Do gravitational waves carry energy-momentum and angular momentum?

Janusz Garecki

Institute of Physics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, POLAND

(February 27, 2001)

Abstract

In the paper we show that the real gravitational waves which have $R_{iklm} \neq 0$ always carry energy-momentum and angular momentum. Our proof uses canonical superenergy and supermomentum tensors for gravitational field.

KEY WORDS: gravitational waves, gravitational energy, gravitational superenergy

04.20.Me.04.30.+x
I. INTRODUCTION

In General Relativity (GR) the gravitational field $\Gamma_{kl}^i$ does not possess any energy-momentum tensor but it only possesses the so-called “energy-momentum pseudotensors”. It is a consequence of the Einstein Equivalence Principle (EEP). So, many authors [1,4,8,18] doubt in reality of the energy-momentum and angular momentum transfer by gravitational waves. As an argumentation some of these authors use the fact that for the majority exact solutions of the vacuum Einstein equations which, as we think, can represent gravitational waves, pseudotensors *globally vanish* in some coordinates. In consequence, they give in these coordinates “no gravitational energy and no gravitational energy flux”. Other of these authors [1,8] try to use vanishing of the components $g_{t^{ok}}$ (or $g_{t^{i0}}$) of a gravitational energy-momentum pseudotensor $g_{t^{ik}}$ (or $g_{t^{i0}}$) as a coordinate condition coupled to the Einstein equations and then also get (in special coordinates), as they assert, “global vanishing of the pure gravitational energy and gravitational energy flux”.

However, such conclusions are physically uncorrect. Firstly, these authors neglect an important role of the four-velocity $\vec{v}, \vec{v} \cdot \vec{v} = 1$ of an observer $O$, which is measuring gravitational (or other) field, in definition of the energy density $\epsilon$ and energy flux $P^i$ of the field. Namely, the correct definitions of the energy density and its flux for such observer are the following

$$\epsilon = T^{ik}v_iv_k = T_{ik}v^iv^k,$$

(1)

$$P^i = (\delta^i_k - v^i v_k)T^{kl}v_l;$$

(2)

not

$$\epsilon = T^{00}, \ P^\alpha = T^{0\alpha}.$$  

(3)

$i, k, l, ..., = 0, 1, 2, 3; \ \alpha, \beta, \gamma, \delta, ..., = 1, 2, 3$. $T^{ik}$ mean here the components of an energy-momentum tensor.
$T^{00}$ and $T^{0\alpha}$ give energy density $\epsilon$ and energy flux $P^i$ for the observer $O$ iff in the used global coordinates we have for the observer $v^i = \delta^i_0$, $v_k = \delta^0_k$.\(^1\)

The above facts are also true for a gravitational energy-momentum pseudotensor $g^{tik}$ (or $g^{tk}$).

In consequence, even if $g^{t0k} = 0$ (or $g^{tk} = 0$) globally in the used coordinates then, as one can easily see, *not for all observers* $\epsilon = P^i = 0$. All depends on the four-velocity $\vec{v}$ of the observer $O$.

Moreover, one coordinate condition of the kind $g^{t0k} = 0$ (or $g^{tk} = 0$) does *not* suffice for the all variety of the gravitational energy-momentum pseudotensors. In fact, we need here an own coordinate condition for every gravitational energy-momentum pseudotensor.

Secondly, and it is the most important fault, these authors forget that gravitational energy-momentum (and gravitationam angular momentum) pseudotensors, as being functions of the Levi-Civita connection coefficients\(^2\) describe energy-momentum of the total gravitational field which is a combination of the real gravitational field (for which $R_{iklm} \neq 0$) and inertial forces field (for which $R_{iklm} = 0$). The inertial forces field is generated by the used coordinates. This is also a consequence of the EEP.

So, if in some coordinates $g^{tik}$ or $(g^{tk})$ (or only $g^{t0k}$ or $g^{tk0}$) globally vanish, this does not mean that in these coordinates we have no pure gravitational energy and no pure gravitational energy flux.\(^3\) Simply, it means only that in such coordinates energy and energy flux of the real gravitational field cancel with the energy and energy flux of the inertial forces field.\(^4\)

---

\(^1\)This means physically that the observer $O$ is at rest with respect to the used coordinates.

