Schrödinger equation from an exact uncertainty principle

Michael J. W. Hall$^1$ and Marcel Reginatto$^2$,*

$^1$Theoretical Physics, IAS, Australian National University,
Canberra ACT 0200, Australia

$^2$U. S. Department of Energy,
Environmental Measurements Laboratory,
New York, New York 10014-4811, USA

*Present address: Physikalisch-Technische Bundesanstalt, Bundesallee 100, 38116 Braunschweig,
Federal Republic of Germany

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Abstract

An exact uncertainty principle, formulated as the assumption that a classical ensemble is subject to random momentum fluctuations of a strength which is determined by and scales inversely with uncertainty in position, leads from the classical equations of motion to the Schrödinger equation.

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I. INTRODUCTION

The uncertainty principle is generally considered to be a fundamental conceptual tool for understanding differences between classical and quantum mechanics. As first argued by Heisenberg in 1927 [1], the fact that quantum states do not admit simultaneously precise values of conjugate observables, such as position and momentum, does not necessarily imply an incompleteness of the theory, but rather is consistent with not being able to simultaneously determine such observables experimentally to an arbitrary accuracy.

Corresponding uncertainty relations such as \( \Delta x \Delta p \geq \hbar / 2 \) “give us that measure of freedom from the limitations of classical concepts which is necessary for a consistent description of atomic processes” [2]. The uncertainty principle provides the basis of the Copenhagen interpretation of quantum mechanics, famously used by Bohr in defending the completeness of the theory against critics such as Einstein [3].

If regarded as merely asserting a physical limit on the degree to which classical concepts can be applied, the uncertainty principle is not sufficiently restrictive in content to supply a means for moving from classical mechanics to quantum mechanics. Thus Landau and Lifschitz write that “this principle in itself does not suffice as a basis on which to construct a new mechanics of particles” [4]. In particular, uncertainty relations expressed as imprecise inequalities are not enough to pin down the essence of what is nonclassical about quantum mechanics. Authors have tended to point instead, for example, to the commutation relation \([\hat{x}, \hat{p}] = i\hbar\) for quantum observables [5], or to the principle of superposition of wavefunctions [4,6], in this regard.

However, it will be shown here that an \textit{exact} form of the uncertainty principle may in fact be formulated, which provides the \textit{single} key element in moving from the equations of motion of a classical ensemble to those of a quantum ensemble. In particular, if it is assumed that a classical ensemble is subject to random momentum fluctuations, \textit{where the strength of these fluctuations is precisely determined by and scales inversely with uncertainty in position} (as characterised by the position probability density), then the resulting modified equations of motion are equivalent to the Schrödinger equation. Thus, surprisingly, there is an exact formulation of the uncertainty principle which does in fact capture the essence of what is “quantum” about quantum mechanics.

In the following section we recall the description of a classical ensemble in terms of a pair of equations in configuration space (the Hamilton-Jacobi equation and the continuity equation), and provide the corresponding Lagrangian from which these equations follow.
In Section III we show that the above exact uncertainty principle leads to a modification of this Lagrangian (essentially incorporating the kinetic energy of the random momentum fluctuations), the form of which yields equations of motion equivalent to the Schrödinger equation. Further, an exact uncertainty relation for position and momentum uncertainties is derived, corresponding to the exact uncertainty principle, from which the usual Heisenberg inequality follows as a consequence.

Of course, equations of motion equivalent to the Schrödinger equation do not in themselves imply the full quantum formalism. Accordingly, in Section IV a Hamiltonian formulation is provided for the equations of motion, which leads naturally to the usual wavefunction representation as corresponding to the normal modes of the modified system. Conclusions are presented in Section V.

II. CLASSICAL MECHANICS

For simplicity, we limit ourselves to the case of a single particle, described in a configuration space of \( n \) dimensions. In the Hamilton-Jacobi formulation of classical mechanics, the equation of motion takes the form [7]

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V = 0. \tag{1}
\]

The velocity field \( \mathbf{u}(\mathbf{x}, t) \) that describes the motion of the particle is related to \( S(\mathbf{x}, t) \) by

\[
\mathbf{u} = \frac{1}{m} \nabla S. \tag{2}
\]

We assume that the initial conditions are not known exactly, and that the probability of finding the particle in a given volume of the configuration space is described by a probability density \( P(\mathbf{x}, t) \). The probability density must satisfy the following two conditions: it must be normalized,

\[
\int P d^n x = 1,
\]

and it must satisfy a continuity equation,

\[
\frac{\partial P}{\partial t} + \nabla \cdot \left( P \frac{1}{m} \nabla S \right) = 0. \tag{3}
\]

Eqs. (1) and (3), together with (2), completely determine the motion of the classical ensemble. Eqs. (1) and (3) can be derived from the Lagrangian

\[
L_C = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V \right\} d^n x \, dt \tag{4}
\]

by fixed end-point variation (\( \delta P = \delta S = 0 \) at the boundaries) with respect to \( S \) and \( P \).
III. THE TRANSITION FROM CLASSICAL MECHANICS TO QUANTUM MECHANICS

A. Momentum fluctuations

Consider now the possibility that the classical Lagrangian is not quite right, because \( \nabla S \) is actually an average momentum: one also has a fluctuation \( N \) about \( \nabla S \). Thus the physical momentum is \( p = \nabla S + N \) with

\[
< N > = \int P N d^3x = 0.
\]

It is simplest to assume that these “nonclassical” momentum fluctuations are random, in the sense of being linearly uncorrelated with average momentum \( \nabla S \), i.e., \( < N \cdot \nabla S > = 0 \).

