QCD sum rules for the pseudoscalar decay constants
— To constrain the strange quark mass

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Abstract

We study the higher order corrections of quark masses to the Gell-Mann–Oakes–Renner (GOR) relation by constructing QCD sum rules exclusively for pseudoscalar mesons from the axial-vector correlation function, $i \int d^4 x \ e^{ip \cdot x} \langle 0 | T[A_\mu(x) A_\nu(0)] | 0 \rangle$. To project out the pseudoscalar meson contributions, we apply $-p^\mu p^\nu/p^2$ to this correlation function and construct sum rules for the decay constants of pseudoscalar mesons, $f_\pi$, $f_K$ and $f_{\eta_8}$. The OPE is proportional to quark masses due to PCAC. To leading order in quark mass, each sum rule reproduces the corresponding GOR relation. For kaon and $\eta_8$, the deviation from the GOR relation due to higher orders in quark mass is found to be substantial. But the deviation gives better agreements with the phenomenology. Our sum rule provides a sensitive relation between $f_K$ and $m_s$, which stringently constrain the value for $m_s$. To reproduce the experimental value for $f_K$, $m_s$ is found to be 186 MeV at 1 GeV scale. The $f_{\eta_8}$ sum rule also supports this finding.

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According to PCAC [1], the decay constant of a pion measures the coupling strength of the corresponding axial-vector current to the pion. One useful constraint which can be derived by combining PCAC with the soft-pion theorem is the Gell-Mann–Oakes–Renner (GOR) relation
\[ -4m_q\langle \bar{q}q \rangle = m_q^2 f_\pi^2, \quad (f_\pi = 131 \text{ MeV}). \] (1)

As the pion decay constant \( f_\pi \) is well-known [2], this GOR relation can be used to constrain either the quark condensate or the quark mass. In particular, the small quark mass \( m_q = (m_u + m_d)/2 \sim 7 \text{ MeV} \) [3,4] at 1 GeV scale restricts the quark condensate to be \( \langle \bar{q}q \rangle \sim -(225 \text{ MeV})^3 \), the value being widely used in practical QCD sum rule calculations. Corrections to the GOR relation are an order of \( \mathcal{O}(m_q^2) \) or higher, which should be small.

When a strange quark is involved, the corresponding GOR relation may not be a reliable constraint. For kaon, one may start from the kaon PCAC relation
\[ \langle 0|\bar{s}\gamma_\mu\gamma_5 u|K^+(p)\rangle = ip_\mu f_K. \] (2)

By taking the divergence and using the soft-kaon theorem, one arrives at the kaon GOR relation
\[ -(m_q + m_s)[\langle \bar{q}q \rangle + \langle \bar{s}s \rangle] = m_K^2 f_K^2. \] (3)

Due to the large strange quark mass \( m_s \), one may expect nonnegligible corrections from \( \mathcal{O}(m_s^2) \) or higher order, and this kaon GOR relation, though exact to leading order in quark mass, may not be a good constraining equation for \( m_s \) and \( f_K \). Indeed, the conventional QCD parameters [for example, see Ref. [5].]
\[ m_s = 150 \text{ MeV}; \quad \langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle, \] (4)
yield \( f_K = 116 \text{ MeV} \), which is much smaller than its experimental value \( f_K^{\text{expt}} \sim 160 \text{ MeV} \) [2]. Much smaller \( m_s \) from the baryon sum rules [6] leads to much larger discrepancy. To reproduce \( f_K^{\text{expt}} \) from this kaon GOR relation, the strange quark mass (assuming that \( \langle \bar{s}s \rangle \) is fixed as above) needs to be around 300 MeV, which is evidently too large for the current mass of strange quark. Consideration for the \( \eta_8 \) case leads to a similar (even larger) discrepancy from the \( \eta_8 \) GOR relation.

Then one may naturally attempt to improve the GOR relations especially when strange quark is involved. We look for the improvement that provides a reliable relation between \( f_K \) and \( m_s \). This improvement of course should not affect the well-established GOR relation in \( u, d \) quark sector, i.e. Eq.(1). We then look for the \( m_s \) in this improvement that reproduces \( f_K^{\text{expt}} \). For this prediction to be reliable, this \( m_s \) must be tested in the other case such as \( \eta_8 \).

