Fuzzy Sphere and Hyperbolic Space from Deformation Quantization

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Abstract

We explicitly construct noncommutative * products on circularly symmetric two dimensional space by using the technique of Fedosov’s deformation quantization. Especially, on constant curvature spaces i.e., $S^2$ and $H^2$, we get $su(2)$ and $su(1,1)$ algebra respectively. These are candidates of * products applicable to noncommutative field theories or noncommutative gauge theories on spaces with nontrivial symplectic structure.
1 Introduction

Since the relation between string theory and noncommutative geometry was discussed in [1], noncommutative field theories and noncommutative gauge theories have been investigated enthusiastically from various viewpoints.

Many authors use the Moyal product as noncommutative associative $\star$ product for explicit calculations. It corresponds to a constant NS-NS $B$-field background in flat space in the context of string theory. On the other hand, at least formally, more general $\star$ products which may correspond to string theory on nonconstant $B$-field background in curved space are defined by some authors. However, explicit form of $\star$ products other than the Moyal product has been scarcely discussed in physical context.

In this paper, we use the technique of Fedosov’s deformation quantization [3] to get explicit forms of $\star$ products on nontrivial backgrounds. For simplicity, we investigate $\star$ products on circularly symmetric two dimensional spaces. Specifically, we focus on constant curvature spaces $S^2$, $H^2$ and $\mathbb{R}^2$, and explicitly construct $\star$ products which are different from the Moyal product. We also discuss some physical applications of our $\star$ products.

2 Construction of $\star$ product

Here we review the construction of Fedosov’s $\star$ product very briefly, and apply this procedure to circularly symmetric two dimensional spaces.

First, for a given symplectic manifold $(M, \Omega_0)$, we define the Weyl algebra bundle $W$ which has $\circ$ product of the Moyal type and its Abelian connection $D$ with some input parameter. For $\text{Ker}D \subset W$ (which is called flat section $W_D$), we get a one to one correspondence with $C^\infty(M)[[h]]$, where $h$ is the deformation parameter. We denote the map from $C^\infty(M)[[h]]$ to $W_D$ as $Q$, and its inverse map as $\sigma$. Then Fedosov’s $\star$ product on $C^\infty(M)[[h]]$ is defined by

$$a_0 \star b_0 := \sigma(Q(a_0) \circ Q(b_0)), \quad a_0, b_0 \in C^\infty(M)[[h]].$$

(1)

This is a solution of the problem of deformation quantization, i.e.,

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1Here we call $\star = \exp \left( i 2 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \theta^{ij} \right)$ with constant $\theta^{ij} = -\theta^{ji}$ the Moyal product.
2[2],[3], for example.
3In [4], nonassociative star product which generalizes [2],[3] is discussed to describe D-brane in curved backgrounds.
4See [3],[5] for details.
* is associative and its commutator $[\, , \,]_*$ is expanded as

$$[\, , \,]_* = i\hbar \{ \, , \, \} + \mathcal{O}(\hbar^2)$$

(2)

where $\{ \, , \, \}$ is the Poisson bracket with respect to the symplectic form $\Omega_0$.

Now, we apply this procedure to a two dimensional space $M$ with metric

$$ds^2 = e^{\Phi(r)}(dr^2 + r^2 d\theta^2),$$

(3)

where $\Phi(r)$ is some function of $r$ only (i.e. circularly symmetric space) for simplicity. Its volume form is given by

$$\Omega_0 = e^{\Phi(r)}rdr \wedge d\theta,$$

(4)

and we identify it with symplectic form. Using Fedosov’s procedure with the input

$$\Omega_0 = \theta^1 \wedge \theta^2 = -12\omega_{ij}\theta^i \wedge \theta^j,$$

$$\theta^1 = e^{\Phi(r)}dr, \quad \theta^2 = rd\theta,$$

$$\omega_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\Omega_1 = 0, \quad \nabla = d,$$

$$\mu = 13e^{-\Phi(r)}r^{-1}(y^1)^2 y^2,$$

(5)

we get an Abelian connection $D$ as

$$Da = da - \delta a + i\hbar(r \circ a - a \circ r), \quad a \in W,$$

$$r = e^{-\Phi(r)}r^{-1}y^1 y^2 \theta^1,$$

$$\circ := \exp \left( -i\hbar 2 \frac{\partial}{\partial y^i} \omega^{ij} \frac{\partial}{\partial y^j} \right), \quad \omega^{ij} := (\omega^{-1})^{ij}.$$  

(6)

For this Abelian connection $D$, we solve the equation $Da = 0$ and get the map $Q : C^\infty(M)[[\hbar]] \to W_D$ as

$$a = Q(a_0(r, \theta)) = a_0 \left( G(r, y^1), \theta + y^2 r \right),$$

(7)

where $G(r, y^1)$ is given by

$$\int_r^{G(r,y')} e^{\Phi(r')} r' dr' = y^1 r.$$  

(8)

Then we can define a * product on $M$ by eq.(1).