It follows that when these fluctuations are significant, the kinetic energy term \( \frac{1}{2m} \nabla S \cdot \nabla S \) in the Lagrangian should be replaced by \( \frac{1}{2m} (\nabla S + N) \cdot (\nabla S + N) \), which yields the modified Lagrangian

\[
L = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S + N) \cdot (\nabla S + N) + V \right\} d^3x dt
\]

\[
= \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V \right\} d^3x dt + \frac{1}{2m} \int P \{N \cdot N\} d^3x dt \tag{5}
\]

\[
\approx L_C + \frac{1}{2m} \int (\Delta N)^2 dt
\]

where \( \Delta N \) denotes the average rms momentum fluctuation. Thus the consequence of taking into consideration the momentum fluctuations is to add a positive term to the Lagrangian, arising from the additional kinetic energy due to the fluctuations.

B. Exact uncertainty principle

How can we estimate the magnitude of this additional term, if we don’t know anything else about the system except the probability density \( P \) and the average momentum \( \nabla S \)? To estimate the magnitude of the momentum spread, we will assume that an exact uncertainty principle holds, in the sense that the strength of the momentum fluctuations at a given time are inversely correlated with uncertainty in position at that time, where the uncertainty of position is characterized by \( P \). Clearly, this assumption is an additional hypothesis that is independent of classical mechanics.

To make this assumption precise, consider the general case of an \( n \)-dimensional space and a one-parameter family of probability distributions (which we label with a parameter \( k > 0 \)) at time \( t_0 \), related by a rescaling of variables

\[
P(x) \rightarrow P_k(x) \equiv k^n P(kx).
\]

These transformations preserve the normalization,

\[
\int P(x) d^n x \rightarrow \int k^n P(kx) d^n x = \int P(y) d^n y
\]
where we have introduced the change of variables \( y = kx \). We also have

\[
\nabla P(x) \cdot \nabla P(x) \rightarrow k^{2n+2} \nabla_y P(y) \cdot \nabla_y P(y) \\
x \cdot \nabla P(x) \rightarrow k^n y \cdot \nabla_y P(y).
\]

Notice that under such a transformation, any direct measurement of position uncertainty \( \delta x \) such as the rms uncertainty \( \Delta x \) changes according to the rule

\[
\delta x \rightarrow \delta x_k \equiv \frac{1}{k} (\delta x).
\]

Probability densities with different values of \( k \) represent physical systems that only differ in how well we know the location of the particle, since the shape of the probability densities are the same except for the rescaling (more general transformations that depend on position would allow for more severe distortions of the probability density, for example, creating two peaks in the distribution where before there was only one). The exact uncertainty principle that we want to make use of can then be in part reformulated as follows: if we consider two physical systems that are identical except with respect to the localizability of the particle (i.e., if they are connected by a \( k \) transformation), then the uncertainty in momentum must be such that the product of the position and momentum uncertainties is invariant under \( k \) transformations. In other words, scaling of position by a factor \( 1/k \) scales the momentum fluctuation by a factor \( k \).

The assumption of an exact uncertainty principle is then equivalent to the statement that the momentum fluctuation \( \Delta N \) is determined by the uncertainty in position, where the latter is characterised by the probability density \( P \), and where

\[
\Delta N \rightarrow k\Delta N \quad (6)
\]

under \( k \) transformations.

To apply this assumption, note first that for the Lagrangian formalism to be applicable to \( L \), the additional term must be a spacetime integral over a scalar function of \( x \), \( P \) and \( S \) and their derivatives. Moreover, since \( \langle (\Delta N)^2 \rangle \) is determined solely by position fluctuations (where the latter are characterised by \( P \)), then this additional term is in fact independent of \( S \). Finally, for causality to be preserved (i.e., the equations of motion require only \( P \) and \( S \) to be specified on an initial surface), second and higher order derivatives of \( P \) must be excluded\(^1\). Hence, the additional term in the Lagrangian (5) can be written in the form

\[
\int \langle (\Delta N)^2 \rangle dt = \int P f(x, P, x \cdot \nabla P, \nabla P \cdot \nabla P) d^n x dt. \quad (7)
\]

The exact uncertainty principle requires \( f \) to transform under \( k \) transformation as follows,

\(^1\)Requirements of causality do not exclude a term linear in \( \nabla^2 P \), but since it can be shown that such a term does not lead to a different result we will not consider it here.
\[
\int P(x) f [x, P(x), x \cdot \nabla P(x), \nabla P, \nabla P(x)] d^n x dt
\to \int P(y) f(k^{-1} y, k^n P(y), k^n y \cdot \nabla_y P(y), k^{2n+2} \nabla_y P(y) \cdot \nabla_y P(y)) d^n y dt
\equiv k^2 \int P(y) f(y, P(y), y \cdot \nabla_y P(y), \nabla_y P(y) \cdot \nabla_y P(y)) d^n y dt.
\]