\(^2\)Physically, this connection plays the role of the total gravitational strenghts.

\(^3\)Even if we confine to the observers which are at rest in the used global coordinates and for which $g^{t00}$ (or $g^{t0}$) is “energy density” and $g^{t0\alpha}$ (or $g^{t0\alpha}$) give the “energy flux”.

\(^4\)Energy-momentum of the inertial forces field gives contribution to energy-momentum pseudotensors.
Energy and energy-momentum flux of the real gravitational field which has $R_{klm} \neq 0$ always exist and are non-null. In order to show this very important fact one can use the canonical superenergy and canonical angular supermomentum tensors for gravitational field [10-14].

The canonical superenergy tensor $gS^{(k}_i$ and canonical angular supermomentum tensor $gS^{k_l} = (-)gS^{k_l}$ are constructed in such a way that they extract covariant information about real gravitational field which is hidden in canonical energy-momentum and canonical angular momentum pseudotensors. Namely, these superenergy and angular supermomentum tensors are obtained from suitable gravitational pseudotensors by some kind of averaging and they are functions of the curvature tensor and its covariant derivatives only. So, they really describe only real gravitational field, which has $R_{klm} \neq 0$.

The paper is organized as follows. In Section II we remember canonical superenergy and supermomentum tensors for gravitational field in GR. In Section III we give a summary about applications of the canonical superenergy and supermomentum tensors for gravitational field to analysis of the gravitational waves and in Section IV we give some concluding remarks.

II. CONSTRUCTIVE DEFINITION OF THE CANONICAL SUPERENERGY TENSOR AND CANONICAL ANGULAR SUPERMOMENTUM TENSOR

We remember here the constructive, general definition of the superenergy tensor $S_{a}^{(b}(P)$ applicable to gravitational field and to matter fields.

In the normal coordinates NC(P) [see, e.g., 20,21,22], we define

$$S_{a}^{(b}(P) := (-) \lim_{\Omega \to P} \frac{\int_{\Omega} \left[ T_{(a}^{(b)}(y) - T_{(a}^{(b)}(P) \right] d\Omega}{1/2 \int_{\Omega} \sigma(P;y) d\Omega},$$

(4)

where

5 Or other superenergy and angular supermomentum tensors.
\[ T_{(a)}^{(b)}(y) := T_i^k(y)e_{(a)}^i(y)e_k^{(b)}(y), \]  

\[ T_{(a)}^{(b)}(P) := T_i^k(P)e_{(a)}^i(P)e_k^{(b)}(P) = T_a^b(P) \]  

are the so-called physical or tetrad components of the pseudotensor (or tensor) field \( T_i^k(y) \) which describes energy-momentum. \( \{ y' \} \) are the normal coordinates. \( e_{(a)}^i(y) \), \( e_k^{(b)}(y) \) mean here an orthonormal tetrad \( e_{(a)}^i(P) = \delta_a^i \) and its dual \( e_k^{(a)}(P) = \delta_k^a \) parallelly propagated along geodesics through \( P \) (\( P = \text{origin of the NC}(P) \)).

We have

\[ e_{(a)}^i(y)e_k^{(b)}(y) = \delta_a^b. \]  

\( \Omega \) is a sufficiently small ball

\[ y_0^2 + y_1^2 + y_2^2 + y_3^2 \leq R^2 \]  

for an auxilliary positive-definite metric \( h^{ik} := 2v^iv^k - g^{ik} \), i.e., \( \Omega \) can be given as

\[ h_{ik}y^iy^k \leq R^2. \]  

\( v^i \) denotes here the components of the four-velocity \( \vec{v} : v^iv_l = 1 \) of an observer \( O \) which is at rest at the beginning \( P \) of the used normal coordinates \( \text{NC}(P) \).

