Abstract

We obtain a complete set of one-loop RG's for a set of combinations of neutrino parameters.

Three-Neutrino Models with Two-Fold Degeneracy

Implications of the Renormalization Group Equations in

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It has been a puzzle that the mixing in the leptonic sector is so large while the mixing in the quark sector is so small. Many attempts have been made to explain this fact. One possible scenario is to utilize the flavour symmetry combined with GUT symmetry at some high energy scale $\Lambda$ [1]. Viable symmetries are those giving rise to large mixing in the lepton mass matrices. Most models of this kind suffer from fine-tunning and the difficulty of constructing a viable superpotential in the flavour symmetry sector that gives rise to the required vacua. An alternative to this scenario is the idea of infrared fixed point (IRFP) [2]. Contrary to the idea of flavour combined with GUT symmetry, in the IRFP scenario, the low energy physics is governed by the low energy dynamics, namely, the renormalization group equations below the scale $\Lambda$. Physics above the scale $\Lambda$ plays no role in the predictions at low energies. Therefore, if there exists any IRFP which leads to viable phenomenology, one does not have to deal with the fine-tuning problem and the difficulty of finding the correct vacua. The focus of our attention in this note is the hierarchical three neutrino models with two-fold denegeracy, and the implications of the exact solutions to one-loop renormalization equations (RGE’s) to these models.

In the flavour basis where the charged leptons are diagonal, the neutrino flavour eigenstates and mass eigenstates are related by $|\nu_\alpha> = U_{\alpha i}|\nu_i>$, where $\alpha$ and $i$ are the flavour and mass eigenstate indices respectively. The mass matrix $m_\nu$ can be diagonalized as follows

$$U^T m_\nu U = \text{diag}(m_1, m_2, m_3)$$

We adopt the usual parametrization for the leptonic mixing matrix $U$

$$U = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) \cdot V \cdot \text{diag}(e^{-i\phi/2}, e^{-i\phi'/2}, 1)$$

$$V = \begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 e^{-i\delta} \\ -c_1 s_3 - s_1 s_2 c_3 e^{i\delta} & c_1 c_3 - s_1 s_2 s_3 e^{i\delta} & s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 e^{i\delta} & -s_1 c_3 - c_1 s_2 s_3 e^{i\delta} & c_1 c_2 \end{pmatrix}$$

where $c_i \equiv \cos \theta_i$ and $s_i \equiv \sin \theta_i$ and $0 \leq \theta_i \leq \pi/2$. The $\delta_{e,\mu,\tau}$ are three unphysical phases which can be absorbed by phase redefinition of the neutrino flavour eigenstates. There are three physical phases: $\delta$ is the universal phase (analog of the phase in the CKM matrix), and $\phi$ and $\phi'$ are the Majorana phases. By properly choosing the phases $\phi$ and $\phi'$ all three mass eigenvalues $m_i$ can be made positive. We therefore assume, without loss of
generality, that \((m_1, m_2, m_3)\) are positive. If any of these three phases is not zero or not \(\pi\), CP violation in the lepton sector is implied. Note that in the limit \(\theta_2 = 0\), \(\theta_1\) is identified as the atmospheric mixing angle, \(\theta_{\text{atm}}\), and \(\theta_3\) is identified as the solar mixing angle, \(\theta_{\odot}\). In general, the mixing matrix elements are related to the physical observables, the atmospheric and solar mixing angles, by \(\sin^2 \theta_{\text{atm}} \equiv 4 |U_{\mu 3}|^2 (1 - |U_{\mu 3}|^2)\), and \(\sin^2 \theta_{\odot} \equiv 4 |U_{e 2}|^2 (1 - |U_{e 2}|^2)\). Recent results indicate [3] that for atmospheric neutrino oscillations, \(\Delta m^2_{\text{atm}} = 3.1 \times 10^{-3} \text{eV}^2\), \(\sin^2 \theta_{\text{atm}} = 0.972 \) [4]; for solar neutrino anomaly problem, there exists four solutions: (i) VO: \(\Delta m^2_{\odot} = 8.0 \times 10^{-11} \text{eV}^2\), \(\sin^2 \theta_{\odot} = 0.75\), (ii) LOW: \(\Delta m^2_{\odot} = 7.9 \times 10^{-8} \text{eV}^2\), \(\sin^2 \theta_{\odot} = 0.96\), (iii) LAMS: \(\Delta m^2_{\odot} = 1.8 \times 10^{-5} \text{eV}^2\), \(\sin^2 \theta_{\odot} = 0.76\), (iv) SAMS: \(\Delta m^2_{\odot} = 5.4 \times 10^{-6} \text{eV}^2\), \(\sin^2 \theta_{\odot} = 6.0 \times 10^{-3}\) [5]; and the matrix element \(|U_{e 3}| = \sin \theta_2\) is constrained by the CHOOZ experiment to be \(|U_{e 3}| < 0.16\) [6].

The observed relation \(\Delta m^2_{\text{atm}} \equiv |m_3^2 - m_\tau^2| \gg \Delta m^2_{\odot} \equiv |m_2^2 - m_1^2|\) in the two-fold degenerate, hierarchical model implies \(m_3 \gg m_2 \simeq m_1\), and this in turn implies \(\nabla_{21} \gg \nabla_{32} \simeq \nabla_{31} \simeq 1\) with \(\nabla_{ij} \equiv (m_i + m_j)(m_i - m_j)^{-1}\).

We assume that the neutrino masses are generated by a dimension-5 effective Majorana mass operator in the MSSM

\[
\mathcal{L} \supset -k_{ij} (H_u L_i)(H_u L_j) + \text{h.c.} \tag{4}
\]

The neutrino mass matrix \((m_\nu)_{ij}\) is related to \(k_{ij}\) by \(m_\nu = k_{ij} v^2 / 2\), where \(v^2 = v^2_\nu + v^2_\phi = (246 \text{eV})^2\) is the squared vacuum expectation value of the SM Higgs. The effective dimension-5 operator is generated by some mechanism at the high energy scale \(\Lambda\). The seesaw mechanism is the most common way to generate this operator. Since we are only interested in physics below the scale \(\Lambda\), we will start with the effective Lagrangian Eq. (4) without specifying the origin of this effective operator.

The general one-loop RGE of the effective left-handed Majorana neutrino mass operator is given by [7]

\[
\frac{\text{d} m_\nu}{\text{d} t} = - \left\{ \kappa_u m_\nu + m_\nu P + P^T m_\nu \right\} \tag{5}
\]
where \( t \equiv \ln \mu \). In the MSSM, \( P \) and \( \kappa_u \) are given by,

\[
P = -\frac{1}{32\pi^2} \frac{Y_u^\dagger Y_e}{\cos^2 \beta} \simeq -\frac{1}{32\pi^2} \frac{h_t^2}{\cos^2 \beta} \text{diag}(0, 0, 1) \quad (6)
\]

\[
\kappa_u = \frac{1}{16\pi^2} \left[ \frac{6}{5} g_1^2 + 6 g_2^2 - \frac{6}{\sin^2 \beta} \text{Tr}(Y_u^\dagger Y_u) \right] \\
\simeq \frac{1}{16\pi^2} \left[ \frac{6}{5} g_1^2 + 6 g_2^2 - \frac{6 h_t^2}{\sin^2 \beta} \right] \quad (7)
\]

where \( g_1^2 = \frac{5}{3} g_2^2 \) is the \( U(1) \) gauge coupling constant, \( Y_u \) and \( Y_e \) are the \( 3 \times 3 \) Yukawa coupling matrices for the up-quarks and charged leptons respectively, and \( h_t \) and \( h_\tau \) are the SM \( t \)- and \( \tau \)-Yukawa couplings. Since \( \kappa_u \) gives rise to an overall rescaling of the mass matrix, it has no effects on the running of the mixing matrix \( U \). Eq. (5) can be solved analytically by integrating out its right-hand side \([8]\). Note that at one-loop level, since the evolutions of the gauge coupling constants \( g_{i,i}(t) \) and of the diagonal Yukawa couplings \( h_{i,i}(t) \) are known, it is indeed possible to carry out the integrations on the right-hand side without making any further assumptions. However, the diagonalization procedure of the resulting \( 3 \times 3 \) complex symmetric matrix, \( m_\nu(t) \), is very complicated. It is thus hard to infer analytically the behaviours of the physical observables, the mixing angles and phases. An alternative to this “run-and-diagonalize” procedure is the “diagonalize-and-run” procedure. It is convenient to work with the RGE’s of mass eigenvalues and the diagonalization matrix, given by \([9]\)

\[
\frac{dm_i}{dt} = -2m_i \dot{P}_{ii} - m_i \text{Re}\{\kappa_u\} \quad (8)
\]

\[
\frac{dU}{dt} = UT \quad (9)
\]

where

\[
T_{ii} \equiv i \dot{Q}_{ii} \quad (10)
\]

\[
T_{ij} \equiv \left( \frac{1}{m_i^2 - m_j^2} \right) \left\{ (m_i^2 + m_j^2) \dot{P}_{ij} + 2m_i m_j \dot{P}_{ij}^* \right\} + i \dot{Q}_{ij} \\
= \nabla_{ij} \text{Re}\{\dot{P}_{ij}\} + i \nabla_{ij}^{-1} \text{Im}\{\dot{P}_{ij}\} + i \dot{Q}_{ij} \quad (11)
\]