This leads to the homogeneity condition
\[
f(k^{-1} x, k^n u, k^n v, k^{2n+2} w) = k^2 f(x, u, v, w)
\]
where we have introduced the more compact notation
\[
u = P, \quad v = x \cdot \nabla P, \quad w = \nabla P \cdot \nabla P.
\]

From this requirement we derive the first order partial differential equation
\[
- \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} + nu \frac{\partial f}{\partial u} + nv \frac{\partial f}{\partial v} + (2n + 2)w \frac{\partial f}{\partial w} = 2f.
\]

The problem of finding the general integral of such an equation is equivalent to the problem of finding the general integral of a system of ordinary differential equations [8], which in our case is given by
\[
- \frac{dx_1}{x_1} = \ldots = - \frac{dx_n}{x_n} = \frac{du}{nu} = \frac{dv}{nv} = \frac{dw}{(2n + 2)w} = \frac{df}{2f}.
\]

This system of ordinary differential equations has \((n+3)\) independent integrals, which can be chosen as
\[
u^{-1} w^{1/2} x_i = \text{const.},
\]
\[
u^{-1} v = \text{const.},
\]
\[
u^{2/n} x \cdot x = \text{const.},
\]
\[
u^2 w^{-1} f = \text{const.}
\]

and the solution of the first order partial differential Eq. (10) is then of the general form
\[
f = (u^{-2} w) g(u^{-1} w^{1/2} x, u^{-1} v, u^{2/n} x \cdot x)
\]
where \(g\) is an arbitrary function.

C. Independent subsystems

To determine \(f\) completely, we need one further condition. We therefore introduce a condition of system independence by requiring that the extra term in the Lagrangian \(L\)
be such that it decomposes into additive subsystem contributions for the case of a system that is composed of independent subsystems. This is equivalent to the condition that the momentum fluctuations $N_1$ and $N_2$ are linearly uncorrelated for two such subsystems, and hence can equivalently be interpreted as a further randomness assumption for the momentum fluctuations.

To investigate the requirements of system independence, it will be sufficient to consider the case where we have a system consisting of two uncorrelated particles of mass $m$ that do not interact, one particle described by a set of coordinates $x_1$ and the other by $x_2$. Then, $P$ is of the form

$$P(x_1, x_2) = P_1(x_1)P_2(x_2)$$

which leads to

$$u = u_1u_2$$

$$v' = u^{-1}v = u_1^{-1}v_1 + u_2^{-1}v_2 = v_1' + v_2'$$

$$w' = u^{-2}w = u_1^{-2}w_1 + u_2^{-2}w_2 = w_1' + w_2'.$$

where subscripts 1 and 2 refer to quantities corresponding to subsystems 1 and 2 respectively.

From Eqs. (5) and (7), system independence requires

$$Pf = P_1P_2(f_1 + f_2).$$

Using Eq. (11), we find that

$$f = (w_1' + w_2')g\left(\sqrt{w_1'^2 + w_2'^2}x, \; v_1' + v_2', \; (u_1u_2)^{2/n}x \cdot x\right)$$

where $x = (x_1, x_2)$. From Eq. (13), this form of $f$ must decompose into the sum of a function of $u_1$, $v_1'$, $w_1'$ and $x_1$, and a function of $u_2$, $v_2'$, $w_2'$ and $x_2$. Since the factor that multiplies $g$ and the second and third arguments of $g$ are such functions (with respect to $w'$, $v'$, and $x$ respectively), these terms cannot be mixed by the functional form of $g$. Taking into consideration that the first argument of $g$ is given by $\sqrt{w_1'^2 + w_2'^2}x$, we find that $g$ must be of the form

$$g(a, b, c) = C + g_0(a) + bg_1(a) + cg_2(a)$$

where $C$ is a constant, and the functions $g_j$ must satisfy the condition

$$g_j(\lambda a) = \lambda^{-2}g_j(a), \; j = 0, 1, 2$$

(14)

to allow cancellation of the factor $w_1' + w_2'$ that multiplies $g$.

Hence $f$ has the general form

$$f = C(w_1' + w_2') + g_0(x_1, x_2) + (v_1' + v_2')g_1(x_1, x_2) + (u_1u_2)^{2/n}(x_1 \cdot x_1 + x_2 \cdot x_2)g_2(x_1, x_2).$$

The independence condition Eq. (13) places strong conditions on the $g_j$. First, $g_0$ is required to be a sum of a function of $x_1$ and a function of $x_2$. Hence it only represents a classical additive potential term (satisfying the homogeneity condition (14) above), and will
be ignored as having no nonclassical role (it can be absorbed into the classical potential \( V \) in the Lagrangian). Second, to avoid subsystem cross terms, \( g_1 \) must be constant. But then the homogeneity condition (14) can only be satisfied by the choice \( g_1 = 0 \). Third, cross terms in \( u_1 \) and \( u_2 \) can only be avoided by choosing \( g_2 = 0 \). Thus the form of \( f \) reduces to
\[
f = C (w_1' + w_2')
\]
which from Eq. (13) is to be identified with the sum of \( f_1 \) and \( f_2 \), thus yielding the general form
\[
f = Cw' = C \frac{1}{P^2} \nabla P \cdot \nabla P
\]
where \( C \) is a universal constant.

