Virtual Quantum Subsystems

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The physical resources available to access and manipulate the degrees of freedom of a quantum system define the set of operationally relevant observables. The algebraic structure of a preferred tensor product structure i.e., a partition into subsystems. The notion of compoundness for quantum system is accordingly relativized. Universal control over virtual subsystems can be achieved by using quantum noncommutative holonomies

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In the last few years we witnessed a strong reviving of the interest about the notion of quantum entanglement [1]. This is mainly due to the essential role that such a concept is supposed to play in quantum information processing (QIP) [2]. Whenever one has a compounded (or multi-partite) quantum system, in the space of admissible states there exist states which display uniquely quantum correlations. These states are referred to as entangled and correspond algebraically to the existence, in a vector space obtained by a tensor product, of vectors $|\psi\rangle$ that are not expressible by a simple product e.g., $|\psi_1\rangle \otimes |\psi_2\rangle$.

Given a physical system $S$, the way to subdivide it in subsystems is in general by no means unique. On the contrary it is a widespread praxis in theoretical physics as well as in everyday life to consider different partitions into subsystems in dependence of both the physical regime and the the necessities of the description. It is indeed a quite common experience to refer sometimes to a system e.g., an atom, as elementary and sometime as composite e.g., made out electrons and nucleons. The emergence of a distinguished multi-partite structure is strongly dependent of the physical regime e.g., the energy-scale, at which one is working and on the set of observations (experiments) the observer is interested in. This is of course a well-known lesson from history of physics e.g., fundamental vs composite particles, weak-strong coupling dualities, renormalization group etc.

Clearly even the notion of entanglement is affected by some ambiguity being relative to the selected multi-partite structure. States that are entangled with respect to a given partition in subsystems can be separable with respect to another. Or the other way around: states of a system $S$ that is regarded as elementary can be viewed as entangled once $S$ is endowed with a multi-partite structure. In this case one is in the, somehow paradoxical, situation of having entanglement seemingly without entanglement.

The above ambiguity is removed as soon as, according to some criterion, a preferred multi-partite structure is selected among the family of all possible partition into subsystems. This selection has in most cases a well defined meaning: the system $S$ is viewed as composed by $S_1, S_2, \ldots$ if one has some operational access (is able to "access", "control", "measure") the individual degrees of freedom of $S_1, S_2, \ldots$. In other terms it is the set of "available" interactions that individuates the relevant multi-party decomposition and not an a priori, god-given partition into elementary subsystems. In this letter we shall make an attempt towards a formalization of the ideas brought about by these simple remarks. Our final goal is to provide a satisfactory algebraic definition of what a quantum subsystem is in an operationally motivated framework.

Let us stress that the notion of virtual subsystem that we shall introduce admits as a particular instance the one of quantum code [3], [4] and noiseless subsystems [5], [6]. This remark should make clear that virtual subsystems already play an important role in QIP. In particular error avoiding quantum codes i.e., decoherence-free [4] have been recently also experimentally observed [7], [8].

Compoundness and tensor products. Let us begin by recalling the basic algebraic structures associated with compoundness. Let $S_1$ and $S_2$ be two classical systems with configuration manifolds $M_i$ ($i = 1, 2$). Roughly speaking the associated quantum systems have state-spaces given by $\mathcal{H}_i = \mathcal{F}(M_i)$ where $\mathcal{F}$ denotes some suitable (complex-valued) function space over the $M_i$'s e.g., $L^2$-summable functions. Notice that these spaces (actually abelian $C^*$-algebras [9]) are the classical "observable" spaces; the quantum ones are given by the operator (non-abelian) algebras $\text{End}(\mathcal{H}_i)$. In the classical realm the manifold associated with the joint systems $S_1 \vee S_2$ is given by the cartesian product $M_1 \times M_2$. It follows that at the quantum level one has $\mathcal{H}_{S_1 \vee S_2} = \mathcal{H}_1 \otimes \mathcal{H}_2$ indeed $\mathcal{F}(M_1 \times M_2)$ is given by a suitable closure of $\mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$. This basic functorial identity is the algebraic ground for the quantum theory axiom associating to a bi-partite systems a state-space given by the tensor products of the state-spaces describing the subsystems. The extension to $N$-partite systems is obvious. One has another elementary, yet remarkable,
functorial relation given by the canonical isomorphism \( \text{End}(\mathcal{H}_1 \otimes \mathcal{H}_2) \cong \text{End}(\mathcal{H}_1) \otimes \text{End}(\mathcal{H}_2) \). Even in the quantum realm the observable algebra associated with a joint systems is given by the tensor product of the subsystem subalgebras.

Our key observation is that different kinds of compoundness can emerge in the same system when one considers different sets of observables as the physical ones. Indeed quite often it makes sense to refer to a subalgebra \( \mathcal{A} \) (rather than the full operator algebra) as to the physical observable algebra. Limitations of physical resources may lead to select a specific class of operators to be considered as realizable. For instance energy supply limitations lead naturally to restrict to operators \( X \) which have vanishing matrix elements between energy eigenstates whose energy difference exceeds some bound \( E \). At the dynamical-algebraic level the selection of a particular multi-party decomposition means that the algebra of (operationally relevant) observables \( \mathcal{A} \) has a tensor product structure (TPS) i.e., \( \mathcal{A} \cong \otimes \mathcal{A}_i \) such that all the observables belonging to the individual \( \mathcal{A}_i \)'s can be effectively implemented.

Before passing to general constructions it is useful to consider a very simple example in which one has a set of subsystems (degrees of freedom) associated with a rapidly growing sequence of energy scales. Starting from the ground state and increasing the energy available one is able to excite more and more subsystems that at lower energy were frozen. This situation is realized, for instance, in systems in which one has confined directions or in the cases in which an adiabatic decoupling between fast and slow degrees of freedom has been performed: the effective dimensionality of the system is a function of the energy scale.

The TPS manifold. Let us consider an Hilbert space \( \mathcal{H} \cong \mathcal{H}^n \) with a priori no tensor product structure. A first very natural question is how many inequivalent TPS’s can be assigned over \( \mathcal{H} \)? Or more physically: in how many different ways \( \mathcal{H} \) can be viewed as the state-space of a multipartite quantum system? If \( n \) is a prime number there are no possibilities: the system is elementary. If \( n \) is not prime it has a non-trivial prime factorization: \( n = \prod_{i=1}^s p_i^{n_i} \) (\( p_i < p_{i+1} \)). If the exponent \( n_i \) of the \( i \)-th prime factor of \( n \) is not one then several regroupings are possible e.g., \( r = 1, p_1 = 2, n_1 = 3 \Rightarrow 3 = 1 + 1 + 1 = 1 + 2 \) corresponding to the state-space factorizations \( \mathcal{H}^n \cong \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \) and \( \mathcal{H}^n \cong \mathcal{H}^2 \otimes \mathcal{H}^4 \). When more than one \( p_i \) appear in the decomposition of \( n \) we see that many other possibilities of writing \( n \) as a product of integers arise. In general, given \( n \), we introduce the set of factorizations \( \mathcal{P}_n = \{ P \subset \mathbb{N} / \prod_{m \in P} m = n \} \) where \( \mathbb{N} \) denotes the set of natural numbers.

Given a factorization \( P = \{ n_1 \leq n_2 \leq \ldots \leq n_{|P|} \} \in \mathcal{P}_n \) of \( n \) is assigned one has the (non-canonical) isomorphisms \( \varphi: \mathcal{H} \mapsto \otimes_{j=1}^{|P|} \mathcal{H}^j \). In the following such isomorphisms will be referred to as tensor product structures (TPS) over \( \mathcal{H} \), and subsystems of the associated multi-party decomposition as virtual.

Given a distinguished TPS, say \( \varphi_0 \), one can identify the group of unitaries \( \mathcal{U}(\mathcal{H}) \) and \( \mathcal{U}(\otimes \mathcal{H}^j) \) via the algebra isomorphism \( U \mapsto \varphi_0^{-1} \circ U \circ \varphi_0 \). A suitable quotient of this latter unitary group parametrizes the space of inequivalent TPS’s. Indeed two elements \( U \) and \( W \) of \( \mathcal{U}(\otimes \mathcal{H}^j) \) define equivalent TPS’s if either \( U = U_1 W U_2 \) where the \( U_i \)'s are multi-local transformation i.e., \( U_i \in \prod_{k=1}^{n_i} \mathcal{A}_i \) (\( i = 1, 2 \)); or the \( U_i \)'s are permutations of factors with equal dimension. In the first case the TPS’s differ just by a change of the basis in each factor, in the second by the order of the factors that in turn amounts simply to a relabelling of the subsystems. The space of inequivalent TPS’s over \( \mathcal{H}^n \) will be denoted by \( \mathcal{T}_n \).

Once a given multiplicative partition \( (n_i) \) of \( n \) is chosen along with a particular \( \varphi \) one has \( \mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i \), \( \mathcal{H}_i := \varphi(\mathcal{H}^j) \) then \( \text{End}(\mathcal{H}) \cong \otimes_{i=1}^n \text{End}(\mathcal{H}_i) \). For any set of unitaries in \( \mathcal{H} \) labelled by the elements \( \lambda \) of some manifold \( \mathcal{M} \), e.g., external fields, one can define \( \mathcal{A}_i(\lambda) := U_\lambda \mathcal{A}_i U_\lambda^\dagger \) (\( i = 1, \ldots, N \)) that describes a family of multipartite structures over \( \mathcal{H} \) parametrized by points of \( \mathcal{M} \). As noticed above not all the points of \( \mathcal{M} \) necessarily correspond to different TPS’s. Indeed it can happen that different \( \lambda \)'s can result in the same structure e.g., \( U_\lambda \mathcal{A}_i U_\lambda^\dagger = \mathcal{A}_i \). If a state is entangled (product) with respect to a TPS labelled by \( \lambda \in \mathcal{T}_n \) it will be referred to as \( \lambda \)-entangled (\( \lambda \)-product).

If \( E: \mathcal{H} \mapsto \mathbb{R}_0^+ \) denotes an entanglement measure over \( \mathcal{H} \) with respect to a given TPS, say \( \lambda = 0 \), one has that \( E_\lambda := E \circ U_\lambda \) is a \( \lambda \)-entanglement measure. In turn the latter provides a natural measure of the "distance" between the TPS at \( \lambda \neq 0 \) and that at \( \lambda = 0 \). Indeed it appears quite natural to say that the more the \( \lambda = 0 \) product states are \( \lambda \)-entangled the more the TPS at \( \lambda \) differs from the one at the origin. To make this idea quantitative one has to make it independent on the particular state; this can be done either by maximizing or by taking the average over all the 0-product states. In this latter case one finds that the distance one is looking for is nothing but the (square root of) entangling power of \( U_\lambda \) \cite{10}: \( e(U) = \int d\psi_1 d\psi_2 E(U |\psi_1\rangle \otimes |\psi_2\rangle) \). Here the integral is done with respect to the uniform e.g., Haar, measure over the pure product state manifold.

In order to exemplify the notion of TPS manifold of we now introduce a family of TPS’s over an infinite dimensional state-space parametrized by a group of \( N \times N \) matrices. Let us consider \( N \) harmonic oscillators. The global state-space is given by \( \mathcal{H}_N := \otimes_{i=1}^N \mathcal{H}_i \) where each of the factors is the single boson Fock space i.e., \( \mathcal{H}_i = \text{span} \{ |n_i\rangle \} \in \mathbb{N} \) associated with the annihilation and creation operators \( a_i \) and \( a_i^\dagger \). Let \( U \in \mathcal{U}(N) \) be a complex \( N \times N \) unitary matrix. The
operators \( a^i_U := \sum_{j=1}^N U_{ij} a_j \) (\( i = 1, \ldots, N \)) represents new bosonic modes i.e., \([a^i_U, a^j_U] = \delta_{ij}, [a^i_U, \dagger a^j_U] = 0\), moreover one has \( \mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}^U_i \) where the \( \mathcal{H}^U_i \)'s are the Fock spaces associated with the \( a^i_U \)'s. Notice that the Fock vacuum \( |0\rangle := \otimes_i |0\rangle_i \) is \( U \) independent i.e., \( a^i_U |0\rangle = 0 (\forall U, j) \). One has \( \mathcal{H}^U_j \cong \mathcal{A}^U_j |0\rangle \) where \( \mathcal{A}^U_j \) is the algebra generated by \( a^i_{U \dagger} \) and \( a^i_{U \dagger \dagger} \). States like \( a^i_{U \dagger} |0\rangle \) are unentangled with respect to the TPS defined by the given \( U \) but entangled with respect to the one associated with e.g., \( U = 1 \).

**Virtual bi-partitions.** Now we address the following issue: when is it legitimate to consider a pair of observable algebras as describing a bi-partite quantum system? Suppose that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are two commuting subalgebras of \( \mathcal{A} := \text{End}(\mathcal{H}) \) such that the subalgebra \( \mathcal{A}_1 \vee \mathcal{A}_2 \) they generate i.e., the minimal \( * \)-subalgebra containing both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), amounts to the whole \( \mathcal{A} \) and moreover one has the (non-canonical) algebra isomorphism

\[
\mathcal{A}_1 \vee \mathcal{A}_2 \cong \mathcal{A}_1 \otimes \mathcal{A}_2.
\]

(1)

The standard, genuinely bi-partite, situation is of course \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{A}_1 = \text{End}(\mathcal{H}_1) \otimes \mathbb{I}, \mathcal{A}_2 = \mathbb{I} \otimes \text{End}(\mathcal{H}_2). \) If \( \mathcal{A}'_1 := \{X \mid [X, \mathcal{A}_1] = 0\} \) denotes the commutant of \( \mathcal{A}_1 \), in this case one has that \( \mathcal{A}'_1 = \mathcal{A}_2 \).

It is important to mention that a prototypical and ubiquitous situation described by Eq. (1) is when \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are local observable algebras associated to disjoint regions of space at equal time. More generally such an independence of local degrees of freedom e.g., quantum fields, is encoded in terms of commutativity between observables supported on causally disconnected domains [11]. Notice also the spatial separation between parties e.g., Alice and Bob, is a common assumption in protocols for quantum communication e.g., teleportation [2].

The point of view advocated in this letter is to consider condition (1) as the definition of bi-partite system, regardless the "real" compoundness or not of the underlying state-space. Accordingly we shall consider as "real" entanglement the one occurring in that case. The (nearly obvious) point is that: in order to computational advantage from this virtual entanglement one must have access to i.e., to be able to control the subalgebras \( \mathcal{A}_1, \mathcal{A}_2 \). As far as the operations in \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are easily realizable (accessible) in the lab we shall consider them as primitive and local, regardless how they look at the original level.

The theory of *noiseless subsystems* [5], [6] provides an important exemplification as well as source of inspiration for the approach to compoundness advocated here. Let us consider a system made of \( N \) "real" subsystems e.g., qubits. Suppose that the algebra of relevant interactions is given by \( \mathcal{A}_1 \cup \mathcal{A}'_1 \) where \( \mathcal{A}_1 \) [9]

\[
\mathcal{A}_1 \cong \bigoplus_{j} \mathbb{I} \otimes M_{d_j}(\mathbb{C})
\]

This decomposition reads at the state-space level as \( \mathcal{H} \cong \bigoplus_{j} \mathbb{C}^{n_j} \otimes \mathbb{C}^{d_j} \). For a fixed label \( \mathcal{J} \) one has that the elements of \( \mathcal{A}_1 (\mathcal{A}'_1) \) act as the identity on the \( \mathbb{C}^{n_j} (\mathbb{C}^{d_j}) \) factor. This means that the system is viewed, for all practical purposes, as a bipartite one, in which the observables of the first (second) subsystems are given by \( \mathcal{A}_1 (\mathcal{A}'_1) \). For collective decoherence, \( \mathcal{A}_1 \) is the interaction algebra generated by couplings with the environment invariant under qubit permutations. While \( \mathcal{A}'_1 \) is given by any linear combinations of permutation operators [6]. In particular the latter algebra is generated by exchange i.e., Heisenberg like operators between the different pairs of qubits [12].

Generally speaking Eq. (2) shows in which sense an observable algebra \( \mathcal{A}_1 (\mathcal{A}'_1) \) is associated with a collection of virtual subsystems, i.e., the \( \mathbb{C}^{n_j} (\mathbb{C}^{d_j}) \) factors, labelled by its spectrum. It is worth to observe that when \( \mathcal{A}_1 \) is abelian all the \( d_j \)'s are equal to one. In this case, if \( n > 1 \), the \( J \)-th factor of the state-space decomposition describes a sort of hybrid bi-partite system in which one of the factors is quantum whereas the other represents a classical system with a one-point configuration space. This is exactly the situation one meets in the case of quantum codes, both error correcting [3] and error avoiding [4]. In this latter case the algebra \( \mathcal{A}_1 \) is generated by the operators coupling the computing system with its environment and \( \mathcal{A}'_1 \) is the set of interactions necessary to perform computations entirely within the decoherence-free sector [6].

To make clear the connection with quantum error correction let us consider a set \( \{X_i\}_{i=1}^k \) of \( k \leq n \) linear independent traceless "parity" operators over \( \mathcal{H} \cong (\mathbb{C}^2)^{\otimes n} \), such that \( X_i = X_i^1 \otimes X_i^2 = \mathbb{I}, [X_i, X_j] = 0, (i, j = 1, \ldots, k) \). Following standard arguments of quantum error correction [3] one can show that the \( X_i \)'s generate an abelian algebra \( \mathcal{A} \cong \mathbb{C} \mathbb{Z}_2^k \). The associated state-space decomposition is given by

\[
\mathcal{H} \cong \bigotimes_{j} \mathbb{C}^{2^{n_j-k}} \otimes \mathbb{C} \cong \mathbb{C}^{2^{n_j-k}} \otimes \mathbb{C}^{2^k}.
\]

(3)

It easy to see that the commutant of \( \mathcal{A} \) contains the algebra of operators over the first factor in the decomposition above This means that the set of operators with well-defined parities defines and controls a virtual subsystem of \( n - k \) bits. Analogously the set of "odd" operators \( \{O \mid \exists i \{X_i, O\} = 0\} \) defines and controls the second \( k \)-qubit subsystem. For instance the parity \( X_1 := \sigma_z \otimes \mathbb{I} \) defines the natural bi-partite structure over \( (\mathbb{C}^2)^{\otimes 2} \) whereas \( X_1^1 = \sigma_z^2 \) defines TPS such that states like 2\(^{-1/2}\)(|00⟩ ± |11⟩) are unentangled. Notice that in error correction theory the first (second) subsystem is related to the code (syndrome). For any unitary \( U \), the operators \( X(U) := U X_i U^\dagger \) span an algebra isomorphic to \( \mathcal{A} \) above. Again one has a continuous set of TPS’s parametrized by points of a unitary group [13].

Turning back to the characterization of pairs of (finite-dimensional) subalgebras satisfying Eq. (1) by using Eq. (2) it is easy to prove the following [14]
Proposition Let $A_1$ and $A_2$ be two commuting *-subalgebras of a finite dimensional *-algebra $A$. Necessary and sufficient condition for the validity of (1) is that $A_1 \cap A_2 = C \mathbb{1}$ i.e., $A_1$ is a factor.

Holonomic control on subsystems. In this paragraph we show that the Holonomic approach to QC [15] provides a natural setting for the issue of information processing within a (virtual) subsystem.

Let $X \in \text{End}(\mathcal{H}) \cong M_{nd}(\mathbb{C})$ be an hermitean operator with a spectrum of $d$ iso-degenerate eigenvalues i.e.,

$$X = \sum_{i=1}^{d} \pi_i \sum_{n=1}^{d} |ki\rangle \langle ki|,$$

and $\{U_{\lambda}\}_{\lambda \in \mathcal{M}} \subset U(\mathcal{H})$ a set of unitaries parametrized by the point of some (control) manifold $\mathcal{M}$. Then the set of $X(\lambda) := U_{\lambda} X U_{\lambda}^\dagger$, is a family that in the generic case, for sufficiently large $D = \text{dim} \mathcal{M}$ satisfies the conditions for (universal) holonomic quantum computation [15] on the $n$-dimensional degenerate eigenspace $C_i = \text{span} \{|ki\rangle\}_{k=1}^{n} \cong \mathbb{C}^n \otimes |i\rangle$, $(i = 1, \ldots, d)$ of $1_n \otimes X$. This implies that the holonomy group $\text{Hol}(A_i)$ associated with the connection $u(n)$-valued 1-forms $A_{i}^{ab} = \langle a \otimes |i\rangle U_{\lambda}^{\dagger} dU_{\lambda} |b\rangle \otimes |i\rangle$, $d := \sum_{i=1}^{d} d\lambda_\mu \partial_\mu$, $(a,b = 1, \ldots, n)$ is the whole $U(C_i) \cong U(n) \otimes |i\rangle \langle i|$. By denoting collectively with $A$ the set of the $A_i$’s one can therefore write that

$$\text{Hol}(A) \cong \oplus_{i=1}^{d} U(n) \otimes |i\rangle \langle i| \supset U(n) \otimes \mathbb{1}_d. \quad (4)$$

The last inclusion tells us that in that in the generic case the holonomy group of $A$ will contain the whole unitary group of the $\mathbb{C}^n$ subsystem. Once the holonomic family $\{X(\lambda)\}_\lambda$ is given, any transformation i.e., computation in the first subsystem can be generated holonomies. Notice that, since for real quantum case one must have $n \geq 2$, the holonomy group is necessarily nonabelian.

Conclusions. We analyzed some the consequences of the non-uniqueness of the decomposition of a given system $S$ into subsystems. Such non-uniqueness implies, at the quantum level, a fundamental ambiguity about the very notion of entanglement that accordingly becomes a relative one. One can parametrize the space of all possible partitions i.e, tensor product structures, of a $n$-dimensional quantum state-space by the points of a set $\mathcal{T}_n$. The fact of considering all the points in $\mathcal{T}_n$ on the same footing (that amounts to establishing a democracy between different TPS’s) provide a relativization of the notion of entanglement. Without further physical assumption, no partition has an ontologically superior status with respect to any other. The subsystems associated with all these possible i.e, potential multi-party decomposition were referred to as virtual. A distinguished point of $\mathcal{T}_n$ is selected i.e., made actual only once the relevant algebra $A$ of “physical” observables is given. Indeed considering a given partition as the privileged has a strong operational meaning, in that it depends on the set of resources effectively available to access and to control the degrees of freedom of $S$. Different sets of resources give rise to different partitions physically relevant. We provided several examples of natural, though hidden, multi-partite structures arising from the given algebraic structure of $A$. We briefly showed that the holonomic approach to quantum computation provides one natural way to address the issue of controllability within virtual subsystems. We believe that this democratic approach to quantum compoundness is, on the one hand, sound from the conceptual point of view, and on the other hand possibly relevant to QIP.

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[12] Let us consider two qubits and the family by $U_\lambda := \exp(\lambda \sigma_z) = \cos \lambda \mathbb{1} + i \sin \lambda \sigma_z$, where $\sigma_z |\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. If $X_1 := \sigma_z \otimes \mathbb{1}$ one has $X_1(U_\lambda) = \cos^2 \lambda X_1 + \sin^2 \lambda \mathbb{1} \otimes \sigma_z + i/2 \sin 2\lambda [S, X_1]$. Clearly $X_1(\pi/2) = \mathbb{1} \otimes \sigma_z$: the TPS associated with $\lambda = \pi/2$ amounts just to the exchange between the two subsystems. All the points of $[0, \pi/2]$ correspond to inequivalent TPSs.

[13] We can assume $A_2 = A_1$. If $A_1$ is a factor one has $A_1 \cong 1_n \otimes M_2(\mathbb{C})$ and then $A_2 = M_2(\mathbb{C}) \otimes 1_n$. There follows that $A_1 \vee A_2 \cong M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = A_1 \otimes A_2$. When $A_1$ is not Eq. (2) implies $A_2 \subset \oplus_{i} M_{n_i}(\mathbb{C}) \otimes 1_n$, then $A_1 \vee A_2 \cong \oplus_{i} \bigotimes M_{n_i}(\mathbb{C}) \otimes M_{n_i}(\mathbb{C})$. A comparison of the dimension of this latter algebra with that of $A_1 \otimes A_2$ completes the proof.