As the discrepancy mentioned above comes from the large strange quark mass, it may be natural to calculate higher quark mass corrections to the GOR relations. For this purpose, we use QCD sum rules [7,8] for the axial-vector correlation function. Specifically, we wish to construct a sum rule whose phenomenological side contains only the pseudoscalar meson contributions while the QCD side is proportional to the quark masses. To leading
order in quark masses, we want it to reproduce the GOR relation precisely so that we can systematically study the corrections from higher orders in quark mass to the GOR relation. Similar QCD sum rule calculations of the kaon decay constant $f_K$ [9] did not quite look into this higher order corrections and our approach in this work is different from them.

To illustrate our approach briefly, we start by noting that the correlation function of the axial-vector current for example $A_{\mu} = \bar{u}\gamma_\mu\gamma_5 d$ contains the two invariant functions, $\Pi_1$ and $\Pi_2$, defined by

$$\Pi_{\mu\nu} = i \int d^4 x e^{ip\cdot x} \langle 0 | T[A_{\mu}(x) A_{\nu}(0)] | 0 \rangle ,$$

$$\equiv -g_{\mu\nu} \Pi_1 + p_{\mu} p_{\nu} \Pi_2 . \quad (5)$$

This is in contrast to the vector correlation function which contains (under the isospin symmetry) one invariant function due to the current conservation. PCAC tells us that pion and its higher resonances contribute only to $\Pi_2$ while the axial-vector meson contributions are contained in both $\Pi_1$ and $\Pi_2$. PCAC further imposes that $p^\mu \Pi^{\mu\nu}_A$ picks up contributions only from the pionic resonances. Therefore, to investigate the properties of a pion without having contamination from the $a_1$ meson, it may be useful to consider the correlation function defined by

$$\Pi(p^2) \equiv -\frac{p^\mu p^\nu}{p^2} \Pi_{\mu\nu} = \Pi_1 - p^2 \Pi_2 . \quad (6)$$

This projected correlation function in the QCD side must be proportional to quark masses. Therefore, the sum rule from $\Pi(p^2)$ may provide a useful constraint which directly relates quark masses to the pion decay constant. By generalizing this sum rule to the strange quark sector, we investigate the quark mass corrections of the order $O(m_s^2)$ or higher to the corresponding GOR relation. This may give stringent constraints for $m_s$.

This paper is organized as follows. In Section II, we present the OPE calculation for the projected correlation function of the general axial-vector current $\bar{q}_1 \gamma_\mu \gamma_5 q_2$. Using this OPE up to dimension 6, we construct QCD sum rules for $f_\pi$ in Section III and estimate how large the higher order corrections in quark mass to the GOR relation. In Section IV, we apply this framework to the kaon sum rule. We use this kaon sum rule to determine the strange quark mass $m_s$ by using $f_K^\text{ext}$ as an input. In Section V, we test this $m_s$ in the $\eta_8$ sum rule and see if it reproduces $f_{\eta_8}$ consistent with the phenomenology. We summarize in Section VI.

**II. THE PROJECTED SUM RULE FOR THE GENERAL AXIAL-VECTOR CURRENT**

In this section, we calculate the operator product expansion (OPE) for the projected correlation function of the general axial-vector current $\bar{q}_1 \gamma_\mu \gamma_5 q_2$

$$\Pi^A(p^2) \equiv -\frac{p^\mu p^\nu}{p^2} \Pi^{A\mu\nu}$$

$$= -\frac{p^\mu p^\nu}{p^2} i \int d^4 x e^{ip\cdot x} \langle 0 | T[\bar{q}_1(x) \gamma_\mu \gamma_5 q_2(x) \bar{q}_2(0) \gamma_\nu \gamma_5 q_1(0)] | 0 \rangle$$

$$= \frac{i}{p^2} \int d^4 x e^{ip\cdot x} \text{Tr} [p^\mu i S_2(x,0) p^\nu i S_1(0,x)] , \quad (7)$$

3
where \( q_j (j = 1, 2) = u, d \text{ or } s \), and \( i S_j(x, 0) \) is the quark propagator of the flavor \( q_j \) in the fixed-point gauge. Once the OPE of this correlation function is calculated, it will be straightforward to obtain the OPE for pion, kaon and \( \eta_8 \). The advantage of using this projected correlation function is that the axial-vector meson contributions are canceled and the continuum threshold starts at larger scale so that the correlation function is well saturated by the low-lying resonance.