D. Equations of motion

The modified Lagrangian follows from Eqs. (5), (7) and (15) as
\[
L = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + \frac{C}{2m} \frac{1}{P^2} \nabla P \cdot \nabla P + V \right\} d^n x dt.
\]
Fixed end-point variation with respect to \( S \) leads again to (3), while fixed end-point variation with respect to \( P \) leads to
\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + \frac{C}{2m} \left[ \frac{1}{P^2} \nabla P \cdot \nabla P - \frac{2}{P} \nabla^2 P \right] + V = 0.
\]
Eqs. (3) and (17) are identical to the Schrödinger equation provided the wave function \( \psi(x, t) \) is written in terms of \( S \) and \( P \) by
\[
\psi = \sqrt{P} \exp \left( i \frac{S}{\hbar} \right)
\]
and the constant \( C \) is set equal to
\[
C = \left( \frac{\hbar}{2} \right)^2.
\]
Just why one would introduce the wavefunction \( \psi \) at all is considered in section IV below.

Note that the classical limit of the Schrödinger theory is not the Hamilton-Jacobi equation for a classical particle, but Eqs. (1) and (3) which describe a classical ensemble.

E. Exact uncertainty relation

The Schrödinger equation has been derived above using an exact uncertainty principle to fix the strength of random momentum fluctuation in terms of the uncertainty in position. Note that no specific measure of position uncertainty was assumed; it was required only that the momentum fluctuations scale inversely with position uncertainty under \( k \) transformations. However, having obtained a unique form, Eq. (15), for the function \( f \) in (7) we are now in a position to write down an exact uncertainty relation relating position and
For simplicity we consider the case of one dimension \((n = 1)\), and define

\[ \delta x = \left[ \int P \left( \frac{1}{P} \frac{dP}{dx} \right)^2 \, dx \right]^{-1/2}. \]

For the case of a Gaussian probability density with rms uncertainty \(\sigma\) one has \(\delta x = \sigma\). More generally, this measure has units of position, scales appropriately with \(x\) \((\delta y = \lambda \delta x\) for \(y = \lambda x)\), and vanishes in the limit that \(P\) approaches a delta function. Hence it represents a direct measure of uncertainty for position. From Eqs. (7) and (15) one then has

\[ \delta x \Delta N = \sqrt{C} = \frac{\hbar}{2}. \]

(18)

Thus we have an exact uncertainty relation between position and momentum. This relation was recently derived in [9], where \(\delta x\) was termed the “Fisher length” of the probability density \(P\), due to its connection with the “Fisher information” of statistical estimation theory [10]. The quantity \(\Delta N\) represents the rms deviation of a nonclassical momentum operator in [9].

The usual Heisenberg uncertainty relation can be derived from this exact uncertainty relation. From the Cramer-Rao inequality of statistical estimation theory [11] one has \(\Delta x \geq \delta x\), while the assumptions in Sec. III.A imply

\[ (\Delta p)^2 = \text{Var}(\nabla S + N) = \text{Var}(\nabla S) + (\Delta N)^2 \geq (\Delta N)^2, \]

and hence it follows immediately from Eq. (18) that \(\Delta x \Delta p \geq \hbar/2\).

IV. HAMILTONIAN FORMULATION AND WAVE FUNCTION REPRESENTATION

A. Hamiltonian formulation

In the previous section, we derived an extension of the classical Lagrangian which yields equations of motion equivalent to the Schrödinger equation. The Lagrangian field formalism was conveniently used because it is well known. However, one can in fact obtain equivalent results using the Hamiltonian form of field theory, with no essential differences in the assumptions and manipulations used.

The Hamiltonian formalism does provide one important advantage: the concept of canonical transformations. In the previous section, the wavefunction representation \(\psi = \sqrt{P} \exp(i\frac{S}{\hbar})\) was simply “magicked out of thin air”, to obtain the Schrödinger equation written in terms of the wavefunction \(\psi\) instead of the hydrodynamical variables \(P\) and \(S\). In contrast, in the Hamiltonian formalism this complex combination of \(P\) and \(S\) arises immediately from asking a natural question about canonical transformations.

The Hamiltonian form corresponding to Lagrangian (16) is given by

\[ H = \int P \left\{ \frac{1}{2m} \nabla S \cdot \nabla S + \frac{\hbar^2}{8mP^2} \nabla P \cdot \nabla P + V \right\} \, dn = \int \mathcal{H} \, dn. \]

(19)
The field $P$ plays the role of a field coordinate, and $S$ the role of the momentum canonically conjugate to $P$. The equations of motion are given by [7]

\[
\begin{align*}
\frac{\partial P}{\partial t} &= \{P, H\} = \frac{\delta H}{\delta S} \\
\frac{\partial S}{\partial t} &= \{S, H\} = -\frac{\delta H}{\delta P}
\end{align*}
\]

where the Poisson brackets of two functions $F$ and $G$ is defined by

\[
\{F(P(x), S(x)), G(P(x'), S(x'))\} = \int \left[ \frac{\delta F(x)}{\delta P(x')} \frac{\delta G(x')}{\delta S(x'')} - \frac{\delta F(x)}{\delta S(x')} \frac{\delta G(x')}{\delta P(x'')} \right] d^n x''.
\]

To simplify the formulae, we will sometimes use the notation $P \equiv P(x)$, $P' \equiv P(x')$, etc. which allows us to write Eq. (20) in the concise form

\[
\{F, G'\} = \int \left[ \frac{\delta F}{\delta P'} \frac{\delta G'}{\delta S'} - \frac{\delta F}{\delta S'} \frac{\delta G'}{\delta P'} \right] d^n x''.
\]

From

\[
\frac{\delta P}{\delta P'} = \frac{\delta S}{\delta S'} = \delta^n(x - x')
\]

we derive the Poisson brackets of the canonically conjugate fields,

\[
\{P, S'\} = \delta^n(x - x').
\]

The equations of motion that correspond to $H$ are

\[
\begin{align*}
\frac{\partial P}{\partial t} &= \frac{\delta H}{\delta S} = -\nabla \cdot \left( P \frac{1}{m} \nabla S \right) \\
\frac{\partial S}{\partial t} &= -\frac{\delta H}{\delta P} = - \left[ \frac{1}{2m} \nabla S \cdot \nabla S + \frac{\hbar^2}{8m} \left( \frac{1}{P^2} \nabla P \cdot \nabla P - \frac{2}{P} \nabla^2 P \right) + V \right].
\end{align*}
\]

These equations are of course identical to (3) and (17) which were derived using the Lagrangian formalism.

**B. Wavefunctions and normal modes**

The Hamiltonian form $H$ has been expressed in (19) in terms of fields which represent important physical quantities: $P$ has the physical interpretation of a position probability density, and $S$ that of an average momentum potential. However, $H$ can be rewritten in terms of any pair of fields $\phi$ and $\chi$ without changing the physical content provided they are related to $P$ and $S$ by a canonical transformation,

\[
\{P, S'\} = \int \left[ \frac{\delta P}{\delta \phi'} \frac{\delta S'}{\delta \chi} - \frac{\delta P}{\delta \chi'} \frac{\delta S'}{\delta \phi''} \right] d^n x'' = \{\phi, \chi'\}.
\]
Of course, such a transformation is generally only of interest if the new fields have some particular physical significance.

One transformation of obvious physical interest, when it exists, is to two fields \( \phi \) and \( \chi \) which have uncoupled equations of motion. Such fields label two independent physical degrees of freedom in the system, and hence have fundamental physical significance as the “normal modes” of the system. It is therefore natural to ask whether such a transformation exists for \( H \), i.e., whether there is a one-one mapping

\[
P = P(\phi, \chi) \\
S = S(\phi, \chi)
\]

such that the fields \( \phi \) and \( \chi \) are uncoupled. It will be seen that this question is sufficient to derive the wavefunction representation \( \psi = \sqrt{P} \exp(i \frac{S}{\hbar}) \) and its complex conjugate from the Hamiltonian \( H \), as corresponding to the physical fields describing the “normal modes” of the system.

To examine the question of whether there is a canonical transformation that will lead to uncoupled equations of motion for \( \phi \) and \( \chi \) we first need to establish the following Lemma.

**Lemma**: a necessary condition for two conjugate fields \( \phi \) and \( \chi \) to be uncoupled is that the corresponding Hamiltonian density \( H' \) has the form

\[
H' = F(x, \phi, \chi) + A_k(x, \phi, \chi) \partial_k \phi + B_k(x, \phi, \chi) \partial_k \chi + G_{jk}(x, \phi, \chi) (\partial_j \phi) (\partial_k \chi)
\]

where \( k = 1, \ldots, n \), repeated indices are summed over, and \( \partial_k \) denotes the partial derivative with respect to \( x_k \). Furthermore, the symmetric part of \( G_{jk} \) is independent of \( \phi \) and \( \chi \), i.e.,

\[
G_{jk}(x, \phi, \chi) + G_{kj}(x, \phi, \chi) = 2G_{jk}(x)
\]

where \( G_{jk}(x) \) is symmetric with respect to \( j \) and \( k \).

**Proof**: For a Hamiltonian

\[
H' = \int \mathcal{H}'(x, \phi, \chi, \partial_k \phi, \partial_k \chi) d^n x
\]

the equations of motion are given by

\[
\frac{\partial \phi}{\partial t} = \frac{\partial \mathcal{H}'}{\partial \chi} - \frac{\partial^2 \mathcal{H}'}{\partial x_k (\partial_k \chi)} - \frac{\partial^2 \mathcal{H}'}{\partial (\partial_l \phi) \partial (\partial_k \chi)} \partial_k \phi - \frac{\partial^2 \mathcal{H}'}{\partial (\partial_l \chi) \partial (\partial_k \chi)} \partial_k \chi
\]

\[
- \frac{\partial^2 \mathcal{H}'}{\partial (\partial_l \phi) \partial (\partial_k \chi)} \partial_{kl} \phi - \frac{\partial^2 \mathcal{H}'}{\partial (\partial_l \chi) \partial (\partial_k \chi)} \partial_{kl} \chi.
\]

A similar expression is obtained for \( \frac{\partial \chi}{\partial t} \). Since we assume that \( \phi \) evolves independently of \( \chi \), then in particular no second derivatives of \( \chi \) can appear in the above equation of motion for \( \phi \), and similarly, no second derivatives of \( \phi \) can appear in the corresponding equation of motion for \( \chi \). Hence \( \mathcal{H}' \) must be linear in both \( \partial_k \phi \) and \( \partial_k \chi \). Hence \( \mathcal{H}' \) has the general form shown in the statement of the Lemma. Substituting this form into the equation of motion for \( \phi \) gives

\[
\frac{\partial \phi}{\partial t} = \frac{\partial F}{\partial \chi} - \partial_k B_k + \left( \frac{\partial A_k}{\partial \chi} - \frac{\partial B_k}{\partial \phi} - \partial_j G_{kj} \right) \partial_k \phi - \frac{\partial G_{jk}}{\partial \phi} (\partial_j \phi) (\partial_k \phi) - G_{jk} \partial_{kl} \phi
\]
and a similar equation for $\frac{\partial \chi}{\partial t}$. Hence, since $G_{jk}$ is the coefficient of $\partial_{kl}\phi$ and $\partial_{kl}\chi$ in the respective equations of motion, the fields are uncoupled only if the symmetric part of $G_{jk}$ is independent of both $\phi$ and $\chi$. ■

If we express the Hamiltonian density $\mathcal{H}$ (19) in terms of the new variables, we find

$$
\mathcal{H} = P \left\{ \frac{1}{2m} \left[ \left( \frac{\partial S}{\partial \phi} \right)^2 + \left( \frac{\hbar}{2} \right)^2 \left( \frac{\partial \ln P}{\partial \phi} \right)^2 \right] \nabla \phi \cdot \nabla \phi \\
+ \frac{1}{2m} \left[ \left( \frac{\partial S}{\partial \chi} \right)^2 + \left( \frac{\hbar}{2} \right)^2 \left( \frac{\partial \ln P}{\partial \chi} \right)^2 \right] \nabla \chi \cdot \nabla \chi \\
+ \frac{1}{m} \left[ \frac{\partial S}{\partial \phi} \frac{\partial S}{\partial \chi} + \left( \frac{\hbar}{2} \right)^2 \frac{\partial \ln P}{\partial \phi} \frac{\partial \ln P}{\partial \chi} \right] \nabla \phi \cdot \nabla \chi + V \right\}.
$$

It follows immediately from the Lemma that the first two terms of $\mathcal{H}$ must vanish, implying

$$
\frac{\partial S}{\partial \phi} = i\alpha \frac{\hbar}{2} \frac{\partial \ln P}{\partial \phi}, \\
\frac{\partial S}{\partial \chi} = i\beta \frac{\hbar}{2} \frac{\partial \ln P}{\partial \chi}
$$

where $\alpha, \beta = \pm 1$. Since $P$ and $S$ are real, the normal modes are therefore complex fields. Substituting into the third term of $\mathcal{H}$, the Lemma then further implies that

$$
G_{jk}(x) = P \left( \frac{\hbar}{2} \right)^2 \left( \frac{1 - \alpha \beta}{m} \right) \frac{\partial \ln P \partial \ln P}{\partial \phi \partial \chi} \delta_{jk} = G \delta_{jk}
$$

where $G$ is a constant and the last equality follows since the second term has no explicit $x$ dependence. Now, if $\alpha = \beta$, the Hamiltonian density reduces to $\mathcal{H} = P(\phi, \chi)V$, and the inverse transformation to the fields $P$ and $S$ then yields a Hamiltonian density proportional to $V$, which is inconsistent with the form of $\mathcal{H}$. Therefore, $\alpha = -\beta$ and

$$
2P \left( \frac{\hbar}{2} \right)^2 \frac{1}{m} \left[ \frac{\partial \ln P \partial \ln P}{\partial \phi \partial \chi} \right] = G.
$$

If we now use Eqs. (23) and (24) to write the Hamiltonian density $\mathcal{H}$ in terms of $\phi$ and $\chi$,

$$
\mathcal{H} = G \nabla \phi \cdot \nabla \chi + P(\phi, \chi)V
$$

and the equations of motion take the simple form

$$
\frac{\partial \phi}{\partial t} = \frac{\partial P}{\partial \chi} V - G \nabla^2 \phi, \\
- \frac{\partial \chi}{\partial t} = \frac{\partial P}{\partial \phi} V - G \nabla^2 \chi.
$$

For these equations to be uncoupled $P$ must be of the form

$$
P = W + X \phi + Y \chi + Z \phi \chi
$$

(25)
for constants $W$, $X$, $Y$ and $Z$. Substituting (25) in (24) yields

$$P = \left(\frac{2}{\hbar}\right)^2 \frac{m}{2} G (\phi + K) (\chi + L)$$

(26)

where $K$ and $L$ are constants related to $X$, $Y$ and $Z$. The general form of $S(\phi, \chi)$ can be found by substitution of (26) in (23) with $\alpha = -\beta$, which leads to a pair of differential equations with solution

$$S = b + i\alpha \frac{\hbar}{2} \ln \frac{\phi + K}{\chi + L}$$

(27)

where $b$ is an arbitrary complex constant.

The previous analysis establishes the functional forms of $P(\phi, \chi)$ and $S(\phi, \chi)$ that will permit uncoupled equations of motion for $\phi$ and $\chi$. We now have to check that these functional forms lead to a canonical transformation. This requires

$$\{P, S'\} = \int \left[ \frac{\delta P}{\delta \phi'} \frac{\delta S'}{\delta \chi''} - \frac{\delta P}{\delta \chi''} \frac{\delta S'}{\delta \phi'} \right] d^n x''$$

$$= -mG \left( \frac{i\alpha}{2} \frac{\hbar}{2} \right) \delta^n (x - x')$$

$$= \delta^n (x - x')$$

which fixes the value of $G$, and leads to

$$P = \frac{i\alpha}{\hbar} (\phi + K) \chi + L).$$

Recalling that $P$ and $S$ are real, and using the property that $P$ is positive, one can show that the inverse transformation follows from (26) and (27) as

$$\phi = a\sqrt{P} \exp \left( -i\alpha S/\hbar \right) - K$$

$$\chi = \frac{\alpha h}{ia} \sqrt{P} \exp \left( i\alpha S/\hbar \right) - L$$

(28)

where $a$ is an arbitrary complex constant (related to $b$).

Thus, a canonical transformation to uncoupled fields $\phi$ and $\chi$ does exist, given by the above relations. We recognise that these fields are, up to a scale factor and additive constant, the usual wavefunction $\psi = \sqrt{P} \exp(i\frac{S}{\hbar})$ and its complex conjugate, and hence the wavefunction has a fundamental physical significance as a “normal mode” of the system. The corresponding Hamiltonian density $\mathcal{H}$ follows as

$$\mathcal{H} = \frac{1}{2m} \frac{\hbar}{2m i\alpha} \nabla \phi \cdot \nabla \chi + \frac{1}{i\alpha \hbar} V (\phi + K) (\chi + L)$$

$$= \frac{\hbar^2}{2m} \left| \nabla \psi \right|^2 + V |\psi|^2$$

for all choices of $a$, $K$ and $L$, and leads directly to the Schrödinger equation and its conjugate.
We point out a quantization condition that follows from Eqs. (27) and (28). If $\Phi(x)$ is a single-valued complex function, $\ln \Phi$ is a multi-valued function which satisfies $\oint_C d\ln \Phi = \pm i2\pi n$, where $n$ is an integer. If we make the assumption that the fields $\phi$ and $\chi$ describing the “normal modes” of the system are single-valued functions, then

$$\oint_C dS = -i\alpha \frac{\hbar}{2} \oint_C d(\ln \psi - \ln \psi^*) = \pm 2\pi \hbar n.$$ 

This is precisely the quantization condition that was introduced by Takabayashi as “a new postulate” in his hydrodynamic interpretation of quantum mechanics [12]. This subsidiary condition is of course compatible with the equations of motion.

C. Expectation values

While the wavefunction and the corresponding wave equation have been obtained, these do not represent the full quantum formalism. For example, the nature of the assumptions about momentum fluctuations in Sec. III.A provide recipes for calculating the first two moments of the momentum distribution in terms of integrals that can now be expressed in terms of the wavefunction, since

$$<p>=\int P \nabla S d^n x = \frac{\hbar}{i} \int \psi^* \nabla \psi d^n x$$

$$<p^2>=\int P \left( \nabla S \cdot \nabla S + \left( \frac{\hbar}{2} \right)^2 \frac{1}{P^2} \nabla P \cdot \nabla P \right) d^n x = \hbar^2 \int |\nabla \psi|^2 d^n x.$$ 

However, it is not immediately clear how within this framework higher-order moments are to be calculated, nor expectation values of functions of position and momentum. We briefly note here a possible approach to this problem, based on a symmetry in the representation of position and momentum displacements, which leads to the usual relations assumed in the Hilbert space formulation of quantum mechanics.

Under a position displacement $T_a : x \rightarrow x + a$, the fields $P$ and $S$ transform as $P(x) \rightarrow P(x - a)$, $S(x) \rightarrow S(x - a)$. Hence in the wavefunction representation one has

$$T_a : \psi(x) \rightarrow \psi(x - a).$$ 

Under a momentum displacement $M_q : p \rightarrow p + q$, the position distribution (which is given by the field $P$) should be unaffected, while the average momentum must change by $q$, $\nabla S \rightarrow \nabla S + q$. Therefore, the fields $P$ and $S$ transform as $P(x) \rightarrow P(x)$, $S(x) \rightarrow S(x) + q \cdot x$ (where an arbitrary additive constant added to $S$ has been ignored, as it has no effect on the equations of motion). Hence in the wavefunction representation one has

$$M_q : \psi(x) \rightarrow \exp(iq \cdot x/\hbar)\psi(x).$$ 

Comparing (29) and (30), one recognises that the transformations $T_a$ and $M_q$ are Fourier-pairs. In particular, if one defines the Fourier transform of $\psi(x)$ by

$$\varphi(p) = \frac{1}{(2\pi\sigma)^{n/2}} \int \psi(x) \exp(i x \cdot p/\sigma) d^n x$$

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where $\sigma$ is a constant with units of action, then one has

$$
T_a : \varphi(p) \rightarrow \exp(i a \cdot p/\sigma)\varphi(p) \\
M_q : \varphi(p) \rightarrow \varphi(p - \sigma q/\hbar).
$$

Comparing with Eqs. (29) and (30), there is a direct symmetry between $\psi$ and $\varphi$ under position and momentum translations, provided one sets $\sigma = \hbar$.

