The intermittent behavior and hierarchical clustering of the cosmic mass field

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Received ________________; accepted ________________

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ABSTRACT

The hierarchical clustering model of the cosmic mass field is examined in the context of intermittency. We show that the mass field satisfying the correlation hierarchy $\xi_n \simeq Q_n(\xi_2)^{n-1}$ is intermittent if $\kappa < d$, where $d$ is the dimension of the field, and $\kappa$ is the power-law index of the non-linear power spectrum in the discrete wavelet transform (DWT) representation. We also find that a field with singular clustering can be described by hierarchical clustering models with scale-dependent coefficients $Q_n$ and that this scale-dependence is completely determined by the intermittent exponent and $\kappa$. Moreover, the singular exponents of a field can be calculated by the asymptotic behavior of $Q_n$ when $n$ is large. Applying this result to the transmitted flux of HS1700 Lyα forests, we find that the underlying mass field of the Lyα forests is significantly intermittent. On physical scales less than about $2.0 \, h^{-1} \, \text{Mpc}$, the observed intermittent behavior is qualitatively different from the prediction of the hierarchical clustering with constant $Q_n$. The observations, however, do show the existence of an asymptotic value for the singular exponents. Therefore, the mass field can be described by the hierarchical clustering model with scale-dependent $Q_n$. The singular exponent indicates that the cosmic mass field at redshift $\sim 2$ is weakly singular at least on physical scales as small as $10 \, h^{-1} \, \text{kpc}$.

Subject headings: cosmology: theory - large-scale structure of universe
1. Introduction

The cosmic mass density field seems to possess two basic features which at first glance appear contradictory. The first is the statistical homogeneity and isotropy of the field. The second is that the mass field consists of isolated density peaks like galaxies and clusters of galaxies, in which the mass densities are much higher than average. Moreover, coherent structures, like filaments, sheets and even cellular or network structures with characteristic lengths have been detected in galaxy distributions.

These two features can be reconciled by assuming the formation of a highly intermittent field from an initially gaussian random field. In this scenario, the growth of structure in the mass field is initially linear and randomness is the mechanism by which structure is formed. At later stages, non-linearities enter the dynamics and prevent infinite growth. As a result, the structures that thereby arise in the random mass field have an interesting character: they are strong enhancements (peaks) of the density field scattered in a space with a low density background. This feature generally is called intermittency and was originally introduced for describing the temperature and velocity distributions in turbulence (Batchelor & Townsend 1949; for review, see Frisch, 1995.) An early attempt of studying the formation of the large scale coherent structures from a random field is the theory of pancakes (Zeldovich 1970). This work is the predecessor of the intermittency approach to the formation of coherent structures (Shandarin & Zeldovich 1984; for a review, see Shandarin & Zeldovich 1989).

The intermittency picture is supported by modelling the cosmic matter distribution in non-linear regime with lognormal(LN) random field, which is a simplest model of intermittency behavior in random fields (e.g. Zeldovich, Ruzmaikin, & Sokoloff 1990). The LN model has been found to reproduce the nonlinear matter distribution evolved from Gaussian initial conditions and gives rise to spatial patterns characterized statistically by
a kind of intermittency ('heterotopicity') (Cole and Jones, 1991; Jones, Coles & Martinez, 1992). The LN model of the baryonic matter distribution (Bi & Davidsen, 1997) has also been successfully applied to explain the non-Gaussian features of the underlying mass field traced by Lyα forests (Feng & Fang 2000).

The intermittency picture is also supported by the singular density profile of massive halos (like $\rho(r) \propto r^{-\gamma}$ with exponent $\gamma \simeq 1$) given by N-body simulations (e.g. Navarro, Frenk & White, 1997, Jing & Suto 2000.) This profile indicates that the events of large differences between the local densities separated by small distances $r$ are more frequent than with a gaussian field. As such the probability distribution function (PDF) of the density difference, $\Delta_r(x) \equiv \rho(x + r) - \rho(x)$, on small scales $r$ is long tailed.

The PDF of $\Delta_r(x)$ can only be effectively detected by a scale-space decomposition because $\Delta_r(x)$ consists of both scale $r$ and position $x$. A scale-space decomposition analysis of the mass density field traced by the Lyα forests has indeed obtained more direct evidence of the intermittency (Jamkhedkar, Zhan, & Fang 2000.) They found that the PDF of $\Delta_r(x)$ is gaussian on scale $r$ larger than about $2 \, h^{-1} \, \text{Mpc}$, but becomes longed tailed as $r$ decreases. Hence, the random mass field traced by the Lyα transmitted flux shows an excess of large fluctuation on small scales in comparison to a Gaussian distribution. This is typical of an intermittent field (Shraiman & Siggia 2000.)

It seems that intermittency is a common feature of the non-linearly evolved cosmic mass field. At the same time, hierarchical clustering is believed to be a common feature of the non-linearly evolved cosmic mass field, and has been widely applied to construct semi-analytic models of gravitational clustering in the universe (e.g. White 1979). Although hierarchical clustered fields show indeed some kind of intermittency, it is unclear whether the correlation hierarchy is consistent with the observed intermittency of the cosmic mass field. For instance, in the LN model, the high order correlations do not satisfy a hierarchical
relation, but obey a well-known Kirkwood scaling relation (Kirkwood, 1935; Peebles, 1980). Therefore, a LN random field is intermittent, but does not satisfy the hierarchical clustering.

The purpose of this paper is to investigate the relationship between intermittency and hierarchical clustering. We will not limit ourselves to special models of intermittency, but will try to calculate the intermittent feature of a hierarchical clustered field in general. We first introduce the intermittent exponent which gives a detailed classification of the non-linear clustering. Specifically it gives a complete and uniform description of the nonlinearity of the cosmic mass field from initially gaussian perturbations, to weak, and eventually strong non-linear clustering. We then study the intermittent exponent predicted by the hierarchical clustering. As expected, the intermittent behavior given by the hierarchical clustering model is found to be in good agreement with observed results on large scales. However, for the highly non-linearly evolved field, we find that the simplest correlation hierarchy, i.e., one with constant coefficients $Q_n$, is no longer adequate to describe the observed features in the density field. In this case, the model of hierarchical clustering with scale dependent $Q_n$ may still work. Moreover, we find the relationship between the intermittent and singular behaviors of the cosmic mass field and the scale dependence of the $Q_n$.

The paper will be organized as follows. §2 introduces the basic statistical measures of the intermittency of random mass fields – the structure function and intermittent exponent. §3 presents the predicted intermittency of a hierarchically clustered field. The singular behavior of the hierarchical clustering with scale-dependent $Q_n$ will be discussed in §4. The comparison of the intermittency and singular behaviors of a hierarchically clustered field with samples of QSO’s Ly-α forests is given in §5. Finally, the conclusions and discussions are in §6.
2. Intermittency

The chief characteristic of an intermittent field is that the interesting stuff happens in a local area. As a consequence, the description of intermittent fields by traditional methods such as the Fourier power spectrum are ineffective because these methods are essentially non-local. For instance, the amplitudes of the Fourier coefficients lose the spatial information. Two fields which have a similar gaussian PDF of the amplitudes of the Fourier coefficient may have very different intermittent features. Though the phases of the Fourier coefficients do contain the spatial information, they are not computationally convenient.

The density profiles described in §1 look to be a good way to the measure mass density peaks. The profiles are, however, only a measure for the individual massive halos and not a statistical measure of the entire random field consisting of these peaks. For example, imagine a homogeneous gaussian field in a cubic box with 128³ grids, which corresponds to 128³ realizations of a gaussian random variable. Rare peaks as high as 4σ will occur at some points. Obviously, the individual density profiles are useless in measuring the excess of the large density fluctuations with respect to a gaussian distribution. Moreover, the exponent γ introduced with the individual density profiles \( \rho(r) \propto r^{-\gamma} \) cannot be used to describe a statistically homogeneous and isotropic field, which requires that proper measures be statistically invariant quantities under translational and rotational transformations.

The proper measure of an intermittent field should satisfy the following conditions simultaneously: 1. its second order statistics should describe the power spectrum of the field, and 2. the correlations of phases should describe coherent structures. We show in this section that the structure functions in the discrete wavelet transform (DWT) representation meet these conditions.
2.1. The local density difference

The basic quantity for describing the intermittency of a random density field $\rho(x)$ is the difference between the densities at the position $x$ with separation $r$, i.e.,

$$\Delta_r(x) \equiv \rho(x + r) - \rho(x) = \frac{1}{\bar{\rho}} [\delta(x + r) - \delta(x)],$$

where $\delta(x) = [\rho(x) - \bar{\rho}]/\bar{\rho}$ is the density contrast. For simplicity, we assume that the density is normalized, $\bar{\rho} = 1$. The density difference is essential to describe the most important intermittent characteristics of a mass field. These features include:

1. The power spectrum and the local power spectrum.

The ensemble average of the second moment of $\Delta_r(x)$ is

$$S^2(r) = \langle |\Delta_r(x)|^2 \rangle.$$  \hspace{1cm} (2)

If the field is homogeneous, $S^2(r)$ is independent of $x$ and depends only on $r$. $S^2(r)$ is the mean power of the density fluctuations at wavenumber $k \simeq 2\pi/r$, and therefore, $S^2(r)$ is the power spectrum of the field (see §2.3). For a given realization of the random field, $|\Delta_r(x)|^2$ is the local power spectrum at $x$. For an intermittent field the spatial distribution of $|\Delta_r(x)|^2$ is highly irregular or spiky. The power of the density fluctuations on small $r$ is highly localized in space with very low powers between these spikes. This is probably the easiest way of identifying intermittency (Jamkhedkar, Zhan, & Fang 2000.)

2. The long-tailed PDF of $\Delta_r(x)$.

One of the defining characteristics of an intermittent field is the long-tailed PDF of $\Delta_r(x)$ on small scales $r$. This trait can effectively be measured by the higher order moments of $\Delta_r(x)$,

$$S^{2n}(r) = \langle |\Delta_r(x)|^{2n} \rangle.$$  \hspace{1cm} (3)
where $n$ is a positive integer. The $S^{2n}(r)$ are called structure functions. When the “fair sample hypothesis” is applicable (Peebles 1980), $S^{2n}(r)$ can be calculated by the spatial average,
\[ S^{2n}(r) = \frac{1}{V} \int |\Delta_r(x)|^{2n} \, dx, \tag{4} \]
where $V$ is the spatial normalization. When $n$ is large, $S^{2n}(r)$ is dominated by the long-tail events. Eqs. (2) and (4) show that the structure function $S^{2n}$ unifies the analysis of intermittency ($n > 1$) with the power spectrum ($n = 1$).

3. Density peaks and scale-scale correlations.

The density profile of mass halos $\rho(|x - x_0|) \propto |x - x_0|^{-\gamma}$ at $x_0$ means that events with large $|\Delta_r(x)|$ on different scales $r$ occur at the same place $x = x_0$. In other words, the density peaks lead to the correlation between $|\Delta_r(x)|$ and $|\Delta_{r'}(x)|$ with $r \neq r'$. As such, one can distinguish the field containing only peaks caused by gaussian fluctuations from an intermittent field by the scale-scale correlations defined as (Pando et al. 1998)
\[ C_{r,r'}^{n,n'} = \frac{\langle \Delta_r(x)^n \Delta_{r'}(x)^{n'} \rangle}{\langle \Delta_r(x)^n \rangle \langle \Delta_{r'}(x)^{n'} \rangle}, \tag{5} \]
where $n$ and $n'$ are even integers. $C_{r,r'}^{n,n'} = 1$ for gaussian fields, while $C_{r,r'}^{n,n'} > 1$ for intermittent fields. When $n$ and $n'$ are large, $C_{r,r'}^{n,n'}$ is dominated by the high density peaks. The excess of high peaks (as compared to gaussian peaks) can be measured by the scale-scale correlations.

4. Singular behavior and Hölder exponent $\alpha$

In the context of random fields, the singular behavior is characterized by the Hölder exponent $\alpha$ defined as (Adler 1981)
\[ \Delta_r(x) = |\delta(x + r) - \delta(x)| \leq (r/L)^{\alpha} \quad \text{as} \quad r \to 0, \tag{6} \]
where the constant $L$ can be taken as the sample size. The exponent $\alpha$ measures the smoothness of the field: for larger $\alpha$, the field is smoother on smaller scales and vice versa.
If $\alpha$ is negative, the field becomes singular. In this case the H"older exponent is dominated by the events with extremely large density difference $|\Delta_r(x)|$ on extremely small scales ($r \to 0$), and therefore, the exponent $\alpha$ should be dependent on the index $\gamma$ of the singular density profile if the object with the singular density profile is typical of the mass field. For a homogeneous and isotropic field, the PDF of the local density (contrast) difference $[\delta(x + r) - \delta(x)]$ for a given scale $r$ has to be independent of position $x$. Therefore, the exponent $\alpha$ defined by eq.(6) is statistically invariant under translation and rotational.

Since the resolution of real and simulation samples is always finite, one cannot practically measure the individual singularity of $|x_0 + r|^{-\gamma}$ at position $x_0$ with $r \approx 0$. However, $\alpha$ defined by eq.(6) can be calculated as the asymptotic behavior of the statistics of local density differences when $1/r \to \infty$. Hence, one can measure the singular exponent $\alpha$ as a statistical property of the random field, rather than using the density profiles of the individual massive halos.

We should make an important distinction between statistics based on $\Delta_r(x)$ and statistics that use the density $\rho(x)$ or density contrast $\delta(x)$. The former is the difference in either the density $\rho(x)$ or the density contrast $\delta(x)$ at different positions in space. It is possible that a large difference $\Delta_r(x)$ on small scales occurs in regions of low density, and also possible that a small difference occurs in regions of high density. To study intermittency, data which samples the entire cosmic mass field is necessary.

Finally, an advantage of using the local density difference is in reducing the contamination of the random velocity field. It is well known that the random velocity field will smooth the mass field in redshift space and repress the power of the density perturbations on scales below that characterized by the velocity dispersion. Because the difference $\Delta_r(x)$ is localized in space and scale, the velocity fluctuations on scales larger than $r$ will lead to offset of the event $\Delta_r(x)$ in redshift space, but keep the magnitude of
\( \Delta_r(\mathbf{x}) \) unchanged. As a consequence, the PDF of \( \Delta_r(\mathbf{x}) \) will not be affected by velocity fluctuations on scales larger than \( r \), and the spiky features of the local power spectrum of the Ly\( \alpha \) forests are still significant on scales of a few tens \( h^{-1} \) kpc (Jamkhedkar, Zhan, & Fang 2000.)

### 2.2. The Intermittent exponent

In the previous section we showed that the intermittency is described by the scale-dependence of the higher order moments of \( \Delta_r(\mathbf{x}) \). In this section, we introduce the intermittent exponent, which is a more effective tool of describing this scale-dependence.

For a homogeneous and isotropic gaussian field, the structure function eq.(3) can be calculated by

\[
S^{2n}(r) = \int_{-\infty}^{\infty} P_g(\Delta_r)[\Delta_r]^{2n} d\Delta_r
= (2n-1)!! \left[ \sigma^2(\rho) \right]^n,
\]

where \( P_g(\Delta_r) \) is the gaussian PDF of \( \Delta_r(\mathbf{x}) \), and \( \sigma^2(\rho) = \langle |\Delta_r|^2 \rangle \) is the variance of \( P_g(\Delta_r) \).

The deviation of the field from a gaussian distribution on scale \( r \) can be measured by the intermittent exponent defined as\(^4\)

\[
\tilde{S}^{2n}(r) \equiv \frac{S^{2n}(r)}{[S^2(r)]^n} \propto \left( \frac{r}{L} \right)^\zeta.
\]

\( \zeta \) is equal to zero if the mass density field is gaussian on all scales. From eq.(8), we have

\[
\frac{\tilde{S}^{2n}(r)}{\tilde{S}^{2n}(r_0)} = \left( \frac{r}{r_0} \right)^\zeta,
\]

\(^4\)In turbulence, \( \tilde{S}^{2n}(r) \) is used to define the so-called anomalous scaling describing the intermittent behavior (Shraiman & Siggia 2000.)
where again, $\zeta$ is equal to zero if the mass density field is gaussian on all scales.

A field is self-similar if the difference $\Delta_r(x)$ as a random variable satisfies

$$\Delta_r(x) = \lambda^h \Delta_{\lambda r}(x),$$

where $h$ is constant. In this case, $\bar{S}^{2n}(r)$ is independent of $r$. Once again, the exponent $\zeta$ defined by eq.(9) is zero. A field is said to be intermittent if the exponent $\zeta$ is non-zero. Hence, an intermittent field is neither gaussian nor selfsimilar (Frisch, 1995.)

If the field contains more “abnormal” events, i.e., large $\Delta_r(x)$ on small scales, $\bar{S}^{2n}(r)$ will be dominated by these events, especially when $n$ is large. In this case, we have $S^{2n}(r) > [S^2(r)]^n$ on small $r$, and therefore, the exponent $\zeta$ is less than zero. Similar to the exponent $\alpha$ [eq.(6)], the intermittent exponent $\zeta$ measures the smoothness of the field: for larger $\zeta$, the field is smoother on smaller scales, and vice versa. If $\zeta$ is negative on small scales, the field is probably singular.

The $n$- and $r$-dependence of $\zeta$ provide a detailed description of the non-linear features of the mass field. $\zeta$ gives an unified criterion for classifying the non-linear features of a random field, from gaussian, to self-similar, to mono- and multi-fractal, and to singular.

### 2.3. The Intermittent exponent in the DWT representation

The density contrast difference $\Delta_r(x)$ [eq.(1)] contains two variables: the position $x$ and the scale $r$, and therefore, the information of $\Delta_r(x)$ can best be extracted by a proper space-scale decomposition.

We will use the discrete wavelet transform (DWT) decomposition [For details on the DWT refer to Mallat (1989a,b); Meyer (1992); Daubechies, (1992), and for physical applications, refer to Fang & Thews (1998)]. To simplify the notation, we consider only
the DWT decomposition for 1-D density field \( \rho(x) \) or \( \delta(x) \) on a spatial range \( L \). It is straightforward to generalize the result to 2- and 3-D fields.

We first divide the spatial range \( L \) into \( 2^j \) segments labelled by \( l = 0, 1, \ldots, 2^j - 1 \). Each of the segments has length \( L/2^j \). The density contrast difference \( \Delta_r(x) = \delta(x + r) - \delta(x) \) defined in eq.(1) can be replaced by

\[
\tilde{\epsilon}_{j,l} = \sqrt{\frac{2^j}{L}} \left[ \int_{lL/2^j}^{(l+1/2)L/2^j} \delta(x)dx - \int_{(l+1/2)L/2^j}^{(l+1)L/2^j} \delta(x)dx \right].
\]  

(11)

\( \tilde{\epsilon}_{j,l} \) measures the difference between the mean density contrasts in the localized segments \( lL/2^j \leq x < (l + 1/2)L/2^j \) and \( (l + 1/2)L/2^j \leq x < (l + 1)L/2^j \). Therefore, it can be identified as \( \Delta_r(x) \) with \( x = lL/2^j \) and \( r = L/2^j \).

\( \tilde{\epsilon}_{j,l} \) is the Haar wavelet function coefficient (WFC), given by the projection of \( \delta(x) \) onto the Haar wavelet basis \( \psi_{j,l}^H(x) \) as

\[
\tilde{\epsilon}_{j,l} = \int \delta(x)\psi_{j,l}^H(x)dx,
\]

(12)

where

\[
\psi_{j,l}^H(x) = \sqrt{\frac{2^j}{L}} \begin{cases} 
1 & \text{from } Ll2^{-j} \text{ to } Ll2^{-j} + L2^{-j-1} \\
-1 & \text{from } Ll2^{-j} + L2^{-j-1} \text{ to } L(l+1)2^{-j} \\
0 & \text{otherwise.}
\end{cases}
\]  

(13)

The factor \( \sqrt{2^j/L} \) insures the orthonormality of the wavelet with respect to both indices \( j \) and \( l \).

For other wavelets \( \psi_{j,l}(x) \), the WFC, \( \tilde{\epsilon}_{j,l} \), also measures the difference of the density contrast on separation \( L/2^j \) at a position \( lL/2^j \). We will use the Daubechies 4 wavelet (Daubechies, 1992) in our numerical calculations below because it is better behaved in scale space than the Haar wavelet (Fang & Thews 1998). The set of wavelets \( \psi_{j,l}(x) \) \( (j = 0, 1, \ldots \) and \( l = 0, 1, \ldots, 2^j - 1 \) is complete and orthogonal. In the DWT representation, the field \( \delta(x) \)
can be expressed as  
\[ \delta(x) = \sum_{j'=0}^{J} \sum_{l=0}^{2^{j'}-1} \tilde{\epsilon}_{j',l} \psi_{j',l}(x). \]  

(14)  

\( J \) is determined by the spatial resolution \( \Delta x \) of the sample, i.e., \( J \approx \log_2(L/\Delta x) \). The WFC \( \tilde{\epsilon}_{j',l} \) is given by eq.(12), replacing the Haar wavelet \( \psi_{j',l}^H(x) \) by \( \psi_{j,l}(x) \). Since wavelets are admissible, the WFC can be calculated with either the density or the density contrast 
\[ \tilde{\epsilon}_{j,l} = \int \rho(x) \psi_{j,l}(x) dx = \frac{1}{\bar{\rho}} \int \delta(x) \psi_{j,l}(x) dx. \]  

(15)  

In the DWT representation, eq.(2) is 
\[ S_{j}^2 = \frac{1}{2^j} \sum_{l=0}^{2^j-1} |\tilde{\epsilon}_{j,l}|^2. \]  

(16)  

This is the power spectrum in the DWT representation and equal to a band-averaged Fourier power spectrum given by (Fang & Feng 2000) 
\[ S_{j}^2 = \frac{1}{2^j} \sum_{n=-\infty}^{\infty} |\hat{\psi}(n/2^j)|^2 P(k), \]  

(17)  

where \( P(k) \) is the Fourier power spectrum with the wavenumber \( k = 2\pi n/L \). \( \hat{\psi}(n) \) is the Fourier transform of the basic wavelet \( \psi(\eta) \).

The structure function (3) in the DWT representation is 
\[ S_{j}^{2n} = \langle |\tilde{\epsilon}_{j,l}|^{2n} \rangle = \frac{1}{2^j} \sum_{l=0}^{2^j-1} |\tilde{\epsilon}_{j,l}|^{2n} \]  

(18)  

and eq. (8) is 
\[ S_{j}^{2n} = \frac{S_{j}^{2n}}{|S_{j}^2|^n} \propto 2^{-j\zeta}. \]  

(19)  

It has been shown that \( \zeta \) given by eq.(19) is the same as (8) (Jaffred 1994.) \( \tilde{S}_{j}^{2n} \) or \( \zeta \) is the basic measure of the intermittency of the cosmic mass fields.
3. Intermittency within hierarchical clustering

Hierarchical clustering provides a model describing the non-linear clustering of the cosmic mass field. It assumes that the correlation functions of the mass density can be described by the linked-pair approximation, i.e., the $n$-th irreducible correlation function $\xi_n$ is given by the two-point correlation function $\xi_2$ as $\xi_n = Q_n \xi_2^{n-1}$, where $Q_n$ is the hierarchical coefficient (White 1979). As the high order correlation functions provide a direct way to investigate the nonlinearity in a random field and test the scaling hierarchy, it follows that one can study the hierarchical clustering predictions pertaining to intermittency and the singular behavior of the mass field. In this section we develop the higher order correlation functions in the DWT representation.

3.1. Linked-pair approximation with the DWT modes

For the $n$-th order correlation function, the correlation hierarchy, or linked pair approximation, is

$$\xi_n(x_1, \ldots, x_n) = \sum_\varpi Q_n^\varpi \sum_{(ab)} \prod_{i=0}^{n-1} \xi_2(x_{ab}),$$

(20)

where coefficients $Q_n^\varpi$ are constant, the sum $\varpi$ is over the types of the tree graphs with $n$ vertices, $x_{ab} = |x_a - x_b|$, and the sum $(ab)$ is over relabelings within $\varpi$ (Fry 1984). With this relation, the higher order behavior of the clustering is completely determined by the two-point correlation $\xi_2(x_{ab})$ and the constants $Q_n^\varpi$. Hence, the intermittent behavior of a hierarchical clustered field can be determined by the coefficients $Q_n^\varpi$ and the non-linear power spectrum.

For instance, in the case of $n = 4$, eq.(20) yields

$$\langle \delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4) \rangle =$$

$$Q_4^\varpi \langle \delta(x_1)\delta(x_2) \rangle \langle \delta(x_2)\delta(x_3) \rangle \langle \delta(x_3)\delta(x_4) \rangle + \text{cyc. 11 terms}$$

(21)
\[ + Q_4^b [ \langle \delta(x_1) \delta(x_2) \rangle \langle \delta(x_1) \delta(x_3) \rangle \langle \delta(x_1) \delta(x_4) \rangle ] + \text{cyc. 3 terms} \].

where \( Q_4^a \) is for snake diagrams, and \( Q_4^b \) is for stars.

We express eq.(21) in the DWT basis, \( \psi_{j1}(x_1) \psi_{j1}(x_2) \psi_{j1}(x_3) \psi_{j1}(x_4) \), where \( \psi_{j1}(x) \) is the 3-D wavelet \( (x = x^1, x^2, x^3) \). \( \psi_{j1}(x) \) is given by a direct product of the 1-D wavelets, i.e., \( \psi_{j1}(x) = \psi_{j1,t1}(x^1) \psi_{j1,t2}(x^2) \psi_{j1,t3}(x^3) \).

Equation (21) yields

\[
\langle \tilde{\epsilon}_{j1}^4 \rangle = Q_4^a \sum_{y', l'} \sum_{j'} \sum_{l''} \sum_{m''} \langle \tilde{\epsilon}_{j,l} \tilde{\epsilon}_{j',l'} \rangle \langle \tilde{\epsilon}_{j',l'} \tilde{\epsilon}_{j''} \rangle \langle \tilde{\epsilon}_{j''} \rangle + Q_4^b \sum_{y'} \sum_{j'} \sum_{l''} \sum_{m''} \langle \tilde{\epsilon}_{j,l} \tilde{\epsilon}_{j',l'} \rangle \langle \tilde{\epsilon}_{j',l'} \tilde{\epsilon}_{j''} \rangle \langle \tilde{\epsilon}_{j''} \rangle + \text{cyc. 11 terms}]
\]

\[
\int \psi_{j1}(x_2) \psi_{j1}(x_2) \psi_{j1}(x_2) \psi_{j1}(x_2) dx_2 \int \psi_{j1}(x_3) \psi_{j1}(x_3) \psi_{j1}(x_3) \psi_{j1}(x_3) dx_3
\]

where the \( j \) and \( l \) terms are the 3-D scales and positions respectively.

The DWT is very efficient in compressing data, i.e., the off \( j \)-diagonal elements of the covariance \( \langle \tilde{\epsilon}_{j,l} \tilde{\epsilon}_{j',l'} \rangle \) generally are much less than the \( j \)-diagonal elements. This property is also consistent with the quasi-locality of mass clustering (Pando, Feng & Fang 2001.) Thus, the r.h.s. of eq.(22) is dominated by the terms \( \langle \tilde{\epsilon}_{j1}^2 \rangle^3 \) and we have

\[
\langle \tilde{\epsilon}_{j1}^4 \rangle \simeq [Q_4^a A_j + Q_4^b B_j] \langle \tilde{\epsilon}_{j1}^2 \rangle^3
\]

where the factors \( A_j \) and \( B_j \) are given by

\[
A_j = \left[ \int \psi_{j1}^3(x) dx \right]^2
\]

\[
B_j = \int \psi_{j1}^4(x) dx.
\]
Because wavelets are constructed by dilating and translating a basic wavelet $\psi(\eta)$, $A_j$ and $B_j$ can be more simply written as

$$A_j = 2^{j^3} \frac{1}{L^3} \left[ \int \psi^3(\eta) d\eta \right]^2$$

and

$$B_j = 2^{j^3} \frac{1}{L^3} \int \psi^4(\eta) d\eta$$

Thus, eq.(23) becomes finally

$$\langle \tilde{\epsilon}^4_{j,l} \rangle \simeq Q_4 2^{j^3} \langle \tilde{\epsilon}^2_{j,l} \rangle^3,$$

where the coefficient $Q_4$ is independent of $j$ and given by

$$Q_4 = \frac{1}{L^3} \left[ Q^a_4 \left[ \int \psi^3(\eta) d\eta \right]^2 + Q^b_4 \int \psi^4(\eta) d\eta \right].$$

Similarly, for $2n$-th order, we have

$$\langle \tilde{\epsilon}^{2n}_{j,l} \rangle \simeq Q_{2n} 2^{(n-1)(j^3)} \langle \tilde{\epsilon}^2_{j,l} \rangle^{2n-1},$$

where again $Q_{2n}$ is independent of $j$. Eq.(30) is the counterpart of correlation hierarchy $\xi_n = Q_n \xi_{n-1}^2$ in the DWT representation.

For a 1-D field, such as Ly$\alpha$ forests, we use a projection of a 3-D distribution $\delta(x)$ onto 1-D as

$$\tilde{\epsilon}_{j,l} = \int_{-\infty}^{\infty} \delta(x) \psi_{j,l}(x^1) \phi_{j,m}(x^2) \phi_{j,n}(x^3) dx^1 dx^2 dx^3,$$

where $x^1$ is the redshift, $x^2$ and $x^3$ are the positions in the sky, and $\phi_{j,l}(x)$ is the scaling function of the DWT analysis. The scaling function $\phi_{j,l}(x)$ plays the role of a window function on scale $j$ at position $l$ (Fang and Thews 1998). With eq.(31), the 1-D field linked-pair approximation (20) yields

$$\langle \tilde{\epsilon}^{2n}_{j,l} \rangle \simeq Q_{2n} 2^{(n-1)(j^3)} \langle \tilde{\epsilon}^2_{j,l} \rangle^{2n-1},$$
where \( Q_{2n} \) is for a 1-D field and is generally not equal to \( Q_{2n} \) for a 3-D field because the difference in the projection (31) for the 1-D case. We use the same notation \( Q_{2n} \) for both 1-D and 3-D and from the context it will be clear whether we are using the 1-D or 3-D quantity.

3.2. The intermittency exponent within hierarchical clustering

Applying eq.(30) for the diagonal case, i.e. \( j^1 = j^2 = j^3 = j \), we have

\[
\langle \tilde{\epsilon}_{j,l}^{2n} \rangle \approx Q_{2n} 2^{3j(n-1)} \langle \tilde{\epsilon}_{j,l}^2 \rangle^{2n-1}. \tag{33}
\]

One can rewrite eqs.(32) and (33) as

\[
\langle \tilde{\epsilon}_{j,l}^{2n} \rangle \approx Q_{2n} 2^{dj(n-1)} \langle \tilde{\epsilon}_{j,l}^2 \rangle^{2n-1}, \tag{34}
\]

where \( d \) is the number of dimensions.

Substituting eq.(34) into eq.(19), we have

\[
\tilde{S}_{j}^{2n} = Q_{2n} 2^{dj(n-1)} \langle \tilde{\epsilon}_{j,l}^2 \rangle^{n-1}. \tag{35}
\]

Therefore, the intermittent exponent of a hierarchical clustered field is

\[
\zeta_h = -(n - 1) \left[ d + \frac{1}{j} \ln_2 P_j \right] - \frac{1}{j} \ln_2 Q_{2n}, \tag{36}
\]

the subscript \( h \) is used to emphasize that \( \zeta_h \) is the predication in the hierarchical clustering model. For small scales (large \( j \)), the last term of eq.(36) is negligible if the \( Q_{2n} \)'s are constant. Hence, for a hierarchically clustered field, the intermittency exponent \( \zeta_h \) is completely determined by the power spectrum \( P_j = \langle \tilde{\epsilon}_{j,l}^2 \rangle \) in the non-linear regime. If the power spectrum is a power law, \( P_j \propto 2^{-j\kappa} \), we have

\[
\zeta_h \approx -(n - 1)(d - \kappa). \tag{37}
\]
Hence, the field is intermittent if $\kappa < d$.

The intermittent exponent eq.(37) is independent of $j$. This is the simplest type of intermittency – monofractal with fractal dimension $\kappa$. That is, the space filling of the density perturbations will be less by a factor $(1/2)^{(d-\kappa)}$ from scale $j$ to $j + 1$, and in effect, the hierarchical clustering model with constant $Q_n$ is equivalent to assuming that the random field $\delta(x)$ is monofractal. This conclusion is consistent with the phenomenological model of the hierarchical relations developed by Soneira & Peebles (1977). The model is essentially based on a dimension $\kappa$ fractal distribution.$^5$

4. Singular behavior of hierarchical clustering

4.1. Singular exponent

We now consider the singular exponent $\alpha$ of eq.(6). If the mass field contains singular structures like $\rho \propto r^{-\gamma}$ with $\gamma > 0$ at position $x_0$ (or $l_0$), the modulus of the diagonal WFCs ($j_1 = j_2 = j_3 = j$) satisfy

$$|\tilde{\epsilon}_{j,l_0}| \leq A 2^j (\gamma - d/2),$$

where the factor $d/2$ is from the normalization $\sqrt{2^j/L}$ in eq.(13). Eq.(38) shows that the singular exponent defined in eq.(6) is $\alpha = \gamma - d/2$. Thus, one can measure $\gamma$ asymptotically as follows.

Because for large $n$, $S_j^{2n}$ is dominated by events with large $\tilde{\epsilon}_{j,l}$, eq.(19) gives

$$\gamma \approx \frac{d}{2} - \frac{1}{2n} \zeta + \frac{1}{2j} \ln_2 P_j, \text{ when } n \text{ is large.}$$

$^5$This model is also similar to the so-called $\beta$-model of the intermittency of turbulence, for which the exponent $\zeta$ has similar terms as eq.(42) (Frisch 1995.)
Thus, a field is singular if the following condition holds
\[
\frac{d}{2} - \frac{1}{2n} \zeta + \frac{1}{2j} \ln_2 P_j > 0, \quad \text{when } n \text{ is large.} \tag{40}
\]

For a hierarchical clustered field, condition (40) is
\[
d - \kappa > 0 \quad \text{when } n \text{ is large.} \tag{41}
\]

Therefore, a hierarchical clustered field cannot be singular if the power index \( \kappa > d \). Under condition (41), eq.(37) leads to \( \zeta < 0 \). This result is consistent with the statement that the field is singular when \( \zeta < 0 \) on small scales (§2.2).

### 4.2. Hierarchical clustering with scale dependent coefficients

In order to have the hierarchical clustering scenario compatible with observed data and N-body simulation samples, it is often assumed that the \( Q_n \) are scale dependent. In this section, we will show that the scale dependence of the \( Q_n \) is given by the intermittent exponents. The validity of the hierarchical clustering model with scale dependent \( Q_n \) can be tested by examining the relationship between the \( Q_n \) scale dependence and the singular exponents.

For the hierarchical clustering models with scale-dependent \( Q_{2n} \), eq.(34) still holds, as all the quantities in eq.(34) are on same scale \( j \). Thus, the scale-dependence of \( Q_{2n} \) is given by
\[
Q_{2n} = \frac{1}{2^{d(n-1)}} \frac{\bar{S}_{2n}^j}{P_{j-1}^{n-1}}. \tag{42}
\]
The scale dependence of \( Q_{2n} \) can be measured by a power law index defined by
\[
Q_{2n} \propto 2^{j \beta_{2n}}. \tag{43}
\]
Using eqs.(19) and (42), eq.(43) yields
\[
\beta_{2n} = -d(n - 1) - \zeta - \frac{n - 1}{j} \ln_2 P_j. \tag{44}
\]
We see that the scale-dependence of $Q_{2n}$ is completely determined by the intermittent exponent $\zeta$ and the non-linear power spectrum.

Using eq.(39), the singular exponent $\gamma$ can be expressed by $\beta_{2n}$ as

$$\gamma \simeq d + \frac{1}{2n} \beta_{2n} + \frac{1}{j} \ln_2 P_j, \quad \text{when } n \text{ is large.}$$  \hspace{1cm} \text{(45)}

Eq. (45) shows that the singular index $\gamma$ is determined by the scale-dependence of the coefficients $Q_{2n}$ and the non-linear power spectrum $P_j$.

The important point of eq.(45) is that the l.h.s. is given by the singular profiles such as $\rho \propto r^{-\gamma}$, independent of $n$, while the r.h.s. consists of $n$-dependent quantities like $\beta_{2n}$. Therefore, a test of the scale-dependent $Q_n$ model is to check whether $\gamma$ given by eq.(45) is $n$-independent when $n$ is large.

It is known that a scale-dependent intermittent exponent indicates that the mass field is no longer monofractal, but is instead, multifractal (Farge at al. 1996.) Galaxy distributions have been shown to be multi-fractal in nature (Jones et.al, 1988) and to explain this distribution, a phenomenological model of non-linear fragmentation with multi-scaling parameters has been proposed (Jones, Coles & Martinez, 1992). Our present study has revealed the intrinsic relationships between the scale-dependence of $Q_{2n}$, multifractal nature, and singular behavior of the mass field.

This completes our derivation of intermittency within hierarchical clustering. With eq.(18) we pick up the DWT power spectrum, eq.(37) gives the intermittency exponent predicted by the hierarchical picture, eq.(42) provides a way to check for the scale dependence of the correlation coefficients, and eq.(45) gives a measure of singular behavior of the field. The versatility of this approach is clear.
5. Intermittent exponents of Lyα forests

Since the intermittent and singular features of the cosmic mass field are measured by the scale-dependence of the structure functions, samples covering a large range of scales are best suited for this analysis. As has been emphasized, the intermittent and singular features are traits of the mass field and not of the individual massive halos. As such, the transmitted flux of QSO’s Lyα forests given by high resolution absorption spectrum are ideal data sets with which to work. The Lyα transmitted flux is due to absorption by gases in cool and low density regions. The pressure gradients are generally less than the gravitational forces. The distribution of cool baryonic diffuse matter is almost point-by-point proportional to the underlying dark matter density (Bi, Ge & Fang 1995). The statistical features of the underlying mass field of the Lyα forests can be detected by the transmitted flux.

5.1. Samples

We use the Lyα transmitted flux of QSO HS1700+64 (z = 2.72) for our analysis. This sample has been employed to study the baryonic matter density (Bi & Davidsen 1997), the Fourier and DWT power spectra and non-gaussian features (Feng & Fang 2000). The recovered power spectrum has been found to be consistent with the CDM cosmogony on scales larger than about 0.1 \( h^{-1} \) Mpc. The data ranges from 3727.012Å to 5523.554Å, for a total of 55882 pixels. On average, a pixel is about 0.028Å, equivalent to a physical distance of \( \sim 5 \ h^{-1} \) kpc at \( z \sim 2 \) for an Einstein-de Sitter universe. These scales weakly depend on cosmological density parameter \( \Omega \). One can safely ignore the effect of the non-linear relation between redshift and distance for our present analysis. In this paper, we use the data from \( \lambda = 3815.6\text{Å} \) to 4434.3Å, which corresponds to \( z = 2.14 \sim 2.65 \). The comoving spatial size of our data is 189.84 \( h^{-1} \) Mpc in the CDM model. This means for scale \( j \), the spatial size is 189.84/2\(^j\) \( h^{-1} \) Mpc.
With Ly\(\alpha\) data there is always the danger of contamination from the presence of metal lines. We use three ways to estimate the error this contamination causes. First, we block out the significant metal line regions. Since the WFC \(\tilde{\epsilon}_{j,l}\) are localized, the metal line regions are easily separated from the rest. Any effect from these regions can be removed by not counting the DWT modes in the blocked regions. The second way is to fill those regions with random data which has the same mean power as the rest of the original data and to smooth the data over the boundaries. Lastly we discard the metal line chunks and smoothly connect the good chunks of data. We find that the results not sensitive to the method of removing metal lines.

Another source of contamination is noise. To estimate the effect of noise we smooth the QSO’s spectrum by filtering out all extremely sharp spikes in the local power spectra on finest scales. These spikes are caused by relatively strong fluctuations between two neighboring pixels. Since such events are on the smallest scales only, the analysis on larger scales does not depend on whether we smooth the sample or not.

The DWT power spectrum of the Ly\(\alpha\) transmitted flux of QSO HS1700+64 is shown in Fig. 1. It is clear that \(\kappa\) is larger than 1 on all small scales \(j \geq 9\) or physical scales \(< 100 \, h^{-1} \text{kpc}\).

5.2. Observed intermittent exponent

We calculate the intermittent exponent with eq.(19),

\[ \zeta = -\frac{1}{j} \ln_2 \tilde{S}_j^{2n}. \]  

(46)

The result is plotted in Fig. 2 and shows that on scales \(j \geq 9\), the intermittent exponents are always significantly less than zero. The underlying mass field of the QSO forests is neither non-gaussian nor non-selfsimilar, but essentially intermittent on physical scales less
than 100 h\(^{-1}\) kpc.

We next test the predicted relation (36) between the intermittent exponent and power spectrum. The results of \(\zeta_h\) are plotted in Fig. 3. Because the power law index \(\kappa\) of the DWT power spectrum is generally larger than 1, \(\zeta_h\) is always positive. This is qualitatively different from \(\zeta\) on small scales. Therefore, only on large scales \(j \leq 5\) or physical scales \(> 2\) h\(^{-1}\) Mpc, does the correlation hierarchy with constant \(Q_n\) match observations.

5.3. Multifractal scaling and singular exponent

To check for hierarchical clustering with scale-dependent coefficients, we calculate \(Q_{2n}\) with eq.(42). The results are shown in Fig. 4, which shows that only for lower \(n\), can the coefficient approximately be considered as constant. At all higher \(n\), \(Q_{2n}\) significantly increases with decreasing scale after \(j > 9\), which implies the breaking of a single scaling hierarchy in the baryonic matter distribution.

Fig. 5 plots the singular exponent \(\gamma\) calculated by eq.(45). It indeed shows that for each \(j\), \(\gamma\) is asymptotically independent of \(n\), when \(n\) is large. That is, models with scale-dependent \(Q_n\) are reasonable and the asymptotic values of \(\gamma\) are probably given by the exponent of the singular mass density profile.

Fig. 5 also show that the singular exponents \(\gamma\) become larger with increasing scales. For instance, \(\gamma \simeq 0.45\) at \(j = 9\), while \(\gamma \simeq 0.18\) at \(j = 13\). This is understandable if we consider the so-called “universal” density profile of massive halos (Navarro, Frenk & White 1997)

\[
\rho(r) \sim \frac{1}{r(r + a)^2}
\]

where \(a\) is the core radius. From this profile it is clear that \(\gamma\) on small scale \(r < a\) will be less than for \(r > a\). Comparing with (47), the \(\gamma\) given by the Ly\(\alpha\) transmitted flux are small.
and implies that the singular behavior of mass field at redshift \( z \simeq 2 \) is much weaker than the present.

Although Fig. 5 shows that \( \gamma > 0 \) on scale \( j = 13 \) or \( \simeq 7 \, h^{-1} \, \text{kpc} \), the field is not necessarily singular. For instance, if the profile (47) is modified as

\[
\rho(r) \sim \frac{1}{(r + b)(r + a)^2}
\]

(48)

where physical scale \( b \sim 5 \, h^{-1} \, \text{kpc} \), \( \gamma \) will no longer be positive on scales \( j \geq 14 \).

6. Conclusion and discussions

The structure functions provide a complete and unified way of studying intermittent fields. With this method, we have shown that intermittency and hierarchical clustering in the non-linear mass field are related. We find that if the field is singular, the intermittent behavior on higher order \( n \) can be completely described by the singular exponents and the non-linear power spectrum. In this case, the scale-dependencies of the hierarchical clustering coefficients \( Q_{2n} \) at large \( n \) are also completely determined by these parameters.

The transmitted flux of HS1700 does show the existence of the asymptotic value of the singular exponents. The cosmic mass field at redshift \( z \sim 2 \) is probably weakly singular at least on physical scales as small as \( 10 \, h^{-1} \, \text{kpc} \). A singular field can be neither self-similar nor mono-fractal. It is well known that under many different conditions the field given by gravitational clustering is self-similar. Therefore, the presence of singular clustering implies a broken self-similarity and/or mono-fractal.

From a dynamical point of view, there is a basis for understanding the relation between the intermittency and hierarchical clustering. The basic dynamical processes of hierarchical clustering are the merging of two halos or the accretion of small halos into bigger ones. Regardless of specific processes, either the infall of small halos into massive ones or the
merging of massive halos, these processes are stochastic. Thus, the hierarchical clustering is
described by equations containing the addition and multiplication of stochastic variables.
But these stochastic processes are typical of the origin of intermittency (e.g. Nakao 1998.).
Even when the initial PDF of the random variables is gaussian, the dynamics involving
additive and multiplicative stochastic variables will evolve to a long tailed PDF. For
instance, the growth of the mass $M$ of a halo is sometimes phenomenologically described
by the rate equation of halo mass growth (e.g. Cole & Lacey 1996, Salvador-Solé, Solanes
& Manrique 1998.) The rate equation is a phenomenological model of the formation of
intermittency via merging.

The so-called merging history tree theory has widely been employed to describe the
mean merging history of individual halos and then to fit with the statistical properties of
galaxies. Many merging tree models have been proposed. In the context of intermittency,
different merging trees have different rules for merging from smaller to larger scales, and
therefore, lead to different scale-scale correlations (§2.1). Thus, discrimination among
merging tree models is possible via a intermittent analysis.

We thank Dr. Wolung Lee for his support to this work. We also thank Dr. D. Tytler
for kindly providing the data of the Keck spectrum HS1700+64. LLF acknowledges support
from the National Science Foundation of China (NSFC) and National Key Basic Research
Science Foundation.
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Fig. 1.— The DWT power spectrum of the Ly$\alpha$ transmitted flux of QSO HS1700+64. The top scale is the comoving scale, i.e., $189.84/2^j \, h^{-1} \, \text{Mpc}$.
Fig. 2.— The intermittent exponent $\zeta$ vs. $j$ of the Ly$\alpha$ transmitted flux of QSO HS1700+64. The top scale is the comoving scale, i.e., $189.84/2^j$ h$^{-1}$ Mpc.
Fig. 3.— The intermittent exponent ($\zeta_h$) calculated by the linked-pair approximation, $\zeta$ is the same as Fig. 2. The top scale is the comoving scale, i.e., $189.84/2^j$ h$^{-1}$ Mpc.
Fig. 4.— The scale-dependence of $Q_{2n}$. The top scale is the comoving scale, i.e., $189.84/2^j h^{-1}$ Mpc.
Fig. 5. — The singular exponent $\gamma$ given by the asymptotic behavior of eq. (45).