It is shown that the cosmological singularity in isotropic minisuperspaces is naturally removed by quantum geometry. Already at the kinematical level, this is indicated by the fact that the inverse scale factor is represented by a bounded operator even though the classical quantity diverges at the initial singularity. The full demonstration comes from an analysis of quantum dynamics. Because of quantum geometry, the quantum evolution occurs in discrete time steps and does not break down when the volume becomes zero. Instead, space-time can be extended to a branch preceding the classical singularity independently of the matter coupled to the model. For large volume the correct semiclassical behavior is obtained.

Structure of isotropic models. According to [8,9] states for isotropic models in the connection representation are distributional states of the full kinematical quantum theory supported on isotropic connections of the form \( A^i_\mu = c \Lambda^i_\mu \omega^_/ \) where \( \Lambda_\mu \) is an internal \( SU(2) \)-dreibein and \( \omega^_/ \) are the left-invariant one-forms on the “translational” part of the symmetry group acting on the space manifold \( \Sigma \). The momenta are densitized triads of the form \( E^i_\mu = p I^i_\mu X^_/ \) with left-invariant densitized vector fields \( X^_/ \) fulfilling \( \omega^_/ (X^_/) = \delta^_/ \). Besides gauge freedom, there are only the two canonically conjugate variables \( \{ c, p \} = \kappa \gamma / 3 \) (\( \kappa = 8 \pi G \) is the gravitational constant and \( \gamma > 0 \) the Barbero–Immirzi parameter) which have the physical meaning of extrinsic curvature and square of radius (\( a = \sqrt{|p|} \) is the scale factor). The kinematical Hilbert space \( H^_/ = L^2(SU(2), d\mu_H) \) is the space of functions of isotropic connections which are square inte-
grable with respect to Haar measure. Orthonormal gauge invariant states are (see [10] for details)
\[
\chi_j = \frac{\sin(j + \frac{1}{2})c}{\sin \frac{j}{2}} , \quad \zeta_j = \frac{\cos(j + \frac{1}{2})c}{\sin \frac{j}{2}}
\]
for \( j \in \frac{1}{2}\mathbb{N}_0 \) together with \( \zeta_{-\frac{1}{2}} = (\sqrt{2}\sin \frac{1}{2})^{-1} \). These states are eigenstates of the volume operator \( \hat{V} \) with eigenvalues [10]
\[
V_j = (\gamma \ell_p^2)^{\frac{3}{2}} \sqrt{\frac{1}{j+\frac{1}{2}}(j+\frac{1}{2})(j+1)} .
\]
Later we will also use a different orthonormal basis of states adapted to the triad by introducing
\[
|n\rangle := \frac{\exp(in\frac{1}{2})}{\sqrt{2\sin \frac{1}{2}}} , \quad n \in \mathbb{Z}
\]
where \( n \) represents the eigenvalues of \( p \) which determines the dreibein. In contrast to \( j \), which is always positive and represents eigenvalues of the square of the scale factor, \( n \) can also be negative. For this it is important that we have not only the character functions \( \chi_j \), but also the additional functions \( \zeta_j \). This concludes the discussion of quantum states.

The inverse scale factor. Classically, the metric of an isotropic spatial slice is given by \( q_{ij} = a^2\delta_{ij} = e^i_je^j_i \) where \( e^i_j \) is the co-triad. From this quantity we can build the expression
\[
m_{ij} := \frac{q_{ij}}{\det q} = \frac{e^i_je^j_i}{|\det e|} = \frac{1}{a} \delta_{ij}
\]
for the inverse scale factor, which we now quantize as a first application of the previously derived calculus. The co-triad is not a fundamental variable, but it can be quantized to \( q_{ij} h_I[h^{-1}_I, \hat{V}] \) due to the classical identity \( e^i_a = 2(\kappa \gamma)^{-1}\{A^a_i, V\} \) [11]. The expression \( \det e \) in the denominator of \( m_{ij} \) can be quantized to the volume operator which then can be absorbed into the commutators. Such a procedure has already been applied in [14] in order to quantize matter Hamiltonians which become densely defined operators, and in the same way we arrive at the bounded operator
\[
\hat{m}_{ij} = \frac{32}{\gamma \ell_p^4} \text{tr}(h_I[h^{-1}_I, \sqrt{\hat{V}}] h_J[h^{-1}_J, \sqrt{\hat{V}}])
\]
\[
= \frac{64}{\gamma \ell_p^4} \left( \left( \sqrt{\hat{V}} \cos \frac{\frac{1}{2}}{2} \sqrt{\hat{V}} \cos \frac{\frac{1}{2}}{2} - \sin \frac{\frac{1}{2}}{2} \sqrt{\hat{V}} \sin \frac{\frac{1}{2}}{2} \right)^2 \right)
\]
\[
- \delta_{ij} \left( \sin \frac{\frac{1}{2}}{2} \sqrt{\hat{V}} \cos \frac{\frac{1}{2}}{2} - \cos \frac{\frac{1}{2}}{2} \sqrt{\hat{V}} \sin \frac{\frac{1}{2}}{2} \right) .
\]
This operator is simultaneously diagonalizable with the volume operator and has the eigenvalues
\[
m_{IJ,J} = \frac{16}{\gamma \ell_p^4} \left( 4 \left( \sqrt{\hat{V}} - \frac{1}{2} \sqrt{\hat{V} + \frac{1}{2}} - \frac{1}{2} \sqrt{\hat{V} - \frac{1}{2}} \right)^2 \right)
\]
\[
+ \delta_{IJ} \left( \sqrt{\hat{V} + \frac{1}{2}} - \sqrt{\hat{V} - \frac{1}{2}} \right)^2 \right) \right)
\]
\[
\sim V^{-\frac{1}{2}} \left( \delta_{IJ} + \frac{\gamma^2}{9} \left( \frac{1}{256} + \frac{37}{192} \delta_{IJ} \right) \frac{\delta_{IJ}}{a^2} \right)
\]
where in the second step we have assumed that \( j - \frac{1}{2} \) and hence \( V_j - \frac{1}{2} \) is large. Thus, for large \( j \), the leading term is the classical value \( V^{-\frac{1}{2}} \delta_{IJ} \), and the corrections (which are not necessarily isotropic) are of only the fourth order. We see that our quantization leading to a bounded operator does not spoil the classical limit. In fact, the \( a^{-1} \)-behavior can be observed in a range which is much larger than expected from the large-\( j \) expansion. As Fig. 1 demonstrates, even for \( j = 1 \) are the eigenvalues very close to the classical expectation, and only the lowest three eigenvalues show large deviations. But this is already deeply in the quantum regime, so such deviations are expected and lead to a finite behavior of the classically diverging \( m_{IJ} \). Note that the volume operator has the eigenvalue zero (three-fold degenerate), but even in the corresponding eigenstates is the quantization of the inverse scale factor perfectly finite. This may be taken as a first indication for a removal of the classical singularity, although only at the kinematical level.

**Discrete time evolution.** Following the basic steps of the quantization in the full theory [11], the Hamiltonian constraints for cosmological models can be quantized with some adaptations to the symmetry [12]. For simplicity we write down here only the key term, the so-called Euclidean term \( H^{(E)} \), of the constraint operator for spatially flat isotropic models. However, all our qualitative results remain true for the full constraint and also for isotropic models with positive curvature. The constraint is of the form

\[
\hat{H}^{(E)} = \frac{4}{\gamma \ell_p^4} \sum_{IJ,K} \epsilon^{IJK} \text{tr}(h_I h_J h^{-1}_K h^{-1}_K[\hat{h}^{-1}_K, \hat{V}])
\]
\[
= -\frac{96i}{\gamma \ell_p^4} \sin^2 \frac{\frac{1}{2}}{2} \cos^2 \frac{\frac{1}{2}}{2} \left( \sin \frac{\frac{1}{2}}{2} \sqrt{\hat{V}} \cos \frac{\frac{1}{2}}{2} - \cos \frac{\frac{1}{2}}{2} \sqrt{\hat{V}} \sin \frac{\frac{1}{2}}{2} \right)
\]
In order to “unfreeze dynamics” and interpret solutions as “evolving states,” as usual [15,16] we have to introduce an internal time which we choose as the dreibein coefficient $p$. Accordingly, we transform states $|s\rangle$ into an adapted representation by expanding $|s\rangle = \sum_n s_n|n\rangle$ in eigenstates $|n\rangle$ of $p$. This will allow us to find an interpretation of physical states as evolving histories. Furthermore, discrete geometry implies that eigenvalues of $p$ are discrete, whence time evolution is now discrete (see [13] for details). Moreover, since we chose a geometrical quantity as time which can be negative and is zero for vanishing volume, we will be able to test the possibility of a quantum evolution through the classical singularity.