In light of this symmetry, it is natural to postulate that, in analogy to $P(x) = |\psi(x)|^2$, the momentum probability density is given by $\tilde{P}(p) = |\varphi(p)|^2$. Under this postulate one finds that

$$
< f(p) > = \int \tilde{P}(p)f(p)d^n p = \int \psi^*(x)f(\frac{\hbar}{i} \nabla)\psi(x)d^n x
$$

which then leads to the natural generalisation

$$
< f(x,p) > = \int \psi^*(x)f(x, \frac{\hbar}{i} \nabla)\psi(x)d^n x \tag{31}
$$

as per standard quantum theory (where in general an operator ordering must be specified for the expectation value to be well defined).

Finally, we point out another approach that is also natural within this framework. Since the equations of motion in the variables $\psi$ and $\psi^*$ are linear, it is natural to investigate the group of canonical transformations that preserve the linearity of these equations. This leads to considering the group of transformations of the form

$$
\rho(y) = \int K(x,y)\psi(x)d^n x \\
\rho^*(y) = \int K^*(x,y)\psi^*(x)d^n x.
$$

Using (21), a simple calculation leads to the following condition for the transformation to be canonical,

$$
\{\rho, \rho^*\} = \int \left[ \frac{\delta \rho}{\delta \psi^m} \frac{\delta \rho^*}{\delta \psi^m} - \frac{\delta \rho}{\delta \psi^m} \frac{\delta \rho^*}{\delta \psi^m} \right] d^n x'' = \int [K(x'',y)K^*(x'',y')] d^n x'' = \delta(y - y').
$$

This is the condition for a transformation to be unitary. Arguments similar to the ones discussed above can then be used to single out the choice

$$
K(x,y) = \frac{1}{(2\pi \hbar)^{-n/2}} \exp(i x \cdot y/\hbar)
$$

which corresponds to the transformation that leads to the momentum space representation.
V. CONCLUSIONS

We have shown that an exact uncertainty principle, formulated in the form that the strength of the momentum fluctuations are inversely correlated with the uncertainty in position, leads from the classical equations of motion to the Schrödinger equation. The assumptions that we used for this fall into three main categories: maximal randomness (i.e., the condition that the nonclassical momentum fluctuations are uncorrelated with the average momentum, and the assumption that fluctuations for independent systems are uncorrelated), an exact uncertainty principle, and causality.

The additional term in the Lagrangian is essentially the Fisher information, originally introduced by Fisher [10] as a measure of “intrinsic accuracy” in statistical estimation theory. This Fisher information term was derived using an information theoretic approach in [13], and in [14] the same approach was shown to be applicable to the Pauli equation. The connection between Fisher information and quantum mechanics has been developed further in [9], where it is shown that the Fisher information is proportional to the difference of a classical and quantum variance (thus providing a measure of nonclassicality), and to the rate of entropy increase under Gaussian diffusion (thus providing a measure of robustness). We point out that in all of these references the Fisher information is defined in the usual way, that is, as a functional of the probability distribution – and therefore, one should not confuse it with the quantity by the same name that appears in Frieden [15], which is essentially a generalized Fisher information defined for wavefunctions and proportional to the quantum kinetic energy.

Note that our approach is very different to that of Bohm [16], in which $\nabla S$ is taken to precisely represent the momentum of a classical particle at a given position (thus there are no momentum fluctuations), and where the particle is acted on by a “quantum potential” that is generated by an associated wave obeying the Schrödinger equation. Similarly, while Bohm and Vigier generalise Bohm’s original formalism to permit fluctuations of momentum about $\nabla S$, this is merely to ensure that an ensemble of such particles will quickly evolve to have a stable distribution given by the modulus-squared of the associated wave [17]. In contrast, our approach is based on a classical ensemble rather than individual systems, no physical wave is assumed, nor the Schrödinger equation for such a wave, nor a quantum potential linking the wave to the motion of individual particles.

In [13] it was suggested that the Fisher information term represented an “epistemological” contribution to the action, which in the context of the present analysis can be interpreted as reflecting a lack of detailed knowledge of nonclassical momentum fluctuations. In our approach we do not attempt to provide a “realistic” model of such fluctuations, which would at any rate require a whole new theory that goes beyond quantum mechanics. Our approach to understanding quantum mechanics is therefore different from other descriptions based on the postulate of an underlying stochastic process, such as stochastic mechanics [18]. What our analysis primarily offers is a new way of viewing the uncertainty principle as the key concept in quantum mechanics. While it is true that no one before quantum mechanics would think of taking an uncertainty principle as a fundamental principle, our analysis is valuable in that it enforces the importance of the uncertainty principle in distinguishing quantum mechanics from classical mechanics – in a sense, it says that the uncertainty principle is the fundamental element that is needed for the transition to quantum mechanics.
REFERENCES