The ball \( \Omega \) surrounds \( P \): \( P \) is centre of the ball.

\[ \sigma(P; y) = \frac{1}{2}(y^0^2 - y_1^2 - y_2^2 - y_3^2) \]  

is the two-points world function which has been introduced by J.L. Synge years ago [23]. The symbol \( \overset{\cdot}{=} \) means that an equation is only valid for special coordinates. The latter can be covariantly defined by the eikonal-like equation

\[ g^{ik}\partial_i\sigma\partial_k\sigma = 2\sigma \]  

together with \( \sigma(P, P) = 0, \ \partial_l\sigma(P, P) \)

The \( \Omega \) can be also given by the inequality

\[ 5 \]
\[ h^{ik} \partial_i \sigma \partial_k \sigma \leq R^2. \] (12)

Since the tetrad components and normal components are equal at the point \( P \), we will write the components of any quantity attached to the point \( P \) without tetrad brackets, e.g., we will write \( S^b_a(P) \) instead of \( S^{(b)}_a(P) \) and so on.

If \( T^i_k \) are the components of a symmetric energy-momentum tensor of matter then we will get from (4)

\[ mS^b_a(P) = \delta^{mn} \nabla_m \nabla_n \hat{T}^b_a. \] (13)

Hat over a quantity denotes its value at \( P \).

By using of the four-velocity \( \vec{v} \) of a fictitious observer \( O \) being at rest at the beginning \( P \) of the used \( \text{NC}(P) \) and local metric \( \hat{g}^{ab} = \eta^{ab} \) one can write this covariantly as

\[ mS^b_a(P; v^l) = (2 \hat{v}^l \hat{v}^m - \hat{g}^{lm}) \nabla_l \nabla_m \hat{T}^b_a. \] (14)

The last formula gives us the canonical superenergy tensor for matter.

For the gravitational field \( \Gamma^i_{kl} = \{^i_{kl}\} \), after substituting into (4) \( T^i_k = E^i_k \), where

\[
E^i_k = \frac{\mathcal{C}^4}{16\pi G} \left\{ \delta^i_k g^{ms} (\Gamma^l_{ms} \Gamma^r_{tl} - \Gamma^r_{ms} \Gamma^l_{rl}) 
+ g^{ms} [\Gamma^k_{ms} - \frac{1}{2} (\Gamma^k_{tp} g^{lp} - \Gamma^l_{tp} g^{kp}) g_{ms} 
- \frac{1}{2} (\Gamma^k_{ml} + \delta^k_m \Gamma^l_{sl})] \right\}
\] (15)

is the canonical Einstein energy-momentum pseudotensor for the gravitational field, we will obtain

\[ gS^b_a(P; v^l) = (2 \hat{v}^l \hat{v}^m - \hat{g}^{lm}) E^b_a \hat{T}^b_a. \] (16)

\[
E T^b_{a lm} = \frac{2\alpha}{9} \left[ B^b_{alm} + P^b_{alm} 
- \frac{1}{2} \delta^b_a R^{ijkl}_{m} (R_{ijkl} + R_{ikjl}) + 2\delta^b_a \beta^2 E_{(l|g} E^g_{|m)} 
- 3\beta^2 E_{a(l|g} E_{|m)} + 2\beta R^b_{(ag)(l|} E^g_{|m)} \right].
\] (17)
In the last formula

$$\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta}$$

(18)

and

$$E^k_i := T^k_i - \frac{1}{2}\delta^k_i T$$

(19)

is the modified energy-momentum of matter. On the other hand

$$B^b_{\text{alm}} := 2R^{b|ik}R_{aik|m} - \frac{1}{2}\delta^b_a R^{i|jk}_l R_{ijk|m}$$

(20)

are the components of the Bel-Robinson tensor, while the tensor

$$P^b_{\text{alm}} := 2R^{b|ik}R_{aki|m} - \frac{1}{2}\delta^b_a R^{i|jk}_l R_{ikjm}$$

(21)

is closely related to the latter.

The tensor $gS^b_a(P; v^l)$ is the **canonical superenergy tensor** for gravitational field.\(^6\)

In vacuum the tensor $gS^b_a(P; v^l)$ takes the simpler form

$$gS^b_a(P; v^l) = \frac{8\alpha}{9}(2\hat{v}^l\hat{v}^m - \hat{g}^{lm})[\hat{R}^{b|ik}_l \hat{R}_{aik|m}]$$

\(-\frac{1}{2}\delta^b_a \hat{R}^{i|kp}_l \hat{R}_{ikp|m}]$$

(22)

and the quadratic form $gS_{ab}(P; v^l)v^a v^b$, where $v^a v_a = 1$, is positive-definite.

The canonical angular supermomentum tensor we define in analogy to the definition of the canonical superenergy tensor. Namely, we define in the normal coordinates $\text{NC}(P)$

$$S^{(a)(b)(c)}(P) = S^{abc}(P) := (-1) \lim_{\Omega \to P} \frac{\int_\Omega \left[ M^{(a)(b)(c)}(y) - M^{(a)(b)(c)}(P) \right] d\Omega}{1/2 \int_\Omega \sigma(P; y) d\Omega},$$

(23)

\(^6\)We must emphasize that the canonical superenergy tensor $gS^b_a(P; v^L)$ is originally a function of the curvature components $R_{iklm}$ and tensor Ricci components $R_{ik}$. We have eliminated the all Ricci components by use of the Einstein equations $R_{ik} = \beta E_{ik}$.
where, as formerly,

\[ M^{(a)(b)(c)}(y) := M^{ikl}(y)\epsilon_i^{(a)}(y)\epsilon_k^{(b)}(y)\epsilon_l^{(c)}(y), \quad (24) \]

\[ M^{(a)(b)(c)}(P) := M^{ikl}(P)\epsilon_i^{(a)}(P)\epsilon_k^{(b)}(P)\epsilon_l^{(c)}(P) \]

\[ = M^{ikl}(P)\delta_i^a\delta_k^b\delta_l^c = M^{abc}(P) \quad (25) \]

are physical (or tetrad) components of the field \( M^{ikl}(y) = (-)M^{kil}(y) \) describing angular momentum.

For matter, as \( M^{ikl}(y) = (-)M^{kil}(y) \), we take

\[ M^{ikl}(y) = \sqrt{|g|}(y^iT^{kl} - y^kT^{il}), \quad (26) \]

where \( T^{ik} = T^{ki} \) are the components of the symmetric energy-momentum tensor of matter\(^7\) and \( \{y^i\} \) denote normal coordinates.

The expression (26) gives us the total angular momentum densities, orbital and spin, because the dynamical tensor \( T^{ik} = T^{ki} \) is obtained from canonical one by means of the Belinfante symmetrization procedure (and, therefore, includes material spin).

For gravitational field, as \( gM^{ikl}(y) \), we favorize and take the expression most closely related to the Einstein canonical energy-momentum complex (See Appendix). We will call this expression canonical also. Namely, as \( gM^{ikl}(y) \), we take the expression given by Bergmann and Thomson [24]

\[ gM^{ikl}(y) = F^{\ast}U^{i[kl]}(y) - F^{\ast}U^{k[il]}(y) + \sqrt{|g|}(y^i_B T^{kl} - y^k_B T^{il}). \quad (27) \]

In the last formula

\[ F^{\ast}U^{i[kl]} := g^{im}F^{\ast}U^{[kl]}_m, \quad (28) \]

where \( F^{\ast}U^{[kl]}_m \) mean von Freud superpotentials and

---

\(^7\)This tensor is the source in the Einstein equations.
\[ B_{T}t_{kl} := g_{E}^{k}t_{i} + g_{m}^{nk}U_{m}^{[lp]} \]  

(29)

is the Bergmann-Thomson gravitational energy-momentum pseudotensor [24].

One can interpret the Bergmann-Thomson expression (27) as a sum of the spinorial part

\[ S^{ikl} := F U^{[kl]} - F U^{k[i]l} \]  

(30)

and orbital part

\[ O^{ikl} := \sqrt{|g|}(y_{BT}^{i}t_{kl} - y^{k}BT_{t}^{il}) \]  

(31)

of the gravitational angular momentum densities.

Substituting (26) into (23) we will get the canonical angular supermomentum tensor for matter

\[ mS^{abc}(P; v^{l}) = 2[(2\hat{\nu}^{a}\hat{\nu}^{p} - \hat{g}^{ap})\nabla_{p}\hat{F}^{bc} - (2\hat{\nu}^{b}\hat{\nu}^{p} - \hat{g}^{bp})\nabla_{p}\hat{F}^{ac}]. \]  

(32)

If we substitute (27) into (23), then we will obtain the gravitational canonical angular supermomentum tensor

\[ gS^{abc}(P; v^{l}) = \alpha(2\hat{\nu}^{p}\hat{\nu}^{t} - \hat{g}^{pt})[\beta(\hat{g}^{ac}\hat{g}^{br} - \hat{g}^{bc}\hat{g}^{ar})\nabla_{(t}\hat{E}_{pr)} \]  

\[ + 2\hat{g}^{ar}\nabla_{(t}\hat{R}^{(b \cdot c)}_{p \cdot r) - 2\hat{g}^{br}\nabla_{(t}\hat{R}^{(a \cdot c)}_{p \cdot r)} \]  

\[ + 2/3\hat{g}^{bc}\nabla_{(t}\hat{R}^{a}_{(p \cdot r) - \beta(\hat{E}_{p\cdot t)) \} \]  

\[ - 2/3\hat{g}^{ac}\nabla_{(t}\hat{R}^{b}_{(r \cdot p) - \beta(\hat{E}_{p\cdot t)) \} \} \]  

(33)

In vacuum, i.e., when \( T_{ik} = 0 \equiv E_{ik} := T_{ik} - 1/2g_{ik}T = 0 \), the gravitational canonical angular supermomentum tensor \( gS^{abc}(P; v^{l}) = (-)gS^{bac}(P; v^{l}) \) simplifies to the form

\[ 8We have also eliminated in (33) the components of Ricci tensor with the help of the Einstein equations.\]
\[ g_{S_{abc}}(P; v^l) = 2\alpha (2\hat{\varphi}^p \hat{v}^t - \hat{g}^{pt}) \left[ \hat{g}^{ar} \nabla_{(p} \hat{R}^{l b c}_{t r)} - \hat{g}^{br} \nabla_{(p} \hat{R}^{l a c}_{t r)} \right]. \] (34)

It is interesting that the orbital part \( 0_{ikl} = \sqrt{|g|} (y_{_{B_T}}^i t_{kl}^j - y_{_{B_T}}^k t_{il}^j) \) of the \( g_{M_{ikl}} \) gives no contribution to the tensor \( g_{S_{abc}}(P; v^l) \). Only spin part \( S_{ikl} = F_{U_{_{[kl]}}^i} - F_{U_{_{[il]}^i}} \) gives non-zero contribution to this tensor. Notice also that the canonical angular supermomentum tensors \( g_{S_{abc}}(P; v^l) \) and \( m_{S_{abc}}(P; v^l) \), gravitation and matter, needn’t any radius vector for their own existing.

### III. SUMMARY ABOUT APPLICATION OF THE SUPERENERGY AND ANGULAR SUPERMOMENTUM TENSORS TO ANALYSIS OF THE GRAVITATIONAL WAVES

One can easily check by a direct calculation that the canonical superenergy tensor \( g_{S_{ik}}(P; v^l) \) and the canonical angular supermomentum tensor \( g_{S_{ikl}}(P; v^m) \) give positive-definite superenergy density, non-vanishing superenergy flux and non-vanishing angular supermomentum flux for every known solution to the vacuum Einstein equations which represents a gravitational wave with \( R_{iklm} \neq 0 \). This is true in any admissible coordinates and it all was showed in our papers [10,12–14]. It results from the above fact that every real gravitational wave, which has \( R_{iklm} \neq 0 \), must carry gravitational energy-momentum and gravitational angular momentum. If not, then there would be a contradiction between “energy level” and “superenergy level” because the canonical superenergy and angular supermomentum tensors originated by some kind of averaging of the suitable pseudotensors. Notice also in this context that, as it follows from the definition (4), the \( \epsilon(\Omega) := (-g_{S_{ik}}(P; v^l)v^i v^k \int_{\Omega} \sigma(P; y) d\Omega \) gives the approximate (relative with respect \( P \)) energy contained in the sufficiently small domain \( \Omega \) and \( P(\Omega) := (-1/2(\delta_k^l - u^i u_k) g_{S_{ik}}(P; v^m)v^l \int_{\Omega} \sigma(P; y) d\Omega \) gives Poynting’s vector of the (relative with respect \( P \)) energy flux contained in \( \Omega \). \( \epsilon(\Omega) \) and \( P(\Omega) \) do not vanish in any admissible coordinates for a real gravitational wave which has \( R_{iklm} \neq 0 \).
So, the non-zero superenergy density and its non-zero flux really demand non-zero gravitational energy and its non-zero flux.