To realize dynamics, we need to extend the model with matter degrees of freedom which can evolve with this internal time. Matter can be incorporated by using coefficients $s_n(\phi)$ depending on the matter field $\phi$ in an appropriate fashion, the details of which is not important for what follows. The Hamiltonian constraint can then be written down using a matter Hamiltonian $\hat{H}_\phi$ (as in [14]) which is diagonal in the gravitational degrees of freedom (and can also contain a cosmological term). The resulting quantum constraint equation can then be regarded as an evolution in discrete time:

$$\left(V_{n+4}/2 - V_{n+4}/2 - 1\right)s_{n+4}(\phi) - 2(V_{n}/2 - V_{n}/2 - 1)s_n(\phi) + (V_{n-4}/2 - V_{n-4}/2 - 1)s_{n-4}(\phi) = \frac{1}{\sqrt{|k|\delta_k}} \hat{H}_\phi s_n(\phi)$$

$$\left(V_{n} - 1 = 0\right)$$

for some negative $n$, we can use (7) in order to determine later values for higher $n$. This, however, is possible only as long as the highest order coefficient $V_{n+4}/2 - V_{n+4}/2 - 1$ is nonzero, which is the case if and only if $n \neq -4$. So all coefficients for $n < -4$ are determined by the initial data. However, (7) does not determine $s_0$ and instead leads to a consistency condition for the initial data. So the quantum evolution appears to break-down just at the classical singularity, i.e. at the zero eigenvalue of $p$. But this is not the case; in fact all $s_n$ for $n > 0$ are determined by (7) from the initial data. This occurs because for $n = 0$ we have: i) $V_{n}/2 - V_{n}/2 - 1 = 0$, and ii) $\hat{H}_\phi s_0(\phi) = 0$; thus $s_0$ completely drops out of the iterative evolution. E.g., $s_4$ is determined solely by $s_{-4}$ because the coefficient of $s_n$ vanishes for $n = 0$. So we can evolve through the singularity and determine all $s_n$ for $n \neq 0$. (The vanishing of $\hat{H}_\phi s_0(\phi)$ follows from the quantization of matter Hamiltonians [14] similarly as described for the inverse scale factor.)

Of course, in order to determine the complete state we also have to know $s_0$, but a closer analysis reveals that $s_0$ is fixed from the outset: The Hamiltonian constraint always has the eigenstate $s_0 = s_0 \delta_{s0}$ with zero eigenvalue which is completely degenerate and not of physical interest. All evolving solutions are orthogonal to this state and have $s_0 = 0$ which already fixes the coefficient $s_0$ left undetermined by using the evolution equation. We see that the complete state is determined by initial data for negative $n$, and so there is no singularity in isotropic loop quantum cosmology. The intuitive picture is as follows: Since for $n < 0$ the volume eigenvalues $V_{|n|}/2$ decrease with increasing $n$, there is a contracting branch for negative $n$ leading to a state of zero volume (in general, $s_{\pm 1} \neq 0$ and the volume vanishes for $n = \pm 1$ which corresponds to $j = 0$) in which the universe bounces off leading to the expanding branch for positive $n$ which only can be seen in the classical theory and in standard quantum cosmology. This conclusion holds true for any kind of matter and cosmological constant, and is a purely quantum gravitational effect. In particular, we do not need to introduce matter violating energy conditions and thereby evade the singularity theorems. However, our result crucially depends on the factor ordering of the constraint which was chosen as one of the standard possibilities ordering all triad components to the right.

