Can Schwarzchildian gravitational fields suppress gravitational waves?

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Abstract

Gravitational waves in the linear approximation propagate in the Schwarzschild spacetime similarly as electromagnetic waves. A fraction of the radiation scatters off the curvature of the geometry. The energy of the backscattered part of an initially outgoing pulse of the quadrupole gravitational radiation is estimated by compact formulas depending on the initial energy, the Schwarzschild radius, and the location and width of the pulse. The backscatter becomes negligible in the short wavelength regime.
I. INTRODUCTION

Backscattering has been investigated for a long time for various wave equations (see, for instance, [1]). In general relativity, this topic has been studied since early 1960’s ([2], [3]). This paper continues the programme that started with the study of the backscatter of scalar [4] and electromagnetic fields ([5] and [6]). Here we investigate the propagation of even-parity gravitational waves in a (fixed) background Schwarzschild spacetime, assuming a nonstationary source. The discussion, however, is done without any reference to the source. We only deal with field quantities. It is assumed that initial data are those of an isolated pulse (burst) of a gravitational wave. The main question that is answered is what fraction of the initially outgoing radiation may undergo backscattering before reaching null infinity? The strength of the backscattering is assessed by bounding the fraction of the initial burst energy that will not reach a distant observer in the main pulse.

The even-parity waves are the only waves which are radiated during the axisymmetric collision of nonspinning black holes [9], and since in this case the Schwarzschild spacetime is a valid starting point for an approximation scheme, it gives us an opportunity to bound the strength of the phenomenon in a fairly realistic astrophysical context.

The following five sections of the paper give a theoretical description of the backscattering effect. The Sec. II brings notation and the Zerilli equation. In Sec. III, the initial data are bounded by the initial energy and the solution is sought in the form of superposition of an outgoing radiation (defined by initial data) and a backscattered term. The evolution of the backscattered term can be bounded by solutions of two differential inequalities. The bounds that are derived in Sec. IV and in the Appendix deal with a general situation; no assumption is made about the initial radiation. In Sec. V, we discuss initial data that are of compact support and in addition, the relative width of the support is small. Such data correspond to radiation that is dominated by short wavelengths. In this case stronger estimates are derived. They imply that in the limit of short wavelengths (relative width of the support tending to zero) the backscattering effect becomes negligible. In Sec. VI, the ”small relative
width condition” of Sec. V is supplemented by the assumption that the initial burst is far away from the horizon.

II. FORMALISM

The space-time geometry is defined by a Schwarzschildian line element,

\[ ds^2 = -(1 - \frac{2m}{R})dt^2 + \frac{1}{1 - \frac{2m}{R}}dR^2 + R^2 d\Omega^2 , \]  

(2.1)

where \( t \) is a time coordinate, \( R \) is a radial coordinate that coincides with the areal radius and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the line element on the unit sphere, \( 0 \leq \phi < 2\pi \) and \( 0 \leq \theta \leq \pi \). Throughout this paper \( G \), the Newtonian gravitational constant, and \( c \), the velocity of light are put equal to 1.

As explained in the Introduction, we restrict ourselves, to the even-parity axial perturbations. Their propagation is ruled, in the linear approximation, by the Zerilli equation [7]. Formulated in terms of the gauge-invariant amplitude \( \Psi \) defined by Moncrief [8], this equation reads, in the case of \( l = 2 \) multipole [9],

\[ (-\partial_t^2 + \partial_\ast^2)\Psi = V\Psi, \]  

(2.2)

where the potential \( V \) is given by

\[ V(R) = 6(1 - \frac{2m}{R})^2 \frac{1}{R^2} + (1 - \frac{2m}{R}) \frac{63m^2}{2R^4}(1 + \frac{m}{R})^2, \]  

(2.3)

and where

\[ r^\ast = R + 2m \ln(\frac{R}{2m} - 1) \]  

(2.4)

is the tortoise radial coordinate.

Consider a set of functions \( \Psi_i(r^\ast - t) \), \( i = 0, 1, 2 \), that satisfy the following linear relations

\[ \partial_t \Psi_1 = 3\Psi_0, \]  

\[ \partial_t \Psi_2 = \Psi_1 - m\partial_t \Psi_1. \]  

(2.5)
The combination

\[ \tilde{\Psi} \equiv \Psi_0(r^* - t) + \frac{\Psi_1(r^* - t)}{R} + \frac{\Psi_2(r^* - t)}{R^2} \]

(2.6)
solves Eq. (2.2) in Minkowski space-time \((m = 0)\); it represents purely outgoing radiation.

