ABSTRACT

This is a short nontechnical note summarizing the motivation and results of my recent work on D-brane categories. I also give a brief outline of how this framework can be applied to study the dynamics of topological D-branes and why this has a bearing on the homological mirror symmetry conjecture. This note can be read without any knowledge of category theory.

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1 Introduction

At the heart of the second superstring revolution one finds a duality in our description of D-brane dynamics. On one hand, D-branes are introduced at the fundamental level as boundary conditions in open string theory, while on the other hand string dualities together with the M-theory interpretation force us to treat them as dynamical objects. There is considerable fuzz surrounding the passage from ‘Dirichlet boundary conditions’ to ‘dynamical objects’. In its most standard incarnation, the argument given takes the following indirect form.

Starting with Dirichlet boundary conditions at the fundamental level, one obtains new open string sectors associated with strings ending on the brane. One next considers the low energy effective action of such strings, and identifies it with an effective description of low energy D-brane kinematics (the DBI action coupled to background fields). This gives us a low energy description of string fluctuations around the D-brane, and not a description of interactions between D-branes, hence our use of the term kinematics.

The DBI action is obviously insufficient for a description of low energy D-brane dynamics. Indeed, the effective action of open strings ending on a D-brane describes the low energy dynamics of strings with prescribed boundary conditions, but the boundary condition itself is not ‘dynamical’ in any fundamental way. To describe D-brane interactions, one can resort to studies of string exchange between D-branes, consider the resulting low energy effective action and treat it as an effective description of D-brane interactions (this of course won’t give anything interesting for collections of mutually BPS D-branes in type II theories, but there is no reason to restrict to type II or BPS saturated D-branes). Then one can study D-brane interactions through the dynamics of this action. However, effective actions do not give a fundamental (microscopic) description, and the way in which ‘boundary conditions’ become dynamical is hard to see from such considerations. What, then, is D-brane dynamics?

A conceptual approach to this issue is afforded by open string field theory. This allows one to answer some dynamical questions at a fundamental level, as demonstrated explicitly by studies of tachyon condensation [2]. In fact, open string tachyon condensation is perhaps the only known example of true \(^1\) D-brane dynamics described in a microscopic manner. Through such a process, D-branes are allowed to annihilate, decay, or form bound states. In a certain sense, passage to string field theory performs their ‘second quantization’.

There are a few obvious lessons to be learned from studies of tachyon condensation. First, a truly dynamical description of D-branes requires second quantization of strings and off-shell techniques, i.e. string field theory. Second, the notion of D-brane has to be extended.

\(^1\) By true dynamics we mean processes involving interaction and decay of branes, which in particular involve ‘second quantization’. In this language, the oscillations of a given D-brane would correspond to its ‘first quantization’.
The second point follows from the observation that the end product of a condensation process is generally not a Dirichlet brane, since it typically cannot be described through boundary conditions on a worldsheet theory. For example, tachyon condensation in superstring compactifications on Calabi-Yau manifolds can produce D-brane composites described by various configurations of bundles and maps\(^2\), for which a direct worldsheet description through a boundary condition is not always available\(^2\). This implies that, at least in geometrically nontrivial backgrounds, tachyon condensation processes can produce genuinely new objects, distinct from the Dirichlet branes originally considered in the theory.

Moreover, consideration of various condensates in a given background shows that they will generally interact through string exchange. It follows that such condensates behave in many respects as ‘abstract D-branes’, even though they do not admit a direct description through boundary conditions. This implies that open string theory must be generalized to allow for a description of such objects.

One is thus lead to the task of formulating open string field theory in the presence of ‘abstract D-branes’. Since these are not simply boundary conditions, one has to find a structure which allows for their systematic description. The main point of \cite{1} is that the correct structure (at least for the ‘associative case’) is a so-called \textit{dG (or differential graded) category}. This mathematical object arises naturally from constructions based on Dirichlet branes, and – in a slightly less direct manner – also in the case of generalized D-branes (D-brane composites). Moreover, it is showed in \cite{1} that D-brane composite formation can be described as a change of this structure. We are lead to the following:

\textbf{Proposal} In first nontrivial approximation, D-brane dynamics is described by certain deformations of a dG category.

By first nontrivial approximation we mean the fact that we only consider tree level dynamics of open strings. Moreover, this approach treats all Dirichlet branes and their condensates on a equal footing (‘bootstrap’), though it also opens the way for finding a ‘system of generators’ which need not be of Dirichlet type. The work of \cite{1}, which I shortly review below, is concerned with formulating and exploring some basic consequences of this proposal.

\section{dG categories on one leg}

I now give a short description of the dG category describing usual (i.e. Dirichlet) D-branes. A category \cite{3} is a collection of objects \(a\) and sets \(\text{Hom}(a,b)\) associated to any ordered pair of objects \(a, b\), together with compositions \((u,v) \rightarrow uv\) for elements \(u\)

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\(^2\)Passage to the derived category as in \cite{20} allows for a representation of some such objects as coherent sheaves, some of which can in turn be identified with bundles living on complex submanifolds. However, not every object of the derived category is a coherent sheaf, and not every coherent sheaf can be represented in the second way.
of $\text{Hom}(b,c)$ and $v$ of $\text{Hom}(a,b)$, where $uv$ belongs to $\text{Hom}(a,c)$. Such compositions are required to be associative, i.e. $(uv)w = u(vw)$, and to admit units $1_a$ (elements of the sets $\text{Hom}(a,a)$) such that $u1_a = 1_vu$ for $u,v$ in $\text{Hom}(a,b)$. The objects $a$ and ‘morphism sets’ $\text{Hom}(a,b)$ can be pretty much anything as long as these requirements are satisfied. Familiar examples are the category $\text{Ens}$ of sets (with the morphism space between two sets $A,B$ given by all functions from $A$ to $B$ and morphism compositions given by composition of functions), the category of vector spaces $\text{Vct}$ (with morphisms given by linear maps), and the category $\text{Vect}(X)$ of vector bundles over a manifold $X$ (with morphisms given by bundle morphisms). In all of these cases the elements are some sets (with extra structure) and the morphisms are maps between these sets which preserve the structure (these are so-called ‘concrete’ categories). A category need not be of this type, however: its objects may not be sets, and its morphisms need not be maps of sets.

As it turns out, Dirichlet branes in an associative oriented open string theory give an example of a (generally non-concrete) category $\mathcal{A}$. This arises by taking Dirichlet branes as objects and the morphism space between two objects to be given by the off-shell state space of open strings stretched between them (figure 1). In general, this space contains the full tower of massive modes, and therefore such morphisms cannot be naturally thought of as maps. The composition of morphisms is given by the string product of [13], which is related to the triple correlator on the disk (figure 1). In an associative string theory, this product is associative off-shell\(^3\). Moreover, one has units $1_a$, related to the boundary vacua of [14]; these are generalizations of the formal unit of cubic string field theory.

\[ \begin{array}{c}
\text{Figure 1. Dirichlet branes define a category whose morphisms are off-shell states of oriented open strings stretched between them (left). Compositions of morphisms are given by the string product, which is related to the triple correlator (right).}
\end{array} \]

As it happens, the resulting category of Dirichlet branes has some extra structure which reflects the basic data of open string field theory. First, the off-shell state spaces

\[^3\text{In more general situations, the product need only be associative up to homotopy; this leads to an } A_\infty \text{ category upon extending the structure discussed in [27] (see also [26]) to the case of backgrounds containing D-branes.} \]
$\text{Hom}(a, b)$ are graded by the ghost degree $^4$. If one uses appropriate conventions for the ghost charge, the composition of morphisms preserves this degree, in the sense that the degree of $uv$ is the sum of degrees of $u$ and $v$. In technical language, this means that we have a graded category. Another essential ingredient is the worldsheet BRST charge, which defines linear operators $Q_{ab}$ on each of the spaces $\text{Hom}(a, b)$; as in [13], these operators act as derivations of the string product. With our conventions, they also have ghost degree +1. A graded category endowed with degree one nilpotent $^5$ operators on its Hom spaces, acting as derivations of morphism compositions, is known as a differential graded (dG, for short) category$^5$[5, 6]. It follows that the Dirichlet branes of any associative string theory form a dG category. In fact, a complete specification of open string field theory requires some more data, for example a collection of bilinear pairings on morphisms and possibly some complex conjugation operations, which are required to satisfy certain properties. I shall neglect this extra structure in order to simplify the discussion; the bilinear forms are treated in detail in [1].