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5See [3],[5] for the meaning of $\nabla, \Omega_1, \mu, \delta$. Here we choose these parameters in such a way that the iteration formula (eq.(21) of [5]) which gives an Abelian connection is satisfied trivially, i.e., $\nabla r + i\hbar r \circ r = 0$. Then we get $r = \delta \mu + \delta^{-1}(d(\omega_{ij}y^i \theta^j) - \Omega_1)$ for the input (5).
3 \( S^2 \) case

In this section we apply the result of §2 to the case \( M = S^2 \). We consider 2-sphere \( S^2 \) with radius \( R \), which is defined as two dimensional surface embedded in \( \mathbb{R}^3 \):

\[
(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2. \tag{9}
\]

We parametrize the coordinate \( X^i, i = 1, 2, 3 \) on \( S^2 \) as

\[
X^1 = 2R^2rr^2 + R^2 \cos \theta, \quad X^2 = 2R^2rr^2 + R^2 \sin \theta, \quad X^3 = Rr^2 - R^2r^2 + R^2, \quad r \geq 0, \quad 0 \leq \theta \leq 2\pi. \tag{10}
\]

Then the metric of \( S^2 \), \( ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \), is given by

\[
ds^2 = 4R^4(r^2 + R^2)^2(dr^2 + r^2d\theta^2), \tag{11}\]

and the conformal factor \( e^\Phi \) of eq.(3) is identified as

\[
e^\Phi(r) = 4R^4(r^2 + R^2)^2. \tag{12}\]

From eqs. (12), (7) and (1), we get the explicit form of our \(*\) product on \( S^2 \):

\[
a_0(r, \theta) \ast b_0(r, \theta)
\]

\[
= \left( a_0 \left( \sqrt{r^2 + y^12R^2r(r^2 + R^2)} - y^12R^4r(r^2 + R^2), \theta + y^2r \right) \exp \left( -i\hbar^2 \left( \overrightarrow{\partial} \partial y^1 \overrightarrow{\partial} \partial y^2 - \overrightarrow{\partial} \partial y^2 \overrightarrow{\partial} \partial y^1 \right) \right) \right. \\
\cdot b_0 \left( \sqrt{r^2 + y^12R^2r(r^2 + R^2)} - y^12R^4r(r^2 + R^2), \theta + y^2r \right) \left. \right|_{y^1=y^2=0}. \tag{13}\]

By using this definition, we can calculate \(*\) product of the \( S^2 \) coordinate \( X^i \) (10). In particular, we have

\[
[X^i, X^j]_* = i\hbar R\varepsilon^{ijk} X^k, \tag{14}\]

\[
X^1 \ast X^1 + X^2 \ast X^2 + X^3 \ast X^3 = R^2 \left( 1 - \hbar^2 4R^4 \right), \tag{15}\]

where \( \varepsilon^{ijk} \) is the antisymmetric tensor with \( \varepsilon^{123} = +1 \). Eq.(14) means that the commutators of \( X^i \)'s form \( su(2) \) algebra which is known as fuzzy sphere algebra, and eq.(15) means that its radius is given by \( R\sqrt{1 - \hbar^2 4R^4} \) which is deformed by \( \mathcal{O}(\hbar^2) \) from the original radius \( R \) of commutative \( S^2 \) (9). Namely, we have obtained a fuzzy sphere by deforming \( S^2 \) using the \(*\) product (13).
4 $H^2$ case

In this section we apply the result of §2 to the case $M = H^2$. Calculation is quite similar to the $S^2$ case (§3). We consider two dimensional hyperbolic space $H^2$ with radius $R$, which is defined as two dimensional surface embedded in $\mathbb{R}^{1,2}$:

$$-(Y^0)^2 + (Y^1)^2 + (Y^2)^2 = -R^2, \quad Y^0 > 0. \quad (16)$$

We parametrize the coordinates $Y^i, i = 0, 1, 2$ on $H^2$ as

$$Y^0 = RR^2 + r^2R^2 - r^2, \quad Y^1 = 2R^2rR^2 - r^2 \cos \theta, \quad Y^2 = 2R^2rR^2 - r^2 \sin \theta,$$
$$0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi. \quad (17)$$