Let the initial data of a solution \(\Psi\) of (2.2) coincide with \(\tilde{\Psi}\) at \(t = 0\). Then, initially, \(\Psi\) represents a purely outgoing wave. It should be noted that the assumption that initial data are (initially) purely outgoing is made in this paper only for the sake of clear presentation. In the linear approximation the propagation of the initially outgoing radiation is independent of whether or not ingoing radiation is present.

We decompose the sought solution \(\Psi(r^*, t)\) into the known part \(\tilde{\Psi}\) and an unknown function \(\delta\)

\[ \Psi = \tilde{\Psi} + \delta. \]

(2.7)

Due to the choice of the initial data made above one has \(\delta = \partial_t \delta = 0\), at \(t = 0\).

III. ENERGY ESTIMATES

Let us assume that the quadrupole initial data are defined by a smooth triad of the functions \(\Psi_k\) \((k = 0, 1, 2)\) with the initial support \([a, b]\) \((b < \infty)\). That guarantees that the initial energy density multiplied by \(R^2\), 

\[ \rho = \left( (\partial_t \Psi)^2 + (\partial_r \Psi)^2 + V \Psi^2 \right) / \eta_R, \]

is smooth and vanishes on the boundary \(a\). Here \(\eta_R = 1 - 2m/R\) holds.

The energy content inside a part of a Cauchy hypersurface \(\Sigma_t\) that is exterior to a ball of a radius \(R\) can be defined as 

\[ E(R, t) \equiv \int_{R}^{\infty} dr \rho(r, t). \]

Let us point out that in order to ensure a proper normalization of the energy flux at infinity, there should be a normalization constant in the definition of the energy \(E(R, t)\). We decided to omit it, since later on we shall be interested only in the relative efficiency of the backscatter and thus the normalization factor cancels out. The total initial energy corresponding to the hitherto defined initial data is equal to \(E(a, 0)\).
Lemma 1. Defining

\[ C_1 \equiv \frac{3}{2} \sqrt{(2 + \sqrt{2/3})E(a, 0)}, \]
\[ C_2 \equiv \sqrt{2E(a, 0)}, \]
\[ C_3 \equiv \frac{2E(a, 0)}{\eta_a(2\sqrt{6} + 1)}. \]

(3.1)

and introducing the two nonnegative functions

\[ g_1(R) = \ln \left( -\frac{2m + R}{a - 2m} \right) + 32m^5 \left( \frac{-1}{5(-2m + R)^5} + \frac{1}{5(-2m + a)^5} \right) + \]
\[ 20m^4 \left( \frac{-1}{(-2m + R)^4} + \frac{1}{(-2m + a)^4} \right) + 80m^3 \left( \frac{-1}{3(-2m + R)^3} + \frac{1}{3(-2m + a)^3} \right) + \]
\[ 20m^2 \left( \frac{-1}{(-2m + R)^2} + \frac{1}{(-2m + a)^2} \right) + 10m \left( \frac{-1}{-2m + R} + \frac{1}{-2m + a} \right) \]

(3.2)

and

\[ g_2(R) = R - a + 16m^4 \left( \frac{-1}{3(-2m + R)^3} + \frac{1}{3(-2m + a)^3} \right) + \]
\[ (16m^3) \left( \frac{-1}{(-2m + R)^2} + \frac{1}{(-2m + a)^2} \right) + \]
\[ (24m^2) \left( \frac{-1}{-2m + R} + \frac{1}{-2m + a} \right) + 8m \ln \left( \frac{-2m + R}{-2m + a} \right), \]