Here \( \dot{P} \) and \( \dot{Q} \) are defined as

\[
\dot{P} \equiv \frac{1}{2} U^\dagger (P + P^\dagger) U, \quad \dot{Q} \equiv \frac{-i}{2} U^\dagger (P - P^\dagger) U \quad (12)
\]
Eq. (5), (8), and (9) have been studied before [9, 10, 11, 12, 13, 14], but the analyses have been done either numerically or perturbatively. Exact solutions to the RGE’s in a two flavour case have been investigated recently in [14]). Due to the large interfamily hierarchy in the charged lepton sector, we keep only the τ-Yukawa coupling. We will further assume that $h_\tau$ does not evolve throughout the entire range of the RG running. This is a valid assumption for the hierarchical case with two-fold degeneracy, as $\nabla_{21}$ is very large.

Under these assumptions, the above quantities are given in terms of the masses and the diagonalization matrix elements as

$$
\hat{P}_{ij} = -\frac{h^2}{32\pi^2} U^*_i U_{3j}, \quad \hat{Q} = 0
$$

(13)

The evolutions of $\nabla_{ij}$, $(\hat{P}_{ii} - \hat{P}_{jj})$ and $Re(\hat{P}_{ij})$ can be derived from Eq. (8) and (9). For $(i,j) = (2,1)$, with $\nabla_{21} \gg \nabla_{31} \simeq \nabla_{32} \simeq 1$, the RGE’s for these three functions form a complete set of coupled differential equations as follows:

\begin{align}
\frac{d\nabla_{21}}{dt} &= \nabla_{21}^2 (\hat{P}_{22} - \hat{P}_{11}) \\
\frac{d(\hat{P}_{22} - \hat{P}_{11})}{dt} &= -4\nabla_{21} [Re(\hat{P}_{21})]^2 \\
\frac{dRe(\hat{P}_{21})}{dt} &= \nabla_{21} (\hat{P}_{22} - \hat{P}_{11}) Re(\hat{P}_{21})
\end{align}

(14a)

(14b)

(14c)

The exact solutions to these coupled differential equations are given by

\begin{align}
\nabla_{21}(t) &= a_0 Z(t)^{-1/2} \\
(\hat{P}_{22}(t) - \hat{P}_{11}(t)) &= (b_0^2 + 4c_0^2(1 - Z(t)^{-1}))^{1/2} \\
Re(\hat{P}_{21}(t)) &= a_0 Z(t)^{-1/2}
\end{align}

(15a)

(15b)

(15c)

where

$$
Z(t) \equiv 1 - 2a_0 b_0 t + a_0^2 (b_0^2 + 4c_0^2) t^2
$$

(16)

and $a_0$, $b_0$ and $c_0$ are the initial values at the high energy scale $\Lambda$:

$$
a_0 \equiv \nabla_{21}(0); \quad b_0 \equiv (\hat{P}_{22}(0) - \hat{P}_{11}(0)) \\
c_0 \equiv Re(\hat{P}_{21}(0)).
$$

(17)

The behaviours of these three functions are shown in Fig. (1)-(3). Note that $\nabla_{21}(t)$ and $Re(\hat{P}_{21}(t))$ flow to zero, while $(\hat{P}_{22}(t) - \hat{P}_{11}(t))$ flows to a constant value of $(b_0^2 + 4c_0^2)^{1/2}$ in
the infrared. This set of parameters, \( (\nabla_{21}(t^*), \text{Re}(\vec{P}_{21}(t^*)), (\vec{P}_{22}(t) - \vec{P}_{12}(t^*)) = (0, 0, (b_0^2 + 4c_0^2)^{1/2}) \) is an infrared stable fixed point; however, it is unrealistic. The function \( \nabla_{21}(t) \) decreases to \( O(1) \) very fast as the energy scale goes down, for any non-vanishing \( b_0 \) and \( c_0 \), however small they are. This is phenomenologically unacceptable. In addition, it contradicts with the assumption \( \nabla_{21} \gg \nabla_{32}, \nabla_{31} \) we made in order to arrive at Eq. (14). For the consistency of the calculations, we thus require the following two conditions at the initial high energy scale \( \Lambda \):

\[
\begin{align*}
    b_0 &\equiv (\vec{P}_{22}(0) - \vec{P}_{11}(0)) = 0 \\
    c_0 &\equiv \text{Re}(\vec{P}_{21}(0)) = 0
\end{align*}
\]