A few technical remarks are in order before we proceed to the detailed OPE calculation. As \( \Pi^A(p^2) \) is a scalar function, the OPE involves only even dimensional operators. No odd dimensional operators can contribute to the sum rule. Also, as the axial-vector current is not conserved by the finite quark mass, all the OPE must be proportional to the quark masses. Thus we can focus on the operators that contain quark masses. This sum rule in the end will give sensitive constraints relating quark mass to the corresponding meson decay constant.

The OPE of the correlator Eq.(7) can be calculated by the standard techniques in the fixed-point gauge [10]. To include the finite quark mass effectively, we calculate the correlator in the momentum space,

\[
\pi^A(p^2) = \frac{i}{p^2} \int \frac{d^4k}{(2\pi)^4} \text{Tr} [\slashed{p} \gamma_5 i S_1(k) \slashed{k} \gamma_5 i S_2(k - p)] .
\]  

(8)

In the coordinate space, one has to expand the propagator in \( m_j \) and hence it is difficult to keep the quark mass terms to all orders at each dimension. The perturbative part can be calculated straightforwardly using the free quark propagator

\[
iS_{j\,\text{free}}(k) = \frac{i}{k^2 - m_j^2} (j = 1, 2).
\]  

(9)

Putting this into Eq. (8) and using the standard techniques of the dimensional regularization and the Feynman parameterization, we readily compute the perturbative part of Eq. (8),

\[
\pi^A_{\text{pert}}(p^2) = \frac{3}{4\pi^2} \int_0^1 du \left[ m_1^2 - u(m_1^2 - m_2^2) + m_1 m_2 \right] \times \ln \left[ -u(1 - u)p^2 + m_1^2 - u(m_1^2 - m_2^2) \right] .
\]  

(10)

Note, this is an order of \( O(m_j^2) \) or higher. For the vector correlation function, the \( m_1 m_2 \) term will have the negative sign so that when the two quark flavors are equal (\( q_1 = q_2 \)), the perturbative part vanishes as can be expected from the current conservation.

For the nonperturbative part, various even dimensional operators can contribute to the sum rule. In this work, we will calculate the OPE up to dimension 6. First we consider the case when one quark propagator in Eq. (8) is disconnected to form quark condensate

\[
iS_{ab}^{\alpha\beta}(k) \rightarrow \int d^4x \ e^{ik} \langle q_\alpha^a(x) q_\beta^b(0) \rangle \text{while the other is remained to be the free propagator. By performing the Tayler expansion around } x_\mu = 0 \text{ and the Fourier transformation to the momentum space, we obtain the expansion for the quark condensate [11],}
\]

\[
^1\text{In our notation, } \alpha, \beta, \gamma \cdots \text{ refer Dirac indices and } a, b, c \cdots \text{ color indices.}
\]
\[
\int d^4 x \, e^{i k \cdot x} \langle q_\alpha^a(x) \bar{q}_\beta^b(0) \rangle = \frac{-\delta^{ab}}{12} (2\pi)^4 \times \left\{ \delta^{\alpha\beta} \left[ \langle \bar{q}q \rangle \delta^{(4)}(k) + \langle \bar{q}D_\mu q \rangle \frac{\partial}{i\partial k^\mu} \delta^{(4)}(k) + \frac{1}{2} \langle \bar{q}D_\mu D_\nu q \rangle \frac{\partial}{i\partial k^\mu} \frac{\partial}{i\partial k^\nu} \delta^{(4)}(k) + \cdots \right] \right. \\
+ \gamma^{\alpha\beta}_\lambda \left[ \langle \bar{q}q \rangle \delta^{(4)}(k) + \langle \bar{q}D_\mu q \rangle \frac{\partial}{i\partial p^\mu} \delta^{(4)}(k) + \cdots \right] \right\}. 
\quad (11)
\]

One technical remark [11] is that the Wilson coefficient of the condensate \( \langle \bar{q}D_{\mu_1} \cdots D_{\mu_n} q \rangle \) is related to the Wilson coefficient of \( \langle \bar{q}q \rangle \) via

\[
C_{\bar{q}D_{\mu_1} \cdots D_{\mu_n} q}(k) = \frac{(-i)^n}{n!} \left[ \frac{\partial}{\partial k_{\mu_1}} \cdots \frac{\partial}{\partial k_{\mu_n}} \right] C_{\bar{q}q}(k),
\quad (12)
\]

and similarly

\[
C_{\bar{q}\gamma_{\mu_1}D_{\mu_1} \cdots D_{\mu_n} q}(k) = \frac{(-i)^n}{n!} \left[ \frac{\partial}{\partial k_{\mu_1}} \cdots \frac{\partial}{\partial k_{\mu_n}} \right] C_{\bar{q}q_\omega q}(k).
\quad (13)
\]