The semiclassical regime. We have seen that the classical singularity is removed in loop quantum cosmology. But we need more for a viable cosmological model, namely we also need the correct behavior in the semiclassical regime. Classical behavior can only be present for large volume and small extrinsic curvature, i.e. if $|n|$ is large, $c$ is small and the wave function does not vary strongly between successive times $n$ (otherwise the state would have access to the Planck scale). In this regime we can interpolate between the discrete labels $n$ and define a wave function $\psi(a) := s_n(a)$, $n(a) := 6a^2/\gamma l_p^2$ with a ranging over a continuous range (using $a = \sqrt{\gamma l_p^2} \approx \sqrt{\gamma l_p^2} \sqrt{|n|/6}$ for large $|n|$ as interpolation points). The difference operator $\Delta$ then becomes $(\Delta s)_n := s_{n+1} - s_{n-1} = \frac{1}{\sqrt{|k|\delta_k}} d\psi/da + O(l_p^2/a^5)$ leading to an approximate constraint operator $\hat{H}^{(E)} = -96(\Delta/2)^2 \cdot a/4 - 6\gamma l_p^2(-\frac{1}{2}d/d(a^2))^2a$ for large $a$. This is exactly what one obtains from the classical constraint $H^{(E)} = -6c^2/\sqrt{\gamma l_p^2}$ in standard quantum cosmology [17] by quantizing $3\epsilon_{ab}$ in the WKB-techniques in order to derive the correct classical behavior.
derive perturbative corrections for an effective Hamiltonian including higher derivative terms. The closer we come to the classical singularity, the more corrections we have to include; and at the singularity we need to know all corrections which, as we know from our non-perturbative solution, have to add up to yield the discrete time behavior. So in these models higher order terms arise from the non-locality in discrete time of the fundamental theory. But even knowing all perturbative corrections, it would be very hard to see the correct behavior without knowing the non-perturbative quantization.

Quantum Euclidean space. In the simplest case, the Euclidean constraint for a spatially flat model without matter, it is possible to find an explicit solution to the constraint. The constraint equation is of order eight with one consistency condition as described above, so one expects seven independent solutions. But we are interested only in solutions which have a classical regime in the previous sense, i.e. no strong dependence on $j$ for large $j$. Under this condition one can see that there is a unique (up to a constant factor) solution

$$\psi(c) = \sum_j \frac{2j+1}{V_j + \frac{1}{2} - V_j - \frac{1}{2}} \chi_j(c)$$

in the connection representation. In standard quantum cosmology the constraint equation is $\hat{c}^2 \sqrt{|p|} \xi(c) = 0$ with a solution $\sqrt{|p|} \xi(c) = \delta(c)$ which is not unique. In order to compare the solutions we quantize $\alpha a \chi_j = 2i (\gamma c^2)^{-1} (V_j + \frac{1}{2} - V_j - \frac{1}{2}) \chi_j$ leading to $\hat{\alpha} \psi \propto \sum_j (2j+1) \chi_j$ which in fact is the delta function on the configuration space $SU(2)$. Therefore, we have a unique solution which incorporates the characterization of Euclidean space to have vanishing extrinsic curvature of its flat spatial slices.

Conclusions. We have shown in this paper that canonical quantum gravity is well-suited to analyze the behavior close to the classical singularity. For this, it is important to use only techniques which are applicable in the full theory. This leads to a discrete structure of space and time which cannot be seen in standard quantum cosmology. In our framework, the standard quantum cosmological description arises only as a limit for large volume where the discreteness is unimportant. For small volume, quantum geometry leads to new effects which are responsible for the removal of the classical singularity. In contrast to earlier attempts this is not achieved by introducing matter which violates energy conditions; it is a pure quantum gravity effect. It also does not avoid the zero volume state present in the classical singularity because in general the wave function is not orthogonal to states with zero volume eigenvalue. Nevertheless there is no sign of a singularity because in quantum geometry it is possible to have vanishing volume but non-diverging inverse scale factor, which in isotropic models dictates all curvature blow-ups. Besides removing the singularity, the fact that an evolution through a state of zero volume is possible without problems could lead to topology change in quantum gravity. Technically, the removal of the singularity is achieved by using Thiemann’s strategy [11] of absorbing inverse powers of $V$ into a Poisson bracket which also lead to densely defined matter Hamiltonians [14]. So it is the same mechanism which regularizes ultraviolet divergences in matter field theories and which removes the classical cosmological singularity. We have also seen that non-perturbative effects are solely responsible for this behavior and a purely perturbative analysis could not lead to these conclusions.

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