We emphasize that these conditions have been obtained by demanding that the exact solutions to the above RGE’s Eq. (14) be consistent with \( \nabla_{21} \gg 1 \). It is to be noted that these conditions have been obtained before numerically and perturbatively [9, 10, 11]. When these conditions are satisfied, all three equations in Eq. (14) do not evolve. The first relation, Eq. (18), gives rise to \( |V_{32}|^2 = |V_{31}|^2 \) which translates into

\[
\frac{c_1^2 s_2^2 - s_1^2}{\sin 2\theta_1 \cdot s_2} = \tan 2\theta_3 \cdot \cos \delta
\]

The second relation, Eq. (19), gives rise to

\[
\frac{\text{Re}(V_{32}^* V_{31})}{\text{Im}(V_{32}^* V_{31})} = \tan(\frac{\phi' - \phi}{2})
\]

\[
= \frac{\sin 2\theta_3 (c_1^2 s_2^2 - s_1^2)}{\sin \delta \cdot \sin 2\theta_1 \cdot s_2} + \frac{\cos 2\theta_3}{\tan \delta}
\]

Combining these two relations, we obtain a very simple relation among \( \theta_3 \) and three CP violating phases \( \delta, \phi, \phi' \):

\[
\cos 2\theta_3 = \frac{1}{\tan \delta} \cdot \frac{1}{\tan(\frac{\phi' - \phi}{2})}
\]

We have studied the RGE’s involving various functions, \( \nabla_{ij}, (\vec{P}_{ii} - \vec{P}_{jj}) \) and \( \text{Re}(\vec{P}_{ij}) \) for the case \( (i, j) = (3, 1) \) and \( (3, 2) \). Upon imposing the above consistency conditions Eq. (18) and (19), we deduce that the functions \( \vec{P}_{11}, \vec{P}_{22}, \vec{P}_{33}, \text{Re}(\vec{P}_{31}) \) and \( \text{Re}(\vec{P}_{32}) \) do not run. These results cannot be tested experimentally at present.

Now we discuss the implications of Eq. (18) and (19) in the limit \( \theta_3 = 0 \). Recently, it has been pointed out that this could be a consequence of the so-called 2-3 symmetry [15].
They imply
\[
\cos\left(\frac{\phi - \phi'}{2}\right) s_1^2 c_3 s_3 = 0, \quad s_1^2 (c_3^2 - s_3^2) = 0
\] (23)

Since the atmospheric angle \( \theta_1 = \pi/4 \) is non-vanishing, these two relations can be satisfied simultaneously only if (i) the solar mixing angle is maximal, i.e. \( \theta_3 = \pi/4 \), and (ii) the Majorana phase difference \( (\phi - \phi') = \pi \). The phases \( \phi \) and \( \phi' \) occur in the matrix element \( \langle M_{ee} \rangle \) for the neutrinoless double beta decay:
\[
\langle M_{ee} \rangle \equiv \left| \sum_{i=1,2,3} U_{ei}^2 m_i \right| \\
\quad = |m_1 e^{-i\phi} c_2^2 c_3^2 + m_2 e^{-i\phi'} c_1^2 s_3^2 + m_3 s_1^2 e^{-2i\theta}| < B
\]
(24)

where index \( i \) denotes the mass eigenstates. Currently, the most stringent bound is given by \( B = 0.2 \text{eV} \) [16]. In the limit \( \theta_2 = 0 \) with nearly degenerate \( m_1 \simeq m_2 \), \( M_{ee} \) becomes
\[
M_{ee} \simeq |m_1 c_2^2 (e^{-i\phi} c_3^2 + e^{-i\phi'} s_3^2)|
\]
(25)

It is obvious that when \( (\phi - \phi') = \pi \) and \( \theta_3 = \pi/4 \) the r.h.s. of Eq. (25) is exactly zero. Thus we conclude that neutrinoless double beta decay is very highly suppressed.

It is interesting to speculate the reason(s) for consistency conditions of Eq. (18) and (19). It could be due to the existence of a symmetry at a high energy scale \( \Lambda \). The other possibility is that these two relations are the fixed point relations of the RGE’s for new physics above the scale \( \Lambda \).
FIG. 2: The function \((\tilde{P}_{22}(t) - \tilde{P}_{11}(t))\). Initial values at \(\Lambda\) are \((a_0, b_0, c_0) = (1000, 1, 1)\).

FIG. 3: The function \(\text{Re}(\tilde{P}_{21}(t))\). Initial values at \(\Lambda\) are \((a_0, b_0, c_0) = (1000, 1, 1)\)

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