Using this technique, we calculate the OPE involving the quark condensate up to dimension 6. Recalling that only even dimensional condensates contribute to the sum rule, the \( \delta^{\alpha\beta} \) part of Eq. (11) gives nonzero contributions when the quark-mass part of Eq. (9) is taken for the remaining propagator. Similarly, the \( \gamma^{\alpha\beta}_\lambda \) part of Eq. (11) contributes when the \( k \) part of the free propagator is taken. Also, since the OPE must be proportional to quark mass we need to extract only the quark mass dependent part from various quark condensates \( (j = 1, 2) \),

\[
\langle \bar{q}_j \gamma_\lambda D_\alpha q_j \rangle \rightarrow -g_\lambda^\alpha \frac{im_j}{4} \langle \bar{q}_j q_j \rangle, \\
\langle \bar{q}_j D_\alpha q_j \rangle \rightarrow -g_\alpha^\beta \frac{m_j^2}{4} \langle \bar{q}_j q_j \rangle, \\
\langle \bar{q}_j \gamma_\lambda D_\alpha D_\beta q_j \rangle \rightarrow (g_\lambda^\alpha g_\beta^\sigma + g_\lambda^\beta g_\alpha^\sigma + g_\alpha^\sigma g_\lambda^\beta) \frac{im_j^3}{24} \langle \bar{q}_j q_j \rangle. 
\quad (14)
\]

In doing so, we obtain the OPE containing the quark condensate up to dimension 6,

\[
\Pi_\bar{q}q^A(p^2) = -(m_1 + m_2) \left[ \frac{\langle \bar{q}_1 q_1 \rangle}{p^2 - m_1^2} + \frac{\langle \bar{q}_2 q_2 \rangle}{p^2 - m_2^2} \right] \\
+ \frac{1}{2} (m_1^2 - m_2^2) \left[ \frac{m_1 \langle \bar{q}_1 q_1 \rangle}{(p^2 - m_1^2)^2} - \frac{m_2 \langle \bar{q}_2 q_2 \rangle}{(p^2 - m_2^2)^2} \right]. 
\quad (15)
\]

The corresponding OPE for the vector correlation function is obtained by replacing \( m_2 \rightarrow -m_2 \) and \( \langle \bar{q}_2 q_2 \rangle \rightarrow -\langle \bar{q}_2 q_2 \rangle \). The first term is the one that leads to the GOR relation at the order \( \mathcal{O}(m_j) \). In our calculation we kept the quark mass in the denominator of the free quark propagator. Thus, when we analytically continue to the time-like region, the imaginary part picks up the pole at \( p^2 = m_j^2 \). This aspect is slightly different from the usual application of QCD sum rules where the pole of the nonperturbative OPE is located at \( p^2 = 0 \). The modification of a sum rule result due to this is marginal in usual practices. However, in our sum rule with \( \Pi(p^2) \), we are looking at small strength both in the phenomenological and
QCD side. In this case, this small distinction due to the quark mass in the quark propagator may be important especially when strange quark is involved.

The contribution from the dimension-4 gluon condensate $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ should be vanished in our sum rule as it is not proportional to quark mass. The other possible dimension 6 operators would be $m_i m_j \langle \frac{\alpha_s}{\pi} G^2 \rangle$ $(i, j = 1, 2)$ and $m_j \langle \bar{q}_j g_s \sigma \cdot G q_j \rangle$. These however do not contribute to our sum rule. The disappearance of the $m_j \langle \bar{q}_j g_s \sigma \cdot G q_j \rangle$ contribution can be easily understood by recalling that the GOR relation is exact at the order $\mathcal{O}(m_j)$. If $m_j \langle \bar{q}_j g_s \sigma \cdot G q_j \rangle$ does contribute, then the GOR relation is no longer valid even at the order $\mathcal{O}(m_j)$. Indeed, one can show by a direct calculation that $m_j \langle \bar{q}_j g_s \sigma \cdot G q_j \rangle$ does not contribute to our sum rule.

Verifying the disappearance of the $m_i m_j \langle \frac{\alpha_s}{\pi} G^2 \rangle$ contribution (more precisely its imaginary part) is technically more involved. For a consistent calculation, one needs to consider the quark propagators with one and two gluons attached. Using the quark propagators interacting with gluons (see Ref. [8] for their explicit expressions), we readily compute this contribution and obtain

$$-\frac{1}{8} \langle \frac{\alpha_s}{\pi} G^2 \rangle \int_0^1 du \left[ (1-u)^2 m_1 + u^2 m_2 \right] \frac{m_1 + m_2}{L^2}$$

where $L = u(1-u)p^2 - m_1^2 + u(m_1^2 - m_2^2)$.

Note, the numerator is already dimension 6. Thus, up to dimension 6, we can safely drop the quark mass dependence in $L$. Now the imaginary part of the first term involving $(1-u)^2 m_1$ is proportional to $\int_0^1 du \delta'(up^2)$ which of course vanishes. By the same reasoning, the second term involving $u^2 m_2$ also does not pick up the imaginary part. Therefore, the dimension 6 contribution involving the gluon condensate does not contribute to our sum rule.

Before closing this section, here we give the imaginary part of the OPE [Eqs.(10) and (15)] up to dimension 6 for the correlation function of the general axial-vector current $A^\mu = \bar{q}_1 \gamma^\mu \gamma_5 q_2$,

$$\frac{1}{\pi} \text{Im}\Pi(s) = -\frac{3}{8\pi^2} \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{s}} + \frac{(m_1^2 - m_2^2)^2}{s^2} \left[ (m_1 + m_2)^2 - \frac{(m_1^2 - m_2^2)^2}{s} \right] \theta \left[ s - (m_1 + m_2)^2 \right]$$

$$+ (m_1 + m_2) \left[ \langle \bar{q}_1 q_1 \rangle \delta(s - m_2^2) + \langle \bar{q}_2 q_2 \rangle \delta(s - m_1^2) \right]$$

$$+ \frac{1}{2} (m_1^2 - m_2^2) \left[ m_1 \langle \bar{q}_1 q_1 \rangle \frac{d}{ds} \delta(s - m_2^2) - m_2 \langle \bar{q}_2 q_2 \rangle \frac{d}{ds} \delta(s - m_1^2) \right].$$

Note, this is symmetric under $1 \leftrightarrow 2$. From this, it will be straightforward to obtain the OPE for the pion, kaon and $\eta$ sum rules to be discussed below.

Now, the advantage of working with the projected correlation function becomes clear. This OPE contains a few sources of uncertainties, quark condensates and quark masses.

To check this easily, it is useful to use the quark propagator in the coordinate space as given in Eq.(8) and Eq.(13) of the first reference in Ref. [12]. The two contributions to this dimension 6 condensate indeed cancel each other in our sum rule.
Other sources of uncertainty such as quark-gluon mixed condensate, gluon condensate, or four-quark operator do not appear in our sum rule, which will certainly reduce the error in our prediction. Furthermore, $\langle \bar{q}q \rangle$ ($q = u, d$) can be determined rather accurately from the pion GOR relation Eq. (1) once $m_q$ is fixed. As we will see in the next section, the higher order corrections in quark mass are small in this $u, d$ sector. When strange quark is involved, we have only the two additional QCD parameters, $m_s$ and $\langle \bar{s}s \rangle$. Thus, as long as it is stable under the modeling of higher resonances or Borel mass variation, this sum rule can give a stringent constraint for $m_s$.

III. THE PION SUM RULE

Having calculated the OPE for the general axial-vector correlator, we can easily obtain the pion sum rule. To do this, we set $q_1 = u$, $q_2 = d$ in Eq.(17). As the normal GOR relation Eq. (1) is well established, our improvement including higher orders in quark masses is not expected to be large for this pion case. In fact, the small quark masses can guarantee this expectation but we will check this for completeness.

Using the isospin symmetry, $m_u = m_d \equiv m_q$ and $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle$, we obtain the spectral density (the imaginary part) of the OPE up to dimension 6 ,

$$\frac{1}{\pi} \mbox{Im} \Pi^{\text{ope}}_\pi(s) = -\frac{3m_q^2}{2\pi^2} \sqrt{1 - \frac{4m_q^2}{s}} \theta(s - 4m_q^2) + 4m_q\langle \bar{q}q \rangle \delta(s - m_q^2).$$

(18)

The phenomenological spectral density has pion and higher resonances of pion, namely,

$$\frac{1}{\pi} \mbox{Im} \Pi^{\text{phen}}_\pi(s) = -f^2_\pi m^2_\pi \delta(s - m^2_\pi) + (\pi' \text{ contribution}) + \cdots.$$ (19)

Note that there is no $a_1$ meson contribution in our projected sum rule. Matching the two expressions under the Borel transformation and invoking QCD duality for higher resonances, we obtain the pion sum rule

$$\int_0^{S_0} ds e^{-s/M^2} \frac{1}{\pi} \mbox{Im} \left[ \Pi^{\text{ope}}_\pi(s) - \Pi^{\text{phen}}_\pi(s) \right] = 0$$

$$\rightarrow f^2_\pi m^2_\pi e^{-m^2_\pi/M^2} = \frac{3m_q^2}{2\pi^2} \int_0^{S_0} ds e^{-s/M^2} \sqrt{1 - \frac{4m_q^2}{s}} \theta(s - 4m_q^2) - 4m_q\langle \bar{q}q \rangle e^{-m^2_q/M^2}.$$ (20)

The first term (the perturbative part) is an order $O(m^2_q)$ or higher. To the order $O(m_q)$, we recover precisely the GOR relation Eq.(1) as expected. Therefore, the perturbative part is the correction to the GOR relation.

We use this sum rule to plot the Borel curve for $f_\pi$ setting $m_\pi = 138$ MeV, $m_q = 7.2$ MeV [3,4]. The GOR relation gives (assuming $f_\pi = 131$ MeV) $\langle \bar{q}q \rangle = -(225 \text{ MeV})^3$. The continuum threshold is taken to be $S_0 = 1.7$ corresponding to the $\pi'$ mass. Figure 1 shows the Borel curve for $f_\pi$, which is quite stable with respect to the variation of the Borel mass $M^2$. Furthermore, the resulting $f_\pi$ is not sensitive to $S_0$. Also shown by the straight line is the $f_\pi$ obtained from the GOR relation. The deviation from GOR relation is about 3 %. Thus our sum rule including the higher mass corrections does not change the established pion GOR relation.
IV. THE KAON SUM RULE–TO CONSTRAIN THE STRANGE QUARK MASS

We now construct a sum rule for kaon. The small deviation from the GOR relation observed in the pion sum rule can not be guaranteed in this case because of the large strange quark mass. In this sum rule, we set \( q_1 = q \) (here \( q = u \) or \( d \)) and \( q_2 = s \) in Eq. (17). The similar procedure as before leads to the kaon sum rule

\[
f_K^2 m_K^2 e^{-m_K^2/M^2} = \frac{3}{8\pi^2} \int_{(m_q+m_s)^2}^{S_0} ds \, e^{-s/M^2} \left[ 1 - \frac{2(m_q^2 + m_s^2)}{s} + \frac{(m_q^2 - m_s^2)^2}{s^2} \right] \\
\times \left[ (m_q + m_s)^2 - \frac{(m_q^2 - m_s^2)^2}{s} \right] \theta \left[ s - (m_q + m_s)^2 \right] - (m_q + m_s) \left( \langle \bar{q}q \rangle e^{-m_q^2/M^2} + \langle \bar{s}s \rangle e^{-m_s^2/M^2} \right) \\
- \frac{1}{2} \left( m_q^2 - m_s^2 \right) \left[ m_q \langle \bar{q}q \rangle \frac{1}{M^2} e^{-m_q^2/M^2} - m_s \langle \bar{s}s \rangle \frac{1}{M^2} e^{-m_s^2/M^2} \right].
\]

(21)

Again, to leading order in \( m_q \) (or \( m_s \)), we recover the kaon GOR relation Eq. (3). The first and third terms are therefore the higher order corrections to the kaon GOR relation. To estimate how large the corrections are, we plot for \( f_K \) with respect to \( M^2 \) in figure 2. In this plot, we use \( m_K = 494 \) MeV and the continuum threshold is set to \( S_0 = 2 \) GeV^2 corresponding to \( K_0^+ (1430) \). For the strange quark condensate, we set \( \langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle \) [8]. As for the strange quark mass, a wide range of its value can be found in literatures [14–19]. It is within the range \( m_s \approx 100 - 200 \) MeV. To cover this range of \( m_s \), we plot the Borel curves for \( m_s = 100 \) (solid), 150 (dashed), 200 (dot-dashed) MeV. The corresponding straight lines are the ones from the kaon GOR relation Eq. (3). We see that all three Borel curves become quite flat for \( M^2 \geq 1 \) GeV^2, but the extracted \( f_K \) is quite different from the \( f_K \) given by the kaon GOR relation, about 20 -25 % level. The corrections shift the Borel curves upward and it is clear that the \( f_K \) extracted from them is closer to its experimental value \( f_K^{\text{expt}} \approx 160 \) MeV. Thus our improvement helps to achieve a better agreement with the experiment. To reproduce \( f_K^{\text{expt}} \) only from the kaon GOR relation, \( m_s \) should be abnormally large like \( m_s \approx 300 \) MeV, which is obviously not acceptable.

Most interesting feature is that the stable Borel curve is shifted noticeably as we change \( m_s \). It means that, once \( f_K \) is fixed, this sum rule can give a stringent constraint for \( m_s \) or vice versa. We therefore look for \( m_s \) which reproduces the experimental kaon decay constant \( f_K^{\text{expt}} \approx 160 \). Since \( m_q \) and \( \langle \bar{q}q \rangle \) are well fixed by the GOR relation for pion, only uncertainty in our sum rule comes from \( \langle \bar{s}s \rangle \). According to the baryon sum rules [5,13], the strange quark condensate at 1 GeV is estimated as

\[
\langle \bar{s}s \rangle = (0.8 \pm 0.1) \langle \bar{q}q \rangle.
\]

(22)

Using this input, we obtain the strange quark mass that reproduces the experimental \( f_K \) at \( M^2 = 1 \) GeV^2,

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\(^3\)However, the sensitivity to \( S_0 \) is very weak. For \( S_0 = 2.5 \) GeV^2, the result is changed only by 2 %.
The error comes from the uncertainty in \( \langle \bar{s}s \rangle \). This \( m_s \) is the value at 1 GeV scale because other parameters used in extracting this is defined at 1 GeV. The Borel stability in extracting this is fairly good. Even if we take into account the additional error of an order 5 MeV from the continuum threshold \( S_0 \), the extracted \( m_s \) from this sum rule is sufficiently accurate. Higher dimensional operators starts at dimension 8, which are expected to be small. Our \( m_s \) is somewhat larger than the conventional value of \( m_s = 150 \text{ MeV} \) or the value from hadronic \( \tau \) decay \( m_s = 159 \text{ MeV} \) [14]. But it is smaller than the one from Ref. [17] and is somewhat consistent with the value from the current algebra ratio \( m_s = 175 \pm 25 \text{ MeV} \) [4] or with the one in Ref. [18].

V. THE \( \eta_8 \) SUM RULE

Having determined \( m_s \) from the kaon sum rule, we now test whether it gives a consistent result for the \( \eta_8 \) case by constructing a sum rule for the current

\[
A_\mu^8 = \frac{1}{\sqrt{6}} \left( \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2 \bar{s} \gamma_\mu \gamma_5 s \right). 
\]

We recall that the \( \eta_8 \) decay constant is defined by the matrix element

\[
\langle 0 | A_\mu^8 | \eta_8(p) \rangle = i f_{\eta_8} p_\mu. 
\]

Ref. [2,20] gives \( f_{\eta_8} \approx 157 - 165 \text{ MeV} \). Using \( m_s \) from the kaon sum rule, we will see if the \( f_{\eta_8} \) from our sum rule is consistent with this range.

Taking the divergence of the axial-vector current and the subsequent use of the soft-\( \eta_8 \) theorem lead to the following \( \eta_8 \) GOR relation

\[
f_{\eta_8}^2 m_{\eta_8}^2 = -\frac{4}{3} m_q \langle \bar{q}q \rangle - \frac{8}{3} m_s \langle \bar{s}s \rangle. 
\]

