T_e$ and $p_e$ are
the mass, the number density, the temperature and the
thermal pressure of electrons. $\rho_g$ and $T_g$ respectively
denote the gas density and temperature, and $\mu_e$ is the
number of electrons per proton mass. Some further
 corrections to this expression can be found in (Rephaeli
1995; Birkinshaw 1999).

In parallel to this spectral distortions, the statistical
determination of the shear field $\kappa$ affecting the images of
background galaxies enable, in the weak lensing regime, to
derive the dominant projected gravitational potential of the
lens (the clustered dark matter) $\Phi_{DM}$ in our general
hypothesis (see (Melier 2000) for details).
3. Method

3.1. Principle

We now answer the question: how should we co-analyze these various data sets? Our first aim is to develop a method which allows us to get maps of projected thermodynamical quantities with as few physical hypothesis as possible.

Our method is the following. Let us suppose we have for a given cluster a set of data a SZ and WL data which enables us to construct a 2-D map of projected gas pressure as well as a 2-D projected gravitational potential map. Let us suppose as well that these maps exhibit an approximate spherical symmetry as is the case for a vast class of experimental observations as e.g. in figure 1. More precisely, let us suppose that the projected gas pressure $y$ as well as the observed projected gravitational potential $\phi_{DM}$ can be well fitted by the following type of functions:

$$y(R, \varphi) = y_0(R) + \varepsilon y_1(R) \, m(\varphi)$$

$$\phi_{DM}(R, \varphi) = \phi_{DM,0}(R) + \varepsilon \phi_{DM,1}(R) \, n(\varphi)$$

where $\varepsilon \ll 1$. $(R, \varphi)$ denotes polar coordinates in the image plane and $m$ and $n$ are some particular functions. This description means first of all that the images we see are linear perturbations from some perfect circularly symmetric images, and second that the perturbation might be described conveniently by the product of a radial function and an angular function. Equivalently we can assert that to first order in $\varepsilon$ our images are circularly symmetric but they admit some corrections to second order in $\varepsilon$.

We then assume that these observed perturbed symmetries are a consequence of an intrinsic 3-D spherical symmetry linearly perturbed too. This point constitutes our key hypothesis. It means that to first order in a certain parameter (e.g. $\varepsilon$) our clusters are regular objects with a strong circular symmetry but they admit some second order linear perturbations away from this symmetry. As a consequence of these assumptions we will make use of this linearly perturbed symmetry to get a map of some complementary projected thermodynamical quantities, the gas density $D_2$ and the gas temperature $T_2$, successively to first and second order in $\varepsilon$.

Formulated this way, the problem yields a natural protocol:

- Looking at some maps with this kind of symmetry, we compute a zero-order map $(y_0(R), \phi_0(R))$ with a perfect circular symmetry by averaging over some concentric annulus. A correction for the bias introduced by perturbations is included. These first order quantities allow us to derive some first order maps of $D_{2,0}(R)$ and $T_{2,0}(R)$ with a perfect circular symmetry.
- We then take into account the first order corrections to this perfect symmetry $(y_1(R)m(\varphi), \phi_1(R)m(\varphi))$ and infer from them first order correction terms to the zeroth order maps: $D_{2,1}(R, \varphi)$ and $T_{2,1}(R, \varphi)$.

Even if for clarity’s sake we formulate our method assuming a perturbed circular symmetry, it applies equivalently to a perturbed elliptical symmetry as it will be shown below. In this more general case, we assume that the cluster exhibit a linearly perturbed spheroidal symmetry.

3.2. The spherically symmetric case: from observations to predictions

Let us now apply the method to the case where the projected gas density (SZ data) and the projected gravitational potential (WL data) exhibit some approximate circular symmetry. These observations lead us to suppose that the 3-D gas pressure, the gravitational potential, the gas density and the gas temperature can be well described by the following equations:

$$\begin{align*}
P_2(r, \theta, \varphi) &= P_{2,0}(r) + \varepsilon \, P_{2,1}(r) \, f(\theta, \varphi) \\
\Phi_{DM}(r, \theta, \varphi) &= \Phi_{DM,0}(r) + \varepsilon \, \Phi_{DM,1}(r) \, g(\theta, \varphi) \\
\rho_2(r, \theta, \varphi) &= \rho_{2,0}(r) + \varepsilon \, \rho_{2,1}(r) \, h(\theta, \varphi) \\
T_2(r, \theta, \varphi) &= T_{2,0}(r) + \varepsilon \, T_{2,1}(r) \, k(\theta, \varphi)
\end{align*}$$

where $(r, \theta, \varphi)$ are spherical coordinates centered on the cluster.
3.2.1. The hydrostatic equilibrium

If we first apply the hydrostatic equilibrium equation
\[ \nabla P_g = -\rho_g \nabla \Phi_{DM} \]
we get the following equations. To first order in \( \varepsilon \) we have
\[ P_{g,1}(r) = -\rho_{g,1}(r) \Phi_{DM,1}(r) \]
and to second order in \( \varepsilon \):
\[
\begin{align*}
P_{g,1}(r) f(\theta, \varphi) &= -\rho_{g,1}(r) \Phi_{DM,1}(r) h(\theta, \varphi), \\
P_{g,1}(r) \partial_\theta f(\theta, \varphi) &= -\rho_{g,2}(r) \Phi_{DM,1}(r) \partial_\theta h(\theta, \varphi), \\
P_{g,1}(r) \partial_\varphi f(\theta, \varphi) &= -\rho_{g,2}(r) \Phi_{DM,1}(r) \partial_\varphi h(\theta, \varphi),
\end{align*}
\]
where \( h \) denotes the derivative with regards to \( r \).

Combining equations (10,b) and (10,c) we get
\[ f(\theta, \varphi) = \lambda_1 h(\theta, \varphi) + \lambda_2 \]
where \( \lambda_{1,2} \) are some constants. Then, using equation (10,a) we can write
\[ f(\theta, \varphi) = \gamma_2 g(\theta, \varphi) + \gamma_2 \]
where \( \gamma_{1,2} \) are some constants as well. At this point, we can get rid of \( \lambda_1 \) and \( \lambda_2 \) by absorbing them in the order 1 mere radial term (i.e. \( \rho_{g,1}(r) \) and \( \Phi_{DM,1}(r) \)). This means we can consider \( \lambda_1 = 0 \) and \( \gamma_1 = 0 \). Similarly we choose to rescale \( \rho_{g,2}(r) \) and \( \Phi_{DM,1}(r) \) so that we can take \( \gamma_1 = \lambda_1 = 1 \).

These simple equalities lead us to assume from now on:
\[ f(\theta, \varphi) = h(\theta, \varphi) = g(\theta, \varphi). \]

This is in no way a restriction since it simply means that we absorb integration constants by redefining some terms. This is possible since the relevant part of \( f \) (and thus \( h \)) will be fitted on observations as will be shown below.

Taking equation (13) into account, equation (10) simplifies to:
\[
\begin{align*}
P_{g,1}(r) &= -\rho_{g,2}(r) \Phi_{DM,1}(r) \\
P_{g,1}(r) &= -\rho_{g,1}(r) \Phi_{DM,1}(r) - \rho_{g,2}(r) \Phi_{DM,1}(r) \\
P_{g,1}(r) &= -\rho_{g,1}(r) \Phi_{DM,1}(r).
\end{align*}
\]

3.2.2. The equation of state

We have now identified the angular part to the first order correction of \( P_{g,1} \), \( \Phi_{DM,1} \) and \( \rho_{g} \). We still have to link those quantities to the angular dependent part of the temperature \( T_g \), namely \( k(\theta, \varphi) \). This is done naturally using the equation of state (1), which directly provide to first and second order in \( \varepsilon \):
\[
\begin{align*}
P_{g,1}(r) &= \beta \rho_{g,1}(r) T_g(r) \\
P_{g,1}(r) f(\theta, \varphi) &= \beta \rho_{g,1}(r) T_g(r) f(\theta, \varphi) \\
+ &\beta \rho_{g,2}(r) T_g(r) k(\theta, \varphi)
\end{align*}
\]
This last equation leads naturally to \( f(\theta, \varphi) = k(\theta, \varphi) \) if we decide once again to absorb any multiplicative factor in the radial part. This way we see that our choice of separating the radial and angular part is in no way a restriction.

We eventually get
\[
\begin{align*}
P_{g,1}(r) &= \beta \rho_{g,1}(r) T_g(r) \\
P_{g,1}(r) &= \beta \rho_{g,1}(r) T_g(r) + \beta \rho_{g,2}(r) T_g(r).
\end{align*}
\]

3.2.3. The observations

Given this description of the cluster hot gas, the experimental SZ and WI data which respectively provide us with the projected quantities \( y(R, \varphi) \) and \( \phi_{DM}(R, \varphi) \) write
\[
\begin{align*}
y(R, \varphi) &= \alpha \int P_{g,1}(r) + \varepsilon \int P_{g,1}(r) f(\theta, \varphi) \, dr \\
\phi_{DM}(R, \varphi) &= \int \Phi_{DM,1}(r) + \varepsilon \int \Phi_{DM,1}(r) f(\theta, \varphi) \, dr
\end{align*}
\]
where \( \alpha = \frac{1}{\rho_{g,1}(r)} \)
and \( \varepsilon = \frac{1}{\rho_{g,1}(r)} \).

Note that in order to get this set of definitions we choose the polar axis of the cluster along the line of sight so that the same azimuthal angle \( \varphi \) is used for \( 2D \) and \( 3D \) quantities.

Our aim is now to derive both a projected gas density map and projected temperature map that we define this way:
\[
\begin{align*}
D_g(R, \varphi) &= \int \rho_g(r, \varphi) \, dR \\
&= \int \rho_{g,1}(r) + \varepsilon \int \rho_{g,1}(r) f(\theta, \varphi) \, dr
\]
and
\[
\begin{align*}
\phi_{DM}(R, \varphi) &= \int \Phi_{DM,1}(r) + \varepsilon \int \Phi_{DM,1}(r) f(\theta, \varphi) \, dr
\end{align*}
\]

3.2.4. A projected gas density map to first order...

Now that we have expressed our observables in terms of \( 3D \) physical quantities, it is easy to infer a gas density map successively to first and second order in \( \varepsilon \). First to order the hydrostatic equilibrium condition (9) states that
\[ P_{g,1}(r) = -\rho_{g,1}(r) \Phi_{DM,1}(r). \]

In order to use it we need to deproject the relevant quantities. From the well known spherical deprojection formula (Binney and Tremaine 1987) based on Abel’s transform we have:
\[
\begin{align*}
\alpha P_{g,1}(r) &= -\frac{1}{\pi} \int_0^\infty y^\prime(R) \frac{dR}{(R^2 - r^2)^{\frac{1}{2}}} \\
&= -\frac{1}{\pi} \int_0^\infty y^\prime(r \cosh u) \, du
\end{align*}
\]
where \( R = r \cosh u \). Thus, we can write
\[
\begin{align*}
\alpha P_{g,1}(r) &= -\frac{1}{\pi} \int_0^\infty \cosh u \, y^\prime(r \cosh u) \, du \\
&= -\frac{1}{\pi} \int_r^\infty \frac{R}{r (R^2 - r^2)^{\frac{1}{2}}} y^\prime(R) \, dR
\end{align*}
\]
Similarly,
\[
\Psi_{D_{M,0}}(r) = -\frac{1}{\pi} \int_0^\infty \frac{1}{r} \frac{R}{(R^2 - r^2)^\frac{3}{2}} \delta_0''(R) dR .
\] (34)

We then get for the projected gas density
\[
D_{g,0}(R) = -2 \int_0^\infty \frac{r dr}{(R^2 - r^2)^\frac{3}{2}} \Phi_{D_{M,0}}(r)
\]
\[
= -\frac{2}{\alpha} \int_0^\infty \frac{r dr}{(R^2 - r^2)^\frac{3}{2}} \left( \int_r^\infty \frac{s ds}{(s^2 - r^2)^\frac{3}{2}} \delta_0''(s) \right) .
\] (36)

### 3.2.5. and a projected gas temperature map to first order

Once we built this projected gas density map, we can recover the projected gas temperature map. If we apply the equation of state (17) we get:
\[
\mathcal{C}_{g,0}(R) = \frac{1}{\beta} \int \frac{P_{g,0}(r)}{P_{g,0}(r)} \rho_0(R) dV
\]
\[
= \frac{1}{\beta} \int \frac{P_{g,0}(r)}{P_{g,0}(r)} \Phi_{D_{M,0}}(r) dV
\]
\[
= -\frac{1}{\pi^2 \beta} \int_0^\infty \frac{P_{g,0}(r)}{P_{g,0}(r)} \Phi_{D_{M,0}}(r) \frac{r dr}{(R^2 - r^2)^\frac{3}{2}} .
\] (37)

Since all the required functions (\(P_{g,0}, P_{g,0}^0, \Phi_{D_{M,0}}\)) have been derived in the previous section (equation (31) and (33)) we can get this way a projected gas temperature map.

#### 3.2.6. Corrections from departure to spherical symmetry: a projected gas density map to second order

We now reach the core of our method, namely we aim at deriving the quantity \(D_{g,1}(R, \varphi)\) defined by (25), i.e. the second order correction to the perfectly circular term:
\[
D_{g,0}(R, \varphi) = D_{g,0}(R) + \epsilon D_{g,1}(R, \varphi)
\]
\[
= \int \rho_{g,1}(r) f(\theta, \varphi) dV + \epsilon \int \rho_{g,1}(r) f(\theta, \varphi) dV .
\] (40)

If we derive equation (16) and combine it with equation (15) we note that
\[
\rho_{g,0}(r) \Phi_{D_{M,0}}(r) = \rho_{g,1}(r) \Phi_{D_{M,1}}(r) .
\]

Therefore we can write
\[
\int \rho_{g,1}(r) f(\theta, \varphi) dV = \int \frac{\rho_{g,0}(r)}{P_{g,0}(r)} \Phi_{D_{M,1}}(r) f(\theta, \varphi) dV .
\] (43)

At this point we want to express this quantity either in terms of WL data or in terms of SZ data depending on the quality of them, or even better in terms of an optimal combination of them.