3 D-brane processes as shifts of the string vacuum

We saw above that Dirichlet branes form a dG category. Does this structure also describe backgrounds containing D-brane condensates? As we shall see in a moment, the answer is affirmative, though the dG category arising after formation of D-brane composites does not admit a direct construction in terms of string worldsheets (in fact, its description requires consideration of off-shell string dynamics). The nontrivial fact that a dG category can be used to describe both Dirichlet branes and generalized branes arising from condensation of boundary operators is what allows us to view the dG category structure as fundamental.

The basic idea behind this approach is that D-brane composites represent new boundary sectors. To understand this, note that a background containing Dirichlet branes can also be described in terms of a ‘total boundary state space’:

$$\mathcal{H} = \oplus_{a,b} \text{Hom}(a, b),$$

(1)

endowed with the multiplication induced by morphism compositions. In this approach, one is given a dG algebra, i.e. a vector space $\mathcal{H}$ endowed with an associative multiplication and a linear operator $Q = \oplus_{a,b} Q_{ab}$, which squares to zero and acts as a derivation of the product. This is precisely the algebraic framework of [13], expressed with our conventions for the ghost grading. The new input represented by the Dirichlet branes can be described as a decomposition property of the product. Namely, we have a decomposition (1) of $\mathcal{H}$ which has the property that the string product vanishes on

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$^4$In a topological A/B string theory, this is replaced by the anomalous $U(1)$ charge of the twisted superconformal algebra.

$^5$Remember that an operator is nilpotent if it squares to zero.
subspaces of the form $\text{Hom}(b', c) \times \text{Hom}(a, b)$ unless $b' = b$, in which case it maps $\text{Hom}(b, c) \times \text{Hom}(a, b)$ into $\text{Hom}(a, c)$. This decomposition also has the property that it is preserved by the total BRST operator $Q$, i.e. $Q$ preserves each boundary sector $\text{Hom}(a, b)$.

At least formally, a decomposition of $\mathcal{H}$ having these properties is the only piece of data distinguishing the open string field theory of [23] from a theory containing various Dirichlet branes. The component spaces $\mathcal{H}_{ab} = \text{Hom}(a, b)$ of such a decomposition will be called boundary sectors. Hence the underlying D-brane category is determined by the properties of the total string product and total BRST charge.

This point of view allows us to recover a category structure after formation of D-brane composites takes place. Indeed, such processes are described by condensation of certain boundary/boundary condition changing operators, which correspond to various states $q_{ab}$ in the boundary sectors $\text{Hom}(a, b)$. From the point of view of string field theory, this amounts to giving VEVs $q_{ab}$ to various components $\phi_{ab} \in \text{Hom}(a, b)$ of the total string field $\phi = \oplus_{a,b} \phi_{ab} \in \mathcal{H}$. It follows that the result of a condensation process can be described by the standard device of shifting the string vacuum. Such a shift $\phi \rightarrow \phi + q$ preserves the total boundary product, but induces a new BRST operator $Q'$. Moreover, the condition that the new vacuum extremizes the string field action imposes constraints on the allowed shifts $q$. The important observation is that the BRST operator $Q'$ for the shifted vacuum will generally fail to preserve the original boundary sectors $\text{Hom}(a, b)$; this signals the fact that the collection of D-branes in the shifted background has changed, which is exactly what one expects from formation of D-brane composites. One can identify the new boundary sectors (and thus the composite D-branes and the state spaces they determine) by looking for a new decomposition of $\mathcal{H}$ into subspaces, which has the required compatibility properties with respect to the modified BRST operator $Q'$ and the boundary product. This analysis is carried out in [1], with the conclusion that the resulting boundary sectors form a new dG category $\mathcal{A}_q$, the so-called contraction of the original category $\mathcal{A}$ along the collection of boundary operators $q$ (figure 2). The objects and morphism spaces of this category are given explicitly in [1].

The conclusion is that D-brane composites can once again be described in terms of a dG category, even though they do not generally correspond to Dirichlet branes. Moreover, D-brane composite formation can be described as a change of the dG category structure. This justifies our proposal that the fundamental objects of interest are not Dirichlet branes per se, but rather abstract dG category structures. This amounts to generalizing D-branes to abstract boundary sectors, the latter being specified by decomposition properties of the total boundary product and BRST charge. As discussed above, this generalization is unavoidable if one wishes to allow for D-brane condensation processes, i.e. describe D-branes as truly dynamical objects.