Here again, we have used the isospin symmetry \( m_u = m_d \equiv m_q \) and \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle \). The OPE for the correlation function

\[
\Pi(p^2) = -\frac{p_\mu p_\nu}{p^2} \int d^4 x e^{ip \cdot x} \langle 0 | T A_\mu^8(x) A_\nu^8(0) | 0 \rangle 
\]

can be easily obtained from the general formula Eq. (17). Assuming that the continuum contribution starts at the \( \eta' \) resonance (i.e. \( S_0 = 0.92 \text{ GeV}^2 \)), we easily obtain the \( \eta_8 \) sum rule,

\[
f_{\eta_8}^2 m_{\eta_8}^2 e^{-m_{\eta_8}^2/M^2} = \frac{m_q^2}{2\pi^2} \int_{4m_q^2}^{S_0} ds \ e^{-s/M^2} \left[ \frac{4}{3} m_q \langle \bar{q}q \rangle e^{-m_q^2/M^2} - \frac{8}{3} m_s \langle \bar{s}s \rangle e^{-m_s^2/M^2} \right] 
\]

Again, to leading order in quark mass, this reproduces the \( \eta_8 \) GOR relation Eq. (26). The perturbative terms that are the second order in quark mass constitute the higher order corrections.
The Borel curve for $\eta_8$ is shown by the thick dashed line in Figure 3. In this plot, we use $m_s = 186$ MeV obtained from the kaon sum rule and assume that $m_{\eta_8} = m_\eta = 547$ MeV. To show the sensitivity to the continuum threshold, the curve for $S_0 = 1.4$ GeV$^2$ is also shown by the thin dashed line. Around $M^2 = 1$ GeV$^2$, the two curves differ by 5%. The Borel stability is not as good as the kaon case but $f_{\eta_8}$ becomes only 5% smaller at $M^2 = 2$ GeV$^2$ than the one at $M^2 = 1$ GeV$^2$. As can be seen, the $f_{\eta_8}$ from this sum rule becomes 30% larger than what the $\eta_8$ GOR relation gives. Therefore, $f_{\eta_8}$ picks up a large correction from higher orders in quark mass to the $\eta_8$ GOR relation. From this sum rule, we extract $f_{\eta_8} = 160$ MeV at $M^2 = 1$ GeV$^2$. This is very close to its value in literatures 157–165 MeV [2,20] even if we take into account the uncertainties due to the continuum threshold or the slight dependence on the Borel mass. Therefore, $m_s = 186$ MeV extracted from the kaon sum rule consistently reproduces $f_K$ and $f_{\eta_8}$.

VI. SUMMARY

In this work, we have constructed QCD sum rules exclusively for decay constants of the pseudoscalar mesons, $\pi$, kaon and $\eta_8$. The phenomenological side of our sum rule contains only the pseudoscalar mesons while the QCD side is proportional to quark masses. Our sum rules give a very stable constraint relating the meson decay constants and the quark masses. To leading order in quark masses, our sum rules produce the GOR relations, which allows us to study the higher order corrections in quark masses. From the kaon sum rule, we have obtained a sensitive constraint relation for $f_K$ and $m_s$, which restricts $m_s \sim 186$ MeV in order to reproduce the experimental $f_K$. Such a $m_s$ when applied to the $\eta_8$ sum rule gives $f_{\eta_8}$ consistent with its value in literatures.

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REFERENCES

FIG. 1. The Borel curve for the pion decay constant. The solid (straight) line is the $f_\pi$ from the GOR relation and the dashed (curved) line is the $f_\pi$ from the Borel sum rule of Eq. (20).
FIG. 2. The kaon decay constants for given $m_s$ are plotted with respect to the Borel mass. For each $m_s$, we plot the curve from the kaon GOR relation and the one from the kaon sum rule. The solid lines are for $m_s = 100$ MeV, the dashed lines are for $m_s = 150$ MeV and the dot-dashed lines are for $m_s = 200$ MeV. The parallel straight lines are obtained from the kaon GOR relation for given $m_s$ while the corresponding curved lines are from the kaon Borel sum rule Eq. (21). The Borel curve (or kaon GOR relation) is shifted noticeably as we change $m_s$. Also, higher order corrections of quark mass give $20\sim 25\%$ shift from the corresponding GOR value. The $f_K$ from the Borel curves is closer to its experimental value of 160 MeV.
FIG. 3. The $\eta_8$ decay constant with respect to the Borel mass. In this plot, we use the best fitting $m_s$ from the kaon sum rule. The solid straight line is obtained from the $\eta_8$ GOR relation and the thick dashed curve is from the $\eta_8$ sum rule (with $S_0 = 0.92$ GeV$^2$) Eq. (28). There is 30% change from the $\eta_8$ GOR relation. The thin dashed line is obtained with $S_0 = 1.4$ GeV$^2$ to show the sensitivity to $S_0$. 