On one hand, WL data provide us with a straightforward access to the function \(\phi_1(R) m(\varphi) = \int \Phi_{D_{M,1}}(r) f(\theta, \varphi) dV \) thus we choose to approximate (43) by
\[
\int \rho_{g,1}(r) f(\theta, \varphi) dV \simeq \frac{\rho_{g,0}(r)}{P_{g,0}(r)} \int \Phi_{D_{M,1}}(r) f(\theta, \varphi) dV
\]
\[
\simeq \frac{\rho_{g,0}(r)}{P_{g,0}(r)} \phi_1(R) m(\varphi)
\]
\[
\simeq \frac{\rho_{g,0}(R)}{P_{g,0}(R)} \left( \phi_{D_{M}}(R, \varphi) - \phi_0(R) \right) .
\] (44)

where we used the definitions of section (3.2.3) and where \(R\) corresponds to the radius observed in the image plane, i.e. the radius \(r\) equal to the distance between the line of sight and the center of the cluster. We will discuss this approximation in more details in section (3.2.8) and validate it through a practical implementation on simulations in section (4). But we already can make the following statements: would the line of sight follows a line of constant \(r\) throughout the domain of the perturbation, this expression would be rigorously exact. Moreover it turns out to be a good approximation because of the finite extent of the perturbation.

On the other hand SZ data provides us with a measurement of the function \(y_1(R) m(\varphi) = \int \Phi_{D_{M,1}}(r) f(\theta, \varphi) dV\) therefore we can use equation (16) and (14) to write
\[
\int \rho_{g,1}(r) f(\theta, \varphi) dV = \int \frac{\rho_{g,0}(r)}{P_{g,0}(r)} \Phi_{D_{M,1}}(r) f(\theta, \varphi) dV
\]
\[
\simeq \frac{\rho_{g,0}(r)}{P_{g,0}(r)} \int \Phi_{D_{M,1}}(r) f(\theta, \varphi) dV
\]
\[
\simeq \frac{\rho_{g,0}(r)}{P_{g,0}(r)} y_1(R) m(\varphi)
\]
\[
\simeq \frac{\rho_{g,0}(R)}{P_{g,0}(R)} \left( y(R, \varphi) - y_0(R) \right) .
\] (45)

Here again we used the same notation and approximation as in equation (44). Note however that as soon as we assumed isothermality, the ratio \(\rho_{g,0}/P_{g,0}\) is constant therefore this last step is exact. Were we not assuming isothermality, the departure from isothermality is expected to be weak thus this last approximation should be reasonable.

This last two alternative steps are crucial to our method since these approximations link the non-spherically symmetric components of various quantities. They are reasonable as will be discussed in section (3.2.8) and will be numerically tested in section (4).

Of course, only well-known quantities appear in equation (44) and (48): \(y, y_1, \phi_{D_{M}}\) and \(\phi_0\) are direct observational data whereas \(P_{g,0}(r)\) and \(\rho_{g,0}(r)\) are zeroth order quantities previously derived.
3.2.7. ...and a projected gas temperature map to second order

The projected temperature map can be obtained the same way as before. Using first the equation of state we can write:

\[
T_{\theta, \varphi}(r) + \varepsilon T_{\theta, \varphi}(r) f(\theta, \varphi) = \frac{1}{\beta} \left( P_{\theta, \varphi}(r) + \varepsilon P_{\theta, \varphi}(r) f(\theta, \varphi) \right)
\]

\[
\approx \frac{1}{\beta} \left( P_{\theta, \varphi}(r) + \varepsilon \frac{P_{\theta, \varphi}(r) - P_{\theta, \varphi}(r)}{P_{\theta, \varphi}(r)} f(\theta, \varphi) \right). \tag{49}
\]

Hence, since

\[
\zeta(R, \varphi) = \zeta_0(R, \varphi) + \varepsilon \zeta_1(R, \varphi) = \int T_{\theta, \varphi}(r) \, dl + \varepsilon \int T_{\theta, \varphi}(r) f(\theta, \varphi) \, dl \tag{50}
\]

we have

\[
\zeta_1(R, \varphi) = \int \frac{P_{\theta, \varphi}(r) - P_{\theta, \varphi}(r)}{P_{\theta, \varphi}(r)} P_{\theta, \varphi}(r) f(\theta, \varphi) \, dl. \tag{52}
\]

Here we choose to approximate the last integral as previously discussed in order to make use of observational SZ data. Therefore we rewrite this last equation as:

\[
\zeta_1(R, \varphi) \approx \frac{P_{\theta, \varphi}(R) - P_{\theta, \varphi}(R)}{P_{\theta, \varphi}(R)} \int P_{\theta, \varphi}(r) f(\theta, \varphi) \, dl \tag{53}
\]

\[
\approx \frac{P_{\theta, \varphi}(R) - P_{\theta, \varphi}(R)}{P_{\theta, \varphi}(R)} y(R, \varphi) m(\varphi) \tag{54}
\]

We obtain this way an expression to second order for the projected temperature in terms of either observed quantities or previously derived functions.

