Finite-size scaling and the deconfinement transition in gauge theories.

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Abstract

We introduce a new method for determining the critical indices of the deconfinement transition in gauge theories. The method is based on the finite size scaling behavior of the expectation value of simple lattice operators, such as the plaquette. We test the method for the case of SU(3) pure gauge theory in (2 + 1) dimensions and obtain a precise determination of the critical index $\nu$, in agreement with the prediction of the Svetitsky-Yaffe conjecture.

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Gauge theories at finite temperature undergo a phase transition from a low temperature phase, in which color sources are confined by a linearly rising potential, to a high temperature phase in which the attractive force is screened at long distances and isolated quarks can exist. This same behavior can be observed even in pure gauge theories, without matter fields, by studying the theory in presence of static sources.

The order parameter of the deconfinement transition in a pure gauge theory is the Polyakov loop [1, 2], representing the world-line of a static source of color charge (that can be thought of as an infinitely massive quark). The Polyakov loop takes values in the center of the gauge group: the \((d+1)\)-dimensional gauge theory generates an effective \(d\)-dimensional theory for the Polyakov loop, having the center of the gauge group as global symmetry group. The deconfinement phase transition coincides with the spontaneous breaking of this global symmetry.

In this work we introduce a new method for the computation of the critical indices of a pure gauge theory with second order deconfinement transition. According to the Svetitsky-Yaffe conjecture [3], such critical indices for the \((d+1)\)-dimensional gauge theory coincide with those of the effective \(d\)-dimensional model, if the latter has also a second order phase transition. We will consider in particular the critical index \(\nu\) of the correlation length. For a general \(d\)-dimensional statistical model a possible way of extracting the value of \(\nu\) from lattice Monte Carlo simulations is to study the finite size scaling (FSS) behavior of the energy operator: FSS theory predicts that, if \(L\) is the lattice size, the expectation value of the energy operator behaves for large \(L\) as

\[
\langle E \rangle_L \sim \langle E \rangle_\infty + k L^{\frac{1}{\nu} - d},
\]

where \(\langle E \rangle_\infty\) is the expectation value of the energy operator in the thermodynamic limit and \(k\) is a non-universal constant. This method was applied \(e.g.\) in Ref. [4] to evaluate the critical index \(\nu\) of the three-dimensional Ising model.

Compared to other methods based on the FSS of fluctuation operators such as susceptibilities or Binder cumulants, the advantage lies in the fact that \(\langle E \rangle_L\) can be computed to high accuracy. The main drawback is that the term containing \(\nu\) in Eq. (1) is subdominant for \(L \to \infty\) with respect to the bulk contribution \(\langle E \rangle_\infty\). The numerical results we will present strongly
suggest that, in the case of gauge theories, the advantages outweigh the drawbacks and the method can give very accurate results.

Consider a gauge invariant operator \( \hat{O} \) that is invariant also under the global symmetry given by the center of the gauge group. Examples of such operators include Wilson loops and those of the form

\[
P(x)P^\dagger(y),
\]

where \( P(x) \) and \( P(y) \) are Polyakov loops at two different sites \( x \) and \( y \) of the \( d \)-dimensional space. It is natural to expect the operator product expansion (OPE) of any such operator to have the same form as the one of \( E \):

\[
\hat{O} = c_I I + c_\epsilon \epsilon + \cdots.
\]

Here \( I \) and \( \epsilon \) are, respectively, the identity and the scaling energy operator\(^1\) in the statistical model, and the dots represent contributions of irrelevant operators. The ansatz of Eq. (3) was introduced and tested in Ref. [5], and used in Ref. [6, 7], to obtain some exact results on correlation functions of \((2 + 1)\)-dimensional gauge theories at the deconfinement transition.

In particular, Eq. (3) implies that the FSS behavior of the expectation value \( \langle \hat{O} \rangle \) will have the form of Eq. (1):

\[
\langle \hat{O} \rangle_L \sim \langle \hat{O} \rangle_\infty + cL^{\frac{1}{\nu} - d}.
\]

The contributions of the irrelevant operators will be subleading for \( L \to \infty \).

We conclude that the FSS behavior of any such operator can be used to determine the value of \( \nu \) through Eq. (4). The obvious advantage is that one can use operators, such as the plaquette, whose expectation value can be computed to high accuracy with relatively modest computational effort.

To test the method in practice, we used it to determine the critical index \( \nu \) for \( SU(3) \) pure gauge theory in \((2 + 1)\)-dimensions. This was evaluated from Monte Carlo simulations in Ref. [8] with the result

\[
\nu_{MC} = 0.90(20).
\]

\(^1\)Not to be confused with the lattice energy operator \( E \), which is just an example of an operator with OPE given by Eq. (3).
However, if the Svetitsky-Yaffe conjecture [3] holds then the 2-dimensional effective theory of the Polyakov loop is in the universality class of the 3-state Potts model, and the value of $\nu$ is known exactly:

$$\nu = \frac{5}{6} = 0.833\ldots$$

(6)

The observables we considered are the expectation values of the plaquette and of the operator defined in Eq. (2), where $x$ and $y$ are taken to be nearest neighbors sites in the 2-dimensional (spatial) lattice. We computed these expectation values on lattices with temporal extension $N_t = 2$ and spatial sizes $L = N_x = N_y$ ranging from 7 to 30. The simulations were performed at the critical coupling

$$\beta_c(N_t = 2) = 8.155$$

(7)

obtained in Ref. [8]. The simulation algorithm we adopted was a mixture of one sweep of a 10-hit Metropolis and four sweeps of over-relaxation consisting in two updates of (random) $SU(2)$ subgroups. For each simulation we collected 400K equilibrium configurations, separated each other by 10 updating steps. The error analysis was performed by the jackknife method applied to data bins at different levels of blocking.

We report in Table 1 the expectation values we obtained. Because of the asymmetry of the lattice, space-like and time-like plaquettes have obviously different expectation values and must be considered as two different operators. To evaluate $\nu$, we performed a single multibranched fit of the three data sets. To avoid cross-correlations we included in the fit only space-like (“magnetic”) plaquettes from lattices with odd $L$ and time-like (“electric”) plaquettes from lattices with even $L$. Polyakov loop correlations were measured in separate runs and therefore are not correlated with the plaquette measurements.

The result of the fit is

$$\nu = 0.827(22) , \quad \chi^2_{\text{red}} = 0.84,$$

(8)

in excellent agreement with the value of Eq. (6), coming from the Svetitsky-Yaffe conjecture, and with remarkably improved accuracy in comparison to the existing Monte Carlo evaluation.

In conclusion, the method we have proposed appears to give very precise results for critical indices, thanks to the fact that only “simple” expectation
values have to be evaluated, such as the plaquette expectation value. The same method can be used to evaluate the critical coupling (that, in our case, was taken from the literature): since Eq. (4) is valid only at the critical point, one should perform the same fit at several couplings and determine the value of the critical one, by comparing the $\chi^2$'s of the different fits.

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References

Table 1: Expectation values of the electric and magnetic plaquette and Polyakov loop correlator.

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