Then, the metric of $H^2$, $ds^2 = -(dY^0)^2 + (dY^1)^2 + (dY^2)^2$, and the conformal factor are given respectively by

$$ds^2 = 4R^4(R^2 - r^2)^2(dr^2 + r^2d\theta^2),$$
$$e^{\Phi(r)} = 4R^4(R^2 - r^2)^2. \quad (18)$$

From eqs. (19), (7) and (1), we get the explicit form of our $\ast$ product on $H^2$:

$$a_0(r, \theta) \ast b_0(r, \theta) = \left(a_0 \left(\sqrt{r^2 + y^1 2R^2r(R^2 - r^2)1 + y^1 2R^4r(R^2 - r^2)}, \theta + y^2r\right) \exp \left(-i\hbar \left(\overleftarrow{\partial} y^1 \overrightarrow{\partial} y^2 - \overrightarrow{\partial} y^1 \overleftarrow{\partial} y^2\right)\right) \right. \cdot \left. b_0 \left(\sqrt{r^2 + y^1 2R^2r(R^2 - r^2)1 + y^1 2R^4r(R^2 - r^2)}, \theta + y^2r\right) \right)_{y^1 = y^2 = 0}. \quad (20)$$

By using this definition, we obtain the following $\ast$ products of the $H^2$ coordinate $Y^i$ (17):

$$[Y^0, Y^1]_\ast = i\hbar R y^2, \quad [Y^2, Y^0]_\ast = i\hbar R y^1, \quad [Y^1, Y^2]_\ast = -i\hbar R y^0, \quad (21)$$
$$-Y^0 \ast Y^0 + Y^1 \ast Y^1 + Y^2 \ast Y^2 = -R^2 \left(1 - \hbar^2 4R^4\right). \quad (22)$$

Eq.(21) means that commutators of $Y^i$'s form $su(1, 1)$ algebra which corresponds to isometry of $H^2$, and eq.(22) means that its radius is given by $R\sqrt{1 - \hbar^2 4R^4}$ which is deformed by $O(\hbar^2)$ from the original radius $R$ of commutative $H^2$ (16). Namely, we get fuzzy hyperbolic space by deforming $H^2$ using the $\ast$ product (20).

5 Large $R$ limit and $\mathbb{R}^2$

Here we consider large radius limit of the results of §3 and §4. The sectional curvature of $S^2$ (9) ($H^2$ (16)) is $1R^2$ ($-1R^2$), which tends to $+0$ ($-0$) in the limit $R \to \infty$. Therefore
they approach the flat space $\mathbb{R}^2$ in the large $R$ limit in the usual commutative picture. How about it from the noncommutative viewpoint?

For comparison, we construct a $\ast$ product on $\mathbb{R}^2$ following the method of §2. We adopt as its flat metric

$$ds^2 = 4(dr^2 + r^2d\theta^2)$$

with its front factor 4 chosen so that (23) coincides with the large $R$ limit of (11) and (18). With $e^\Phi = 4$, we get the explicit form of our $\ast$ product on $\mathbb{R}^2$:

$$a_0(r, \theta) \ast b_0(r, \theta) = \left(a_0\left(\sqrt{r^2 + y^1 r^2}, \theta + y^2 r\right) \exp \left(-i\hbar \left(\vec{\partial} y^1 \vec{\partial} y^2 - \vec{\partial} y^2 \vec{\partial} y^1\right)\right)\right) \cdot b_0\left(\sqrt{r^2 + y^1 r^2}, \theta + y^2 r\right)_{y^1 = y^2 = 0}.$$ (24)

Then, we can calculate the $\ast$ products of the complex coordinate $z := re^{i\theta}$, $\bar{z} := re^{-i\theta}$:

$$z \ast z = \sqrt{r^4 - \hbar^2 16 e^{2i\theta}} = \bar{z} \ast \bar{z}, \quad z \ast \bar{z} = r^2 - \hbar 4, \quad \bar{z} \ast z = r^2 + \hbar 4,$$

$$[z, \bar{z}]_\ast = -\hbar 2.$$ (25)

The commutator $[z, \bar{z}]_\ast$ coincides with that of the usual Moyal product for Cartesian coordinates on $\mathbb{R}^2$, but $\ast$ product itself is different from the Moyal product. This difference comes from ambiguity of deformation quantization.

We can calculate the commutator $[z, \bar{z}]_\ast$ also in the $S^2$ and $H^2$ cases. For $S^2$, from eq.(13) we get

$$[z, \bar{z}]_\ast = -\hbar 2 R^4 (r^2 + R^2)^2 1 - \left(\hbar 4 R^4 (r^2 + R^2)\right)^2 = -\hbar 2 R^4 (R^2 + z \ast \bar{z})(R^2 + \bar{z} \ast z).$$ (26)

And for $H^2$, from eq.(20) we get

$$[z, \bar{z}]_\ast = -\hbar 2 R^4 (R^2 - r^2)^2 1 - \left(\hbar 4 R^4 (R^2 - r^2)\right)^2 = -\hbar 2 R^4 (R^2 - z \ast \bar{z})(R^2 - \bar{z} \ast z).$$ (27)

Both eqs.(26) and (27) are reduced to $[z, \bar{z}]_\ast = -\hbar 2$ (25) as $R \to \infty$. In other words, the $\ast$ product which we obtained in §2 connects $su(2)$ algebra (or fuzzy $S^2$) with $su(1, 1)$ algebra (or fuzzy $H^2$) through $R = \infty$.

### 6 An application

In the previous sections, we explicitly calculated $\ast$ products by using Fedosov’s formulation. They are candidates of $\ast$ product for defining noncommutative field theory or noncommutative gauge theory on fuzzy $S^2$, $H^2$ and $\mathbb{R}^2$. 

As an example, we discuss four dimensional noncommutative U(1) gauge theory with one scalar field which is given by the action

$$S = \text{Tr} \left( 14G^{IJ} G^{KL} F_{IK} \ast F_{JL} + 12G^{IJ} D_I \phi \ast D_J \phi \right).$$  \hspace{1cm} (28)

We assume that only two dimensional space is noncommutative (1,2 direction), and use a general formulation of noncommutative gauge theory of [5]:

$$G^{IJ} = \delta^{IJ}, \ I, J = 1, \cdots, 4,$$
$$F_{IJ} = \partial_I A_J - \partial_J A_I - i[A_I, A_J]_s - J_{IJ} \hbar, \quad J_{12} = -J_{21} = 1, \text{others} = 0,$$
$$\partial_I = i\hbar[-J_{IJ}\tilde{\phi}^J, \ ]_s, \quad I = 1, 2, \quad \partial_3 = \partial \partial x^3, \partial_4 = \partial \partial x^4$$
$$D_I \phi = \partial_I \phi - i[A_I, \phi]_s,$$ \hspace{1cm} (29)

Here, $\tilde{\phi}^J$ is the “canonical” noncommutative coordinate satisfying

$$i\hbar[\tilde{\phi}^1, \tilde{\phi}^2]_s = 1.$$ \hspace{1cm} (30)

Its explicit form is

$$\tilde{\phi}^1 = 2Rr \sqrt{r^2 + R^2} \cos \theta, \quad \tilde{\phi}^2 = 2Rr \sqrt{r^2 + R^2} \sin \theta$$ \hspace{1cm} (31)

for fuzzy $S^2$ (13),

$$\tilde{\phi}^1 = 2Rr \sqrt{R^2 - r^2} \cos \theta, \quad \tilde{\phi}^2 = 2Rr \sqrt{R^2 - r^2} \sin \theta$$ \hspace{1cm} (32)

for fuzzy $H^2$ (20), and

$$\tilde{\phi}^1 = 2r \cos \theta, \quad \tilde{\phi}^2 = 2r \sin \theta$$ \hspace{1cm} (33)

for fuzzy $\mathbb{R}^2$ (24). The action (28) is invariant under noncommutative U(1) gauge transformation:

$$\delta_\lambda A_I = \partial_I \lambda - i[A_I, \lambda]_s, \quad \delta_\lambda \phi = -i[\phi, \lambda]_s.$$ \hspace{1cm} (34)

The equations of motion of (28) are

$$D^I F_{IJ} = -i[\phi, D_J \phi]_s, \quad D^I D_I \phi = 0,$$ \hspace{1cm} (35)

and we obtain a solution by solving the U(1) noncommutative BPS equation:

$$B_I = D_I \phi, \ I = 1, 2, 3, \quad \partial_4 = 0, A_4 = 0, \quad B_I := 12 \varepsilon^{IJK} (F_{JK} + J_{JK} \hbar).$$ \hspace{1cm} (36)

\[^6\text{The symbol Tr is trace for the } \ast \text{ product satisfying Tr} f \ast g = \text{Tr} g \ast f \ [3], \text{ but we can discuss equations of motion without using the explicit form of the trace.}\]
Under the ansatz
\[ A_1 + iA_2 = i f_A(l, x^3)(\tilde{\phi}^1 + i\tilde{\phi}^2), \quad A_3 = 0, \]
\[ \phi = f(l, x^3), \quad l := \sqrt{(\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2 + (x^3)^2}, \] (37)
eq(36) can be rewritten as
\[ \partial_3 G^{(m)} - 4\partial_L f^{(m)} = \sum_{2n+k=m, \ n \geq 1} 4\partial_L^{2n+1} f^{(k)}(2n+1)! + \sum_{2n+k+k' = m-1} 4G^{(k')}\partial_L^{2n+1} f^{(k)}(2n+1)!, \]
\[ \partial_3 f^{(m)} - \partial_L LG^{(m)} = \sum_{2n+k=m, \ n \geq 1} \partial_L^{2n+1}(LG^{(k)})(2n+1)! \] (38)
with
\[ L := (\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2, \quad f = \sum_{k=0}^{\infty} h^k f^{(k)}, \quad (1 + f_A)^2 = 1 + 1h + \sum_{k=0}^{\infty} h^k G^{(k)}. \] (39)
We can solve eq.(38) order by order in \( \hbar \), and we get
\[ f = gl + h g^2 \left( 2x^3 l^4 - l^3 \right) + h^2 \left( -8g^3 x^3 l^6 - 4l^5 + \left( 5g8 + 10g^3 \right) \left( x^3 \right)^2 l^7 \right) + O(h^3), \]
\[ f_A = gl(l + x^3) + h g^2 \left( 2l^4 - 11l^3(l + x^3) - 12l^2(l + x^3)^2 \right) \]
\[ + h^2 \left( -8g^3 l^6 + 4g^3 l^5(l + x^3) + g^3 l^4(l + x^3)^2 + g^3 2l^3(l + x^3)^3 - \left( 5g8 + 10g^3 \right) x^3 l^7 \right) + O(h^3), \] (40)
as a solution such that it becomes the \( U(1) \) Dirac monopole in the commutative limit (i.e., \( \hbar \rightarrow 0 \)). In the fuzzy \( \mathbb{R}^2 \) case (33), the \( O(\hbar) \) terms coincide with those in [6] which solved the equations of motion with the usual Moyal product.

### 7 Conclusion and discussion

In this paper we have presented explicit construction of * products on two dimensional constant curvature spaces \( S^2, H^2 \) and \( \mathbb{R}^2 \). We have found that the algebras of the * products represent fuzzy \( S^2, H^2 \) and \( \mathbb{R}^2 \) because the commutators of the * product form \( su(2), su(1, 1) \) and Heisenberg algebra respectively. The commutators \( [z, \bar{z}]_* \) for fuzzy \( S^2 \) and \( H^2 \) are reduced to that of fuzzy \( \mathbb{R}^2 \) in the large \( R \) limit. In this sense, fuzzy \( S^2 \) and \( H^2 \) approach to fuzzy \( \mathbb{R}^2 \) as \( R \rightarrow \infty \). This is consistent with usual commutative picture.

In §6 we applied explicit form of our * products to \( U(1) \) noncommutative BPS equation (36), and obtained its solution to \( O(\hbar^2) \). In eq.(36) the * product appears only in the
Therefore, eq.(36) is solved unifiedly for fuzzy $S^2, H^2$ and $\mathbb{R}^2$ by using “canonical” noncommutative coordinate $\tilde{\phi}^I$ (30). In other words, we can get a solution of eq.(36) even if the definition of $*$ is different as long as we use “canonical” noncommutative coordinate $\tilde{\phi}^I$ for the $*$ product.

To study the effects of the difference of $*$ products themselves, we should consider noncommutative equations containing “bare” $*$ products. Its typical example is $\phi \ast \phi = \phi$ which is essentially the equation for noncommutative soliton [7]. Even for the $\mathbb{R}^2$ case, the $*$ product which we get here is different from the usual Moyal product, and hence $\phi \sim \exp(-r^2)$ is not a solution$^7$ of $\phi \ast \phi = \phi$. It is a future problem to find an explicit solution of it and to investigate its meaning.

For fuzzy $S^2$, $*$ product is usually defined by using representation matrix of $su(2)$ and spherical harmonic function, and depends on the size of matrix. On the other hand our $*$ product depends on the deformation parameter $\hbar$, so they are very different in appearance. It is also a future problem to study an explicit relation between them. If the relation becomes clear, our $*$ product may give some suggestions to string theory in the literature [8] for example.

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References


$^7$In the case of the Moyal product, this is a solution.