3.2.8. Why the previous approximation is reasonable on intuitive grounds?

Our previous approximations can be justified on intuitive grounds even if we will take care of validating it numerically in section (4) below. It relies on the fact that perturbations have by definition a finite extent, i.e., the first order correction to the perfectly circular (spherical) term is non zero only within a finite range. The typical size and the amplitude of the perturbation can be easily scaled from the SZ and WL data set. This guarantees the validity of our assumptions on observational grounds. The key point is that the perturbation itself has a kind of axial symmetry, whose axis goes through the center of the cluster and the peak of the perturbation. This is reasonable if the perturbation originates in e.g., an incoming filament but not for a substructure. The latter would therefore have to be treated separately by superposition (see section (5)).

This leads naturally to the statement that the typical angle we observe in the image plane is equal to the one we would observe if the line of sight were perpendicular to its actual direction, i.e., the perturbation as intrinsically the same angular extent in the directions along the line of sight and perpendicular to it. This is illustrated schematically in figure (2).

In (b) we represent a schematic slice in the $3-D$ potential responsible for this image. This slice has been performed along the dash-two-dotted plane indicated on figure (a). Here again, the full line corresponds to the perfectly circular $3-D$ term, e.g., $\phi_{DM, 3}$, and the dashed line to the first perturbative correction to it, e.g., $\phi_{DM, 1} m(\varphi)$. The line of sight direction is indicated by the full thin line. Were the line of sight perpendicular to this slice plane, we would observe the angular extent $\Delta \theta$. Giving an axial symmetry to this perturbation leads us to assess that $\Delta \varphi \sim \Delta \theta$.

Given this description we are now in a position to discuss the validity of our approximation. It consists in approximating the line of sight integral $\int \int g(\theta, \varphi) \Phi_{DM, 1}(r) f(\theta, \varphi) \, d\theta \, d\varphi$ by $g(R) \int \Phi_{DM, 1}(r) f(\theta, \varphi) \, d\theta$ where $g$ is any radial function. This approximation would be exact if $g(r)$ were constant in the relevant domain, i.e., if the line of sight had a constant $r$. As mentioned before this is the case in equation (48) if we assume isotropicality. But the functions $g(r)$ we might deal with may scale roughly as $r^2$, e.g., $\rho^2_{\theta, \varphi}(r) / P_{\theta, \varphi}(r)$ in equation (44), thus it is far from being constant. The consequent error committed can be estimated by the quantity $\Delta r g'(|r|$ where $\Delta r$ is the maximum $r$ discrepancy between the value assumed, $g(R)$, and the actual value as it is schematically illustrated in figure (3). In the worst case, $g'(r)$ scales as $r$. Then, using the obvious notations defined in this figure we get

\[
(Dr)_{max} = R(1 - 1/\sin(\theta - \Delta \theta/2)). \tag{55}
\]

Naturally this quantity is minimal for $\theta \approx 90^\circ$ and diverges for $\theta \approx 0^\circ$ when $\Delta \theta = 0^\circ$: the error is minimal when the line of sight is nearly tangential ($\theta \approx 90^\circ$) and so almost radial in this domain, and maximal when it is radial ($\theta = 0^\circ$). Thus in principle it is a very bad behavior, but the fact is that the closer $\theta$ is from $0^\circ$ the weaker the integrated perturbation is since it gets always more degenerate along the line of sight, i.e. the integrated per-
turbations tend to a radial behavior and will therefore be absorbed in the $\Phi_{\text{DM},0}(r)$ term. The extreme situation, i.e., when $\theta = 0^\circ$ will trigger a mere radial image as long as the perturbation exhibits a kind of axial symmetry. This error is impossible to alleviate since we are dealing with a fully degenerate situation and not will fail the method at all since the integrated perturbation will be null. This approximation will be validated numerically below.

3.3. How to obtain a $X$ prediction

The previously derived map offers a great interest that we now aim at exploiting, namely the ability of precise $X$ prediction. Indeed, for a given $X$ spectral emissivity model, the X-ray spectral surface brightness is

$$S_X(E) = \frac{1}{4\pi(1+z)^2} \int n_e^2 \Lambda(E, T_e) \, dl \quad (56)$$

where $\Lambda$ is the spectral emissivity, $z$ is the redshift of the cluster and $E$ is the energy on which the observed band is centered. Hence we can write, assuming a satisfying knowledge of $z$ and $\Lambda$:

$$S_X(E) \propto \int n_e^2 T_e^{1/2} \, dl \quad (57)$$

$$\propto \int \rho_{g,0}^2 T_{g,0}^{1/2} \, dl \quad (58)$$

$$\propto \int \rho_{g,0}^2 T_{g,0}^{1/2} \, dl + 2 \varepsilon \int \rho_{g,0} T_{g,0}^{1/2} \rho_{g,1} f(\theta, \varphi) \, dl$$

$$+ \frac{1}{2} \varepsilon \int \rho_{g,0}^2 T_{g,0}^{1/2} T_{g,1} f(\theta, \varphi) \, dl \quad (59)$$

where we omitted to write the $(r)$s for clarity’s sake. If we now make use of the same approximation as used and discussed before, we can express directly this quantity in terms of observations $y$ and $\phi$. We get indeed

$$S_X(E) \propto \int \rho_{g,0}^2 T_{g,0}^{1/2} \, dl$$

$$+ 2 \varepsilon \int \rho_{g,0} T_{g,0}^{1/2} \rho_{g,1} f(\theta, \varphi) \, dl$$

$$+ \frac{1}{2} \varepsilon \int \rho_{g,0}^2 T_{g,0}^{1/2} T_{g,1} f(\theta, \varphi) \, dl \quad (60)$$

$$\propto \int \rho_{g,0}^2 T_{g,0}^{1/2} \, dl + 2 \varepsilon \int \rho_{g,0} T_{g,0}^{1/2} \rho_{g,1} f(\theta, \varphi) \, dl$$

$$+ \frac{1}{2} \varepsilon \int \rho_{g,0}^2 T_{g,0}^{1/2} T_{g,1} f(\theta, \varphi) \, dl \quad (61)$$

Both the first order terms $T_{g,0}$ and $\rho_{g,0}$, and the second order corrections $D_{g,1}$ and $\rho_{g,1}$ have been derived in the previous sections. We are thus able to generate self-consistently a $X$ luminosity map from our previously derived maps. This is a very nice feature of this method. We will further discuss the approximation and its potential bias in the next section.

This derivation opens the possibility of comparing on the one hand $SZ$ and WL observations with, on the other hand, precise X-ray measurements as done e.g. by XMM or CHANDRA. Note that in the instrumental bands of most of X-ray satellites the $T_d$ dependence is very weak and can be neglected. This can be easily taken into account by eliminating the $T_d$ dependence in the previous formula. Even if the interest of such a new comparison is obvious we will discuss it more carefully in the two following sections. In principle, one could also make some predictions concerning the density weighted X-ray temperature defined by the ratio $\int n_e^2 T_e \, dl / \int n_e^2 \, dl$ but the fact is that since the gas pressure and so the SZ effect tends to have a very weak gradient we are not able by principle to reproduce all the interesting features of this quantity, namely the presence of shocks.

4. Application on simulations

In order to demonstrate the ability of the method in a simplified context we used some outputs of the recently developed N-body + hydrodynamics code RAMSES simulating the evolution of a $\Lambda$CDM universe. The RAMSES code is based on Adaptive Mesh Refinement (AMR) techniques in order to increase the spatial resolution locally using a tree of recursively nested cells of smaller and smaller size. It reaches a formal resolution of 12 $\text{kpc}^{-1}$ in the core of galaxy clusters (see Refregier and Teyssier 2000 and Teyssier 2001, in preparation, for details). We use here the structure of 2 galaxy cluster extracted of the simulation to generate our needed observables, i.e., X-ray emission measure, $SZ$ decrement and projected density (or projected gravitational potential).

The relevant observables, i.e., projected mass density, $SZ$ decrement and for comparison purpose only the X-ray emission measure, of the 2 clusters are depicted using a logarithmic scaling in figure 4 and 5 (upper panels). This
Fig. 4. The upper panel shows the results of simulation, from left to right, all using a logarithmic scaling, the projected mass density (M⊙ Mpc−2), the X-ray emission measure (cm−6 Mpc) and the SZ y parameter. This cluster is a good candidate for our approach since it has a circular core with surrounding perturbations so would be inadequate for an ellipsoidal fit. The lower panel shows, from left to right a zeroth order predicted X emission measure, the first order prediction (the zeroth order term plus the first order correction), both using a logarithmic scaling as well as the the relative error map, i.e. (predicted - simulated)/simulated X emission measure using a linear scaling. The 10 error contours are linearly separated between -1.0 and 1. Each box is 3.5 h−1 Mpc wide. The correlation coefficient between the predicted and the simulated X-ray emission measure is 0.978. The total flux differs only by 0.91%, thus even if the relative error map increases at high R the total error remains small due to the great dynamical range involved.

Clusters have been extracted of the simulation at z = 0.0 and thus tends to be more relaxed. They are ordinary clusters of virial mass (defined by δ200 in our particular cosmology) 4.50 10^{14} h^{-1} M_{\odot} and 4.15 10^{15} h^{-1} M_{\odot}. Both retain fairly regular shape, i.e. they have not undergone recently a major merger. The depicted boxes are respectively 3.5 h^{-1} Mpc and 4.0 h^{-1} Mpc wide. We smooth the outputs using a gaussian of width 120 h^{-1} kpc thus degrading the resolution. We did not introduce any instrumental noise. This clusters are to a good approximation isothermal to the envelope of simplicity we will assume that T_g is constant making the discussion on T_{g,0} and T_{g,1} useless at this point. We apply the method previously described using perturbed spherical symmetry.

We deduce by averaging over concentric annuli a zeroth order circular description of the gas density and then add to it some first order corrections. Note that since we assume isothermality SZ data give us straightforwardly a projected gas density modulo a temperature T_{g,0} coefficient, thus we use the formulation of equation (48), exact in this context. This constant temperature is fixed using the hydrostatic equilibrium and the WL data.

In figure 4 and 5 (lower panels) we show the predicted X-ray emission measure to zeroth and first order as well as a map of relative errors. Note that to first order the shape of the emission measure is very well reproduced. The cross-correlation coefficients between the predicted
and simulated X-ray emission measure are 0.978 and 0.986. Of course this is partly due to the assumed good quality of the assumed SZ data but nonetheless, it demonstrates the validity of our perturbative approach as well as of our approximation. The approximation performed in equation (61), i.e. the multiplication by the function $p_{bg}(R)$ will naturally tend to cut out the perturbations at high $R$. This is the reason why the further perturbation are slightly less well reproduced and the relative errors tend to increase with $R$. Nevertheless, since the emission falls rapidly with $R$ as visible on the lower figures (note the logarithmic scaling) the total flux is well conserved, respectively to 0.9 % and 9 %. This last number might illustrate that the large extent of the perturbations in the second case may limit our method. An ellipsoidal fit could have help decrease this value. Note that moreover the clump visible mainly in X-ray emission measure of figure 5 is not reproduce. This is natural because it does not appear through the SZ effect since the pression remains uniform throughout clumps. If resolved by WL, this substructure should anyway be treated separately, e.g. by considering the addition of a second very small structure. Note that the first cluster showed exhibits a spherical core elongated in the outer region thus it is not actually as ellipsoidal as it looks which may explain why our perturbed spherical symmetry works well.

5. Discussion

5.1. Hypothesis . . . and non hypothesis

Our approach makes several assumptions. Some general and robust hypothesis have been introduced and discussed in section 2.1. Note that we do not need to assume isothermality. Our key hypothesis consists in assuming the validity of a perturbative approach and in the choice of the na-
tecture of this perturbations, *i.e.* with a radial/angular part separation. Theoretical predictions, observations and sim-
ulations show that relaxed clusters are regular and glob-
ally spheroidal objects, which is what initially motivated
our approach. Then in our demonstration on simulations,
this turns out to be reasonable. Such an approach can not
deal properly with sharp features as *e.g.* shocks waves due
to infalling filaments. Then assuming the validity of the
angular and radial separation, leads to the equality of this
angular part for all relevant physical quantities \((P_p, T_\alpha, \phi_{DM} \ldots)\), using to first order in \(\varepsilon\) the hydrostatic equi-
brium and the equation of state. If this is not satisfied in
practice then we could either question the validity of this
separation or the physics of the cluster. Our experience
with simulation shows that for reasonably relaxed clusters,
* i.e., not going through a major merge, the angular part
of the perturbation is constant amongst observables.
Thus it looks like the separation (and thus the equality of
the angular perturbation) is a good hypothesis in general
and its failure is a sign of non-relaxation, *i.e.* non-validity
of our general physical hypothesis.

Then an important hypothesis lies in the validity of the
approximation used. Note first that even if its form is
general, its validity depends on the quantity which is
assumed to be constant along the integral. In the case of
the gas density obtained from the SZ map, it is an exact
statement as soon as we assume the isothermality and
since clusters in general are not too far from isothermality,
this hypothesis is reasonable.

Now, some worth to remember “non hypothesis” are the
isothermality and the sphericity (or ellipsoidality). This
might be of importance. Indeed, in evaluating the
Hubble constant from joint SZ and X-ray measurement
it has been evaluated in (Inagaki et al. 1995; Roettger
et al. 1997; Puy et al. 2000) that both the asphericity
and the non-isothermality of the relevant cluster can yield
some important bias (up to 20%). Even if this measure
is not our concern here, it is interesting to note that this
hypothesis are not required here.

5.2. The equivalent spheroidal symmetry case

So far, we have worked and discussed the perturbed spheri-
oidal symmetry case. If we turn to spheroidal symmetry the
problem is very similar as long as we assume the knowl-
gedge of the inclination angle \(i\) between the polar axis of
the system and the line of sight. This is what we recall in
appendix B which is directly inspired from (Fabricant
et al. 1984): once the projection is nicely parameterised we
get for the projected quantity, *e.g.* for the pressure:

\[
g(\eta) = \frac{2}{\pi} \left( \int_{\eta}^{\infty} P_{\phi,0}(t) \frac{dt}{(t^2 - \eta^2)^{3/2}} \right)
\]

\[
P_{\phi,0}(t) = -\frac{1}{2\pi R E} \left( \int_{\eta}^{\infty} \frac{d\eta}{(\eta^2 - t^2)^{1/2}} \right)
\]

following the notations of appendix B. Since we are dealing
with the same Abel integral we can proceed in two steps
as we did before.