On the “energy-momentum level”, where we use pseudotensors, the very important fact that any gravitational wave which has \( R_{iklm} \neq 0 \) always transfer energy-momentum is camouflaged in some coordinates by energy-momentum of the inertial forces field.

The analogical considerations and conclusion which are based on the definition (23) are valid also for the angular momentum.

**IV. CONCLUDING REMARKS**

If you want to get the correct information about energy-momentum and angular momentum of the real gravitational field by use pseudotensors, then you must use the pseudotensors in very special coordinates only (see eg. [25-27]). Namely, you must use pseudotensors in coordinates in which \( \Gamma^i_{kl} \) describe only the real gravitational field. The examples of such coordinates are given, e.g., by global Bondi-Sachs coordinates for a closed system [7] or, in general, by normal coordinates \( \text{NC}(P) \) [20-22, 28,29].

In order to get information on gravitational energy-momentum and angular momentum in arbitrary admissible coordinates one must use covariant expressions which depend on curvature tensor. Our canonical superenergy and angular supermomentum tensors are exactly the quantities of such a kind. In application to gravitational radiation these quasilocal quantities unambiguously show that any gravitational waves which has \( R_{iklm} \neq 0 \) always transfer energy-momentum and angular momentum. So, the conclusions given in the papers like [1,8,18], where authors use pseudotensors only are uncorrect.\(^9\)

\(^9\)In the paper [18] the Author uses also stationary Tolman integral to analysis of a dynamical system. Of course, this is unjustified physically.
APPENDIX A: THE CANONICAL ENERGY-MOMENTUM COMPLEX IN GENERAL RELATIVITY

One can easily transform 10 Einstein equations to the superpotential form

\[ \sqrt{|g|}(T^k_i + \varepsilon^k_i) = F U_i^{[kl]}_{,l}. \]  \tag{A1} \]

In the last formula $T^k_i = T^i_k$ is the symmetric energy-momentum tensor for matter, $\varepsilon^k_i$ mean the components of the Einstein canonical energy-momentum pseudotensor for gravitational field and $F U_i^{[kl]}$ denote von Freud superpotentials. The sum

\[ \sqrt{|g|}(T^k_i + \varepsilon^k_i) =: E K_i^k \]  \tag{A2} \]

is called the canonical Einstein energy-momentum complex for matter and gravitation in General Relativity and it is usually denoted by $E K_i^k$.

In extended form we have

\[ \varepsilon^k_i = \frac{c^4}{16\pi G} \left\{ \delta^k_i g^{ms}(\Gamma^l_{mr} \Gamma^r_{st} - \Gamma^r_{ms} \Gamma^l_{rt}) ight. \\
+ g^{ms} \Gamma^k_{ms} - 1/2(\Gamma^k_{tp} g^{tp} - \Gamma^l_{tl} g^{kt})g_{ms} \\
- 1/2(\delta^k_s \Gamma^l_{mt} + \delta^k_m \Gamma^l_{st}) \right\}, \]  \tag{A3} \]

\[ F U_i^{[kl]} = \frac{c^4}{16\pi G \sqrt{|g|}} g_{ia} \left[ (-g)(g^{ka} g^{lb} - g^{la} g^{kb}) \right]_{,b}. \]  \tag{A4} \]

From (A.1) there follow the local, differential energy-momentum conservation laws

\[ [\sqrt{|g|}(T^k_i + \varepsilon^k_i)]_k = 0, \]  \tag{A5} \]

for matter and gravitation.

---

10It is the most easily to do by use the formalism of the exterior differential forms.
REFERENCES


