We show that the basic dynamical rules of quantum physics can be derived from its static properties and the condition that superluminal communication is forbidden. More precisely, the fact that the dynamics has to be described by linear completely positive maps on density matrices is derived from the following assumptions: (1) physical states are described by rays in a Hilbert space, (2) probabilities for measurement outcomes at any given time are calculated according to the usual trace rule, (3) superluminal communication is excluded. This result also constrains possible non-linear modifications of quantum physics.

The special theory of relativity is one of the cornerstones of our present scientific world-view. One of its most important features is the fact that there is a maximum velocity for signals, i.e. for anything that carries information, identical to the velocity of light in vacuum.

Another cornerstone of our present understanding of the world is quantum physics. Quantum physics seems to have “nonlocal” characteristics due to the existence of entanglement. Most importantly, it is not compatible with local hidden variables, as shown by the violation of Bell's inequalities [1], which has been experimentally confirmed in many experiments [2–4].

It is very remarkable that in spite of its non-local features, quantum mechanics is compatible with the special theory of relativity, if it is assumed that operators referring to space-like separated regions commute. In particular, one cannot exploit quantum-mechanical entanglement between two space-like separated parties for communication of classical messages faster than light.

On the other hand, if one tries to modify quantum physics, e.g. by introducing non-linear evolution laws for pure states [5], this easily leads to the possibility of superluminal communication [6–8].

This peaceful, but fragile, coexistence between quantum physics and special relativity has led physicists to ask whether the principle of the impossibility of superluminal communication, which we will refer to as the “no-signaling condition”, could be used as an axiom in deriving basic features of quantum mechanics. The answer to this question should at the same time provide insight into what kind of modifications of quantum physics are compatible with the no-signaling condition.

Here we give a positive answer to the above question. If the usual static characteristics of quantum mechanics are assumed, then its dynamical rules can be derived from the no-signaling assumption. By static characteristics we mean the following. The states of our systems are described by vectors in a Hilbert space. Furthermore at any given time we have the usual observables described by projections in the Hilbert space [9], and the probabilities for measurement results are calculated according to the usual trace rule [10]. However, no a priori assumption is made about the time evolution of the system. For example, the states could evolve according to some non-linear wave equation. Note that we also do not assume the projection postulate.

Our result is then that under the stated conditions the dynamics of our system has to be described by completely positive (CP) linear [11] maps on density matrices. This is equivalent to saying that under the given assumptions quantum physics is essentially the only option since according to the Kraus representation theorem [11], every CP map can be realized by a quantum-mechanical process, i.e. by a linear and unitary evolution on a larger Hilbert space. On the other hand, any quantum process corresponds to a CP map. This result is a significant extension of earlier work by one of the authors [7].

Let us first recall how the linearity of standard quantum dynamics prevents the use of entanglement for superluminal communication. Consider two parties, denoted by Alice and Bob, who are space-like separated, such that all operations performed by Alice commute with all operations performed by Bob. Throughout this work we will assume that in this sense locality is implemented in the algebra of observables. For example, this is certainly the case in relativistic quantum field theory. Can the two parties use a shared entangled state $\psi_{AB}$ in order to communicate in spite of their space-like separation?

The short answer is: no, because the situation on Alice’s side will always be described by the same reduced density matrix, irrespective of Bob’s operations. All the effects of his operations (described by linear maps) dis-
appear when his system is traced over. This answer is correct, but not very detailed, and thus it may not be entirely convincing. Let us give a more detailed answer which highlights the essential role played by the linearity of quantum dynamics.

A question that is frequently raised in this context is the following: Bob could choose to measure his system in two different bases and thus “project” Alice’s system into different pure states depending on the basis he chose and his measurement result. Since it is possible to distinguish two different states in quantum mechanics, at least with some probability, shouldn’t it be possible for Alice to infer his choice of basis, at least in some percentage of the cases, which would be dramatic enough?

Of course, the answer is no again, for the following reason. In order to gain information about which basis Bob chose to measure, Alice can only perform some (generalized) measurement on her system. Then she has to compare the conditional probabilities for a given result to occur, for the case that Bob measured in the first or in the second basis. But these conditional probabilities will always be exactly the same for both possibilities.

This can be seen as a consequence of the linearity of the quantum physical time evolution: Suppose that Bob’s first choice projects Alice’s system into states $\psi_j$ with probabilities $p_j$ and his second choice projects it into states $\phi_k$ with probabilities $q_k$. Alice can calculate the probability for her obtained result for every one of the states, and then weight these probabilities with the probability to have that specific state. But because of the linearity of any operation that Alice can perform on her states during her generalized measurement procedure, her final result will only depend on the density matrix of the probabilistic mixtures, which is the same in both cases, because they were generated from the same entangled state. For an example how two such mixtures can become distinguishable through a non-linear (non-quantum-mechanical) evolution, see [6].

Let us note that this argument also implies the non-existence of a perfect cloner in quantum mechanics because such a machine would allow superluminal communication [12] by making it possible for Alice to discriminate between Bob’s choices of basis.

We now show how quantum dynamics can be derived from “quantum statics” and the no-signaling condition. As explained above, by quantum statics we mean that physical systems are described by rays in a Hilbert space, that at any given time we have the usual quantum observables represented by projections in this Hilbert space, and that the trace rule for calculating probabilities holds.

From the trace rule it follows that at any given moment the results of measurements on some system $A$ are determined by its density matrix or reduced density matrix, depending on whether $A$ is in a probabilistic mixture of pure states or entangled with some other system.

A priori we make no assumption about the dynamics of pure states, e.g. it might be described by a non-linear wave equation. Let us note immediately that, if the pure states have a non-linear time evolution, then the density matrix of a probabilistic mixture is not sufficient to determine the dynamics of the system, one has to know the individual pure states and their probabilities.

If we consider a subsystem of the whole Universe it will in general be in an entangled state with other parts of the Universe. In particular, a system $A$ may be entangled with another system $B$ which is space-like separated from $A$, such that their observable algebras commute. This is where the no-signaling constraint comes into play. The dynamics of the systems has to be such that in spite of this entanglement no superluminal communication between $A$ and $B$ is possible.

Suppose that $A$ and $B$ together are in the entangled state $|\psi\rangle_{AB}$ with reduced density matrix $\rho_{AB}$ for system $A$. As a consequence of the trace rule, by performing a measurement of his system the observer $B$ also prepares a certain state in $A$.

To see this, remember that the trace rule tells us how to calculate the (joint) probability for measurement results corresponding to any product of projectors $P_A \otimes P_B$, namely by calculating $\text{Tr}_{AB}\rho_{AB} P_A \otimes P_B$. But this also tells us how to calculate the conditional probability to find any $P_A$, provided that $P_B$ has been found. Namely, we just have to divide the joint probability by the probability to find $P_B$ in the first place. But having a way of calculating the conditional probability for every $P_A$ means that we know the state in $A$ conditional on $B$ having found $P_B$, since a state can be reconstructed from its expectation values for a linearly independent set of projectors. It is given by $\text{Tr}_B \rho_{AB} P_B / \text{Tr}_{AB} \rho_{AB}$. Note that to arrive at this conclusion we did not have to make use of the usual projection postulate.

Actually, every probabilistic mixture of pure states corresponding to the density matrix $\rho_A$ can be prepared via appropriate measurements on $B$ [7,13]. We will give a proof of this statement in the last part of this letter.

Consider two such probabilistic mixtures $\{P_{\psi_i}, p_i\}$ and $\{P_{\phi_j}, q_j\}$, where $P_{\psi_k}$ is the projector corresponding to the pure state $|\psi_k\rangle$ and $p_k$ is its probability, such that

$$\sum_{i} p_i P_{\psi_i} = \sum_{j} q_j P_{\phi_j} = \rho_A. \quad (1)$$

According to the no-signaling principle there should be no way for the observer in $A$ to distinguish these different probabilistic mixtures.

A general dynamical evolution in system $A$ is of the form

$$g : P_{\psi} \rightarrow g(P_{\psi}) \quad (2)$$

where, most importantly, $g$ is not necessarily linear. Furthermore, $g(P_{\psi})$ does not have to be a pure state, since system $A$ could become entangled with its environment,
or $\psi$ could evolve into a probabilistic mixture of pure states. As mentioned above, even if system $A$ is entangled with its environment, the trace rule implies that at any given moment the results of measurements on $A$ will be completely determined by the reduced density matrix of the system. In this case we define $g(P_\psi)$ to be the reduced density matrix of $A$. If $\psi$ evolves into a probabilistic mixture, we define $g(P_\psi)$ to denote the corresponding density matrix.

Under such dynamics the probabilistic mixture $\{P_{\psi_i}, p_i\}$ goes into another probabilistic mixture $\{g(P_{\psi_i}), p_i\}$. Therefore the two final density matrices after the action of $g$ on two different probabilistic mixtures $\{P_{\psi_i}, p_i\}$ and $\{P_{\phi_j}, q_j\}$ are

$$
\rho'_A\{P_{\psi_i}, p_i\} = \sum_i p_i g(P_{\psi_i})
$$

$$
\rho'_A\{P_{\phi_j}, q_j\} = \sum_j q_j g(P_{\phi_j})
$$

(3)

which a priori can be different. Let us recall that according to our assumptions the results of all measurements in $A$ at a given time are determined by the reduced density matrix $\rho'_A$. This means that as a consequence of the no-signaling principle the density matrix $\rho'_A$ at any later time has to be the same for all probabilistic mixtures corresponding to a given initial density matrix $\rho_A$. That is, it has to be a function of $\rho_A$ only.

We can therefore write

$$
\rho'_A = g(\rho_A) = g(\sum_i p_i P_{\psi_i}).
$$

(4)

Eqs. (3) and (4) together imply the linearity of $g$:

$$
g(\sum_i p_i P_{\psi_i}) = \sum_i p_i g(P_{\psi_i}).
$$

(5)

Positivity of $g$ is necessary in order to ensure that $g(\rho_A)$ is again a valid density matrix, i.e. to ensure the positivity of all probabilities calculated from it.

As we have made no specific assumptions about the system $A$ apart from the fact that it can be entangled with some other spacelike separated system, this means that the dynamics of our theory has to be described by linear maps on density matrices in general.

Let us now argue that the linearity and positivity already imply complete positivity in the present framework. To see this, consider again two arbitrary subsystems $A$ and $B$ which may again be in an entangled state $|\psi\rangle_{AB}$. It is conceivable that system $A$ is changed locally (i.e. the system evolves, is measured etc.), which is described by some linear and positive operation $g_A$, while nothing happens in $B$. This formally corresponds to the operation $g_A \otimes \mathbb{1}_B$ on the whole system.

The joint operation $g_A \otimes \mathbb{1}_B$ should take the density matrix of the composite system $\rho_{AB}$ into another valid (i.e. positive) density matrix, whatever the dimension of the system $B$. But this is exactly the definition of complete positivity for the map $g_A$ [11]. If $g_A$ is positive but not CP, then by definition there is always some entangled state $\rho_{AB}$ for which $g_A \otimes \mathbb{1}_B$ applied to $\rho_{AB}$ is no longer a positive density matrix and thus leads to unphysical results such as negative probabilities.

In this way the existence of entangled states and the requirements of positivity and linearity force us to admit only completely positive dynamics. As mentioned already in the introduction, this is equivalent to saying that under the given assumptions quantum dynamics is essentially the only option since any CP map can be realized by a quantum mechanical process, while on the other hand, any quantum-mechanical process corresponds to a CP map [14].

There are three crucial ingredients in our argument: the existence of entanglement, the trace rule, and the no-signaling condition. Specifically, the trace rule leads to the preparation at a distance of probabilistic mixtures and thus, as it were, to the right-hand side of Eq. (5). On the other hand, the no-signaling condition tells us that the dynamics is allowed to depend only on the reduced density matrix, which leads to the left-hand side of Eq. (5). Strictly speaking, in the derivation of complete positivity, we have also used the assumption that the identity operation on a subsystem is a permitted dynamical evolution.

Nonlinear modifications of quantum mechanics [15,16] have to give up at least one of these assumptions. For instance, if the dynamics is allowed to depend on the reduced density matrix $\rho_A$ in a nonlinear way, then it is clear that $\rho_A$ cannot correspond to a probabilistic mixture of pure states, cf. [15]. But $\rho_A$ will correspond to such a mixture whenever the observer in $B$ chooses to make appropriate measurements, as long as we believe in the trace rule, according to our above argument. This seems to imply that, at least for separated systems, the trace rule has to be modified in such a nonlinear theory.

Another example would be a theory where some entangled states are a priori excluded. In this case some non-CP maps might be permissible. An extreme example would be a theory without entanglement. Such a theory would of course be in conflict with experiments. An example for a linear, positive, but non-CP map consistent with the no-signaling condition is the transposition of the density matrix of the whole Universe (physically corresponding to a time reversal). However in this case the identity operation on a subsystem is not an allowed dynamics.

For completeness, let us finally show that any mixture corresponding to a given density matrix can be prepared at a distance from any entangled state with the correct reduced density matrix [7,13]. Let us denote the system under consideration by $A$ and the remote system by $B$. Let us denote the eigenvector representation of $\rho_A$ by
\[ |\psi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k} |v_k\rangle |g_k\rangle, \tag{6} \]

where the \(|g_k\rangle\) are orthonormal states of system B. We want to show that any decomposition of \(\rho_A\) as a mixture of pure states can be prepared from this state by operations on system B only. To this end, consider an arbitrary decomposition \(\rho_A = \sum_{i=1}^{m} x_i |\psi_i\rangle\langle\psi_i|\), where in general \(m > r\). Clearly this decomposition could be obtained from a state

\[ |\phi\rangle_{AB} = \sum_{i=1}^{m} \sqrt{x_i} |\psi_i\rangle |\alpha_i\rangle, \tag{7} \]

with the \(|\alpha_i\rangle\) being an orthonormal basis of a \(m\)-dimensional Hilbert space \(H_m\). It seems that we now require a larger Hilbert space in location B in order to accommodate all the orthonormal \(|\alpha_i\rangle\). But the state \(|\phi\rangle_{AB}\) must also have a Schmidt representation

\[ |\phi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k} |v_k\rangle |h_k\rangle, \tag{8} \]

with \(|h_k\rangle\) being orthonormal states in B. This implies that \(|\phi\rangle_{AB}\) and \(|\psi\rangle_{AB}\) are connected by a unitary transformation on B alone:

\[ |\phi\rangle_{AB} = 1_A \otimes U_B |\psi\rangle_{AB}. \tag{9} \]

The dimension of the support of the reduced density matrix \(\rho_B\) is the same for both states, since it is given by the dimension of the support of \(\rho_A\).

Thus one can prepare \(|\phi\rangle_{AB}\) from any state with the correct reduced density matrix by extending the system B locally to \(m\) dimensions (using an appropriate ancilla), and then perform the required (Von Neumann) measurement in the basis of the \(|\alpha_i\rangle\). This will correspond to a generalized measurement [10] on the original \(r\)-dimensional system. In this way every possible decomposition of \(\rho_A\) can be prepared at a distance.

In conclusion, we have shown that the basic dynamical rule of quantum physics can be derived from its static properties and the condition of no superluminal communication. This result puts significant constraints on non-linear modifications of quantum physics. It is clearly difficult to modify just parts of the whole structure. More universal departures from the formalism may still be possible without violating the no-signaling condition. We would like to mention related recent work by Mielnik [17].

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[9] One can say that a non-linear time evolution leads to the existence of additional observables, which are defined by letting the system evolve for a certain period of time, and then measuring some standard quantum observable.