Even if the inclination angle is a priori not accessible
directly through single observations it has been demon-
strated that it is possible to evaluate it using the de-
projection of an axially symmetric distribution of either
X-ray/SZ maps or SZ/surface density maps (Zaroubi et
al. 1998; Zaroubi et al. 2000). Our approach in this work
try to avoid to exploit the full 3-D structure rather than
building it, and this is done in a simple self-consistent way
therefore we will not get into the details of this procedure
that will be discussed in a coming work (Doré et al. 2001,
in preparation). Note also that axially symmetric config-
uration elongated along the line of sight may appear as
spherical. This is a difficult bias to alleviate without any
prior for the profile. In our case, our method will be biased
in the sense that the deprojected profile will be wrong.
Nevertheless, we might hope to reproduce properly the
global quantities, like abundance of DM or gas and so to
alleviate some well known systematics (see previous sec-
tion), *e.g.* in measuring the baryon fraction.

6. Conclusion and outlook

It this paper we have presented and demonstrated the
efficiency of an original method allowing to perform in a
self-consistent manner the joint analysis of SZ and WL
data. Using it on noise free simulation we demonstrated
how well it can be used to make some X-ray surface
brightness prediction, or equivalently emission measure.
Our choice in this approach has been to hide somehow
the deprojection by using some appropriate approximations.
Thus we do not resolve fully the 3-D structure of clus-
ters, but note that the work presented here is definitely
a first step towards a full deprojection (Doré et al. 2001,
in preparation). Some further refinements of the methods
are under progress as well.

When applying the method to true data, the instru-
mental noise issue is an important matter of concern.
Indeed, whereas the strong advantage of a parametric ap-
proach, *e.g.* using a \(\beta\)-model, is that it allows to adjust
the relevant parameters, *e.g.* \(r_c\) and \(\beta\), on the projected
quantities (the image) itself, which is rather robust to
noise, it might be delicate to determine the profiles and its
derivative by a direct deprojection. Nevertheless, our per-
turbative approach, as it first relies on a zeroth order
quantity found by averaging over some annulus, a noise
killing step (at least far from the center), and then work
on some more projected perturbation should be quite
robust as well. Consequently we hope to apply it very soon
on true data. Furthermore, in this context it should allow a
better treatment of systematics (asphericity, non iso-
thermality, ...). Placing any measure of the baryon fraction
\(f_b\) or the Hubble constant \(H_0\) using X-ray and SZ effect
(Inagaki et al. 1995). These points will be discussed some-
where else (Doré et al. 2001, in preparation).
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Annexe : Deprojection in spheroidal symmetry

In this appendix, we recall some useful results concerning spheroidal projection derived by Fabricant, Gorenstein and Rybicki (Fabricant et al. 1984). In the context of spheroidal systems, cartesian coordinates system are the most convenient for projection. Thus, if the observer’s coordinate system \((x, y, z)\) is chosen such that the line of sight is along the \(z\)-axis and such that the polar axis of the spheroidal system \(x'\) lies in the \(x - z\) plane at an inclination angle \(\alpha\) to the \(z\)-axis, then, in the cartesian coordinate system \((x', y', z')\) the general physical quantities relevant to our problem depends only on the parameter \(t\) defined by

\[
I^2 = x^2 + y^2 + z^2 \over A_e^2
\]

\[
= (x \cos \alpha + y \sin \alpha z)^2 + y^2 + (z \cos \alpha - x \sin \alpha z)^2 \over A_e^2
\]

(64)

(65)

If we project a physical quantity \(G(t)\) on the observer sky plane \(x - y\) then,

\[I(x, y) = \int_{t}^{+\infty} G(t) dt = \int_{t}^{+\infty} \frac{B_e}{R} \int_{t}^{+\infty} \frac{G(t) t dt}{(t^2 - \eta^2)^{\frac{3}{2}}}
\]

where

\[\eta^2 = \frac{x^2}{(RA_e)^2} + \frac{y^2}{(Be)^2}
\]

(69)

and \(R = \sqrt{\frac{B_e^2}{A_e^2} \cos^2 \alpha + \sin^2 \alpha \cdot i}.
\]

(70)

Of course this result shows that if we were to observe a spheroidal system we would map ellipses with an axial ratio equal to \(B = \frac{1}{R A_e} \). But the main result of this appendix is that we obtain at the end an Abel integral similar to the one obtained in the case of spheroidal system, where the radius as been replaced by the parameter \(t\). This simple fact justifies the very analogous treatment developed in this paper for spherical and spheroidal systems.