(3.3)

the following inequalities hold at \( t = 0 \) and for \( R \geq a \):

\[ \frac{|\Psi_1(R)|}{R^{3/2}} \leq C_1 \eta_R^{3/2} \sqrt{g_1(R)}, \]
\[ \frac{|\Psi_2(R)|}{\sqrt{\eta_R R^2}} \leq C_2 \sqrt{g_2(R)} + C_1 \frac{6m}{\sqrt{a}} \sqrt{g_1(R)} \left( 1 - \sqrt{\frac{a}{R}} \right) + \]
\[ 6C_3 \frac{m}{\sqrt{a}} \sqrt{1 - \left( \frac{a}{R} \right)^{2\sqrt{6}+1}} \left( \frac{1}{\sqrt{\eta_a}} - \frac{1}{\sqrt{\frac{R}{a} - \frac{2m}{a}}} \right), \]
\[ \frac{|\tilde{\Psi}(R)|}{\sqrt{R}} \leq C_3 \sqrt{1 - \left( \frac{a}{R} \right)^{2\sqrt{6}+1}}, \]
\[ \frac{|\tilde{\Psi}_0(R)|}{\sqrt{R}} \leq C_3 \sqrt{1 - \left( \frac{a}{R} \right)^{2\sqrt{6}+1}} + \]
\[ C_1 \eta_R^{3/2} \sqrt{g_1(R)} + C_2 \frac{g_2(R)}{R} + C_1 \frac{6m}{\sqrt{aR}} \sqrt{g_1(R)} \left( 1 - \sqrt{\frac{a}{R}} \right) + \]
\[ 6C_3 \frac{m}{\sqrt{aR}} \sqrt{1 - \left( \frac{a}{R} \right)^{2\sqrt{6}+1}} \left( \frac{1}{\sqrt{\eta_a}} - \frac{1}{\sqrt{\frac{R}{a} - \frac{2m}{a}}} \right). \]

(3.4)
Proof.

One can explicitly verify that
\[
-\eta_R \left( \frac{\Psi_1}{R^2} + 2 \frac{\Psi_2}{R^3} \right) = \partial_t \Psi + \partial_r \Psi,
\]
\[
\eta_R \left( 2 \frac{\Psi_0}{R} + \frac{\Psi_1}{R^2} \right) = \partial_t \tilde{\Psi} + \partial_r \tilde{\Psi} + \frac{2 \eta_R \tilde{\Psi}}{R}.
\]
(3.5)

The equations (3.5), using the relations (2.6), result in
\[
\partial_R \left( \frac{\Psi_1}{R^{3/2}} \right) = -\frac{3}{2R^{1/2} \eta_R^{3/2}} \left( \frac{\partial_t \Psi}{\eta_R^{1/2}} + \sqrt{\eta_R \partial_R \Psi} + \frac{2 \tilde{\Psi} \sqrt{\eta_R}}{R} \right).
\]
\[
\partial_R \left( \frac{\Psi_2}{R^2} \right) = \frac{1}{\eta_R} \left( \frac{\partial_t \Psi}{\eta_R^{1/2}} + \sqrt{\eta_R \partial_R \Psi} \right) + \frac{3m}{R \eta_R^{3/2}} \left( \frac{\tilde{\Psi} - \Psi_1}{R} \right).
\]
(3.6)

The integration from \( a \) to \( R \) and the use of the Schwarz inequality yields
\[
\frac{|\Psi_1(R)|}{R^{3/2}} \leq \frac{3 \eta_R^{3/2}}{2} \sqrt{(2 + \sqrt{2/3}) E(a, 0) \left( \int_a^R dr \frac{1}{r \eta_r^2} \right)^{1/2}}.
\]
(3.7)

Integrating \( \int_a^R dr \frac{1}{r \eta_r^2} \), one immediately arrives at the first of the postulated inequalities.

In order to show the third inequality, notice that \(|\tilde{\Psi}(R)| R^{\sqrt{6}} = |\int_a^R dr \partial_r (\tilde{\Psi}(r) r^{\sqrt{6}})|\). The latter expression is bounded from above, using the Schwarz inequality, by
\[
\sqrt{2 \int_a^R dr \left( \eta_r (\partial_r \tilde{\Psi})^2 + 6 \eta_r \tilde{\Psi}^2 / r^2 \right)} \sqrt{\int_a^R dr \eta_r^{-1} r^{2 \sqrt{6}}} \leq \frac{2 E(a, 0)}{\eta_a (2 \sqrt{6} + 1)} R^{\sqrt{6} + 0.5} \left( 1 - \left( \frac{a}{R} \right)^{\sqrt{6} + 1} \right)^{1/2},
\]
(3.8)

where the inequality in Eq. (3.8) follows from the monotonicity of the energy as function of \( R \). The first factor on the left hand side of this inequality is not greater than \( \sqrt{2 E(a, 0)} \) since \( 6 \eta_r / r^2 \leq V(r) / \eta_r \). The replacement of \( \eta_r^{-1} \) by \( \eta_a^{-1} \) and the integration of the other factor leads to the desired result.

The second of the equations of (3.6) can be integrated. The Schwarz inequality and direct integration as well as the \( \tilde{\Psi} \) and \( \Psi_1 \) estimates should be used in order to get the second inequality of Lemma 1. The \( \Psi_0 \)-estimate, in turn, follows from the identity \( \Psi_0 = \tilde{\Psi} - \Psi_1 / R - \Psi_2 / R^2 \) and the preceding estimates.
IV. THE ESTIMATE OF THE DIFFUSED ENERGY

Let us define the strength of the backscattered radiation that is directed inward by

\[ h_-(R, t) = \frac{1}{\eta_R} (\partial_t + \partial_{r^*}) \delta(R, t). \]  

(4.1)

Let the outgoing null geodesic \( \tilde{\Gamma}_{(R,t)} \) originate at \((R, t)\). If a point lies on the initial hypersurface, then we will write \( \tilde{\Gamma}_{(R,0)} \equiv \tilde{\Gamma}_R \). By \( \tilde{\Gamma}_{(R_0,t_0),(R,t)} \) will be understood a segment of \( \tilde{\Gamma}_{(R_0,t_0)} \) ending at \((R, t)\).

A straightforward calculation shows that the rate of the energy change along \( \tilde{\Gamma}_a \) is given by

\[ (\partial_t + \partial_{r^*}) E(R, t) = - \left[ \eta_R^2 h_+^2(R, t) + V \delta^2(a) \right]. \]  

(4.2)

It is necessary to point out that in the case of the initial point \( R_0 > a \) the result would be more complicated; the differentiation of the energy along \( \tilde{\Gamma}_{R_0} \) would depend also on \( \Psi_0, \Psi_1 \) and \( \Psi_2 \). If, however, the outgoing null geodesics is \( \tilde{\Gamma}_a \), then it starts from \( a \) where \( \Psi_0, \Psi_1 \) and \( \Psi_2 \) do vanish. Since these functions depend on the difference \( r^* - t \), their values along outgoing geodesics are constant, and that allows one to conclude that they vanish at \( \tilde{\Gamma}_a \).

The energy loss, that is the amount of energy that diffused inward \( \tilde{\Gamma}_a \) is equal to a line integral along \( \tilde{\Gamma}_a \),

\[ \delta E_a \equiv E(a, 0) - E_\infty = \int_a^\infty dr \left[ \eta_r h_+^2 + \frac{V \delta^2}{\eta_r} \right]. \]  

(4.3)

Our goal is to find an estimate of \( \delta E_a \) of a single pulse of radiation basing only on the information about the position and the energy of the initial pulse. Obviously, \( 0 \leq \delta E_a \leq E(a, 0) \) holds. We are interested in deriving in this section a frequency-independent bound, but later we obtain estimates that are frequency-sensitive.

\( \delta \) is initially zero and its evolution is governed by the following equation
\[-\partial_t^2 - \partial_r^2 \delta = V\delta + \left( V - 6\frac{\eta R}{R^2} \right) \left( \Psi_0 + \frac{\Psi_1}{R} + \frac{\Psi_2}{R^2} \right) + \]
\[\frac{2m\eta R}{R^4} \left[ -3\Psi_1 + 2\frac{\Psi_2}{R} \right]. \tag{4.4}\]

One can define an "energy" $H(R, t)$ of the field $\delta$ which is contained in the exterior of a sphere of radius $R$ as follows

\[H(R, t) = \int_R^\infty dr \left( \frac{(\partial_t \delta)^2}{\eta r} + \eta_r (\partial_r \delta)^2 + \delta^2 \frac{V}{\eta r} \right). \tag{4.5}\]

The rate of change of $H$ along $\tilde{\Gamma}_{(R,t)}$ is given by

\[(\partial_t + \partial_r^*) H(R, t) = \]
\[-\eta R \left[ \eta_R \left( \frac{\partial_t \delta}{\eta R} + \partial_R \delta \right)^2 + \frac{V}{\eta R} \delta^2 \right] - \]
\[4m \int_R^\infty dr \eta r \frac{\partial_t \delta}{r^4} \left[ -3\Psi_1 + 2\frac{\Psi_2}{r} + \frac{63m(1 + \frac{m}{2})}{4(1 + \frac{3m}{2r})^2} \left( \Psi_0 + \frac{\Psi_1}{r} + \frac{\Psi_2}{r^2} \right) \right] \leq \]
\[4m \int_R^\infty dr \eta r \frac{\partial_t \delta}{r^4} \left[ -3\Psi_1 + 2\frac{\Psi_2}{r} + \frac{63m(1 + \frac{m}{2})}{4(1 + \frac{3m}{2r})^2} \left( \Psi_0 + \frac{\Psi_1}{r} + \frac{\Psi_2}{r^2} \right) \right]. \tag{4.6}\]

Herein the inequality follows from the omission of the nonpositive boundary term. This allows one to estimate the maximal value $H_M$ of the $\delta$-energy $H$, namely

\[\sqrt{H_M} \leq 10.43 \frac{m\sqrt{E(a,0)}}{a} + O(m^2). \tag{4.7}\]

The calculation is essentially simple, but the algebra is quite lengthy and some numerical integrations are required. Details are relegated to the Appendix. We would like to point out that the $O(m^2)$ terms become dominant only when the location of the initial radiation pulse is smaller than 6.6$m$. At $a = 15m$ the neglected terms contribute much less than the leading term proportional to $m$.

Now, the integration of the first part of Eq. (4.6) along $\tilde{\Gamma}_{(a,0)}$ yields

\[H(\infty) - H(0) = \]
\[- \int_a^\infty dR \left[ \eta_R \left( \frac{\partial_t \delta}{\eta R} + \partial_R \delta \right)^2 + \frac{V}{\eta R} \delta^2 \right] - \]
\[\int_a^\infty dR 4m \int_R^\infty dr \eta r \frac{\partial_t \delta}{r^4} \left[ -3\Psi_1 + 2\frac{\Psi_2}{r} + \frac{63m(1 + \frac{m}{2})}{4(1 + \frac{3m}{2r})^2} \left( \Psi_0 + \frac{\Psi_1}{r} + \frac{\Psi_2}{r^2} \right) \right]. \tag{4.8}\]
Initially, $H$ vanishes (both $\delta$ and $\partial_t \delta$ vanish) and $H$ is manifestly nonnegative. The first integral on the right hand side of (4.8) is recognized to be just $\delta E_a$. The second integral in turn can be shown to be bounded - using the Schwarz inequality and then the results of the Appendix - by $2\sqrt{H_M} (10.43 \frac{m \sqrt{E(a,0)}}{a} + O(m^2))$. Thus, (4.8) implies
\[
\delta E_a \leq 2 \left( 10.43 \frac{m \sqrt{E(a,0)}}{a} + O(m^2) \right) \sqrt{H_M} \leq \left[ 54.5 \left( \frac{2m}{a} \right)^2 + O(m^3) \right] E(a,0); \quad (4.9)
\]
the right hand side of the first inequality achieves a maximal value when $H$ is maximal and that implies the second inequality.

Thus in summary, for the fraction of the energy that could diffuse through the null cone $C_a$, it holds:

**Theorem.** $\delta E_a/E(a,0)$ satisfies the inequality
\[
\frac{\delta E_a}{E(a,0)} \leq 54.5 \times \left( \frac{2m}{a} \right)^2 + O(m^3/a^3). \quad (4.10)
\]

We would like to point out that the above derivation is more efficient and simpler than the one used in [5] or [6] when $\delta E_a$ was estimated directly on the basis of the estimates of $\delta$ and $h_-$. This alternative approach would require a laborious integration of the field equation and the final estimate would be much worse.

**V. THE WAVELENGTH OF THE INITIAL RADIATION AND THE BACKSCATTER.**

In this section we shall consider the backscatter of the radiation that is initially of compact support and, in addition, the condition $(a-b)/a << 1$ is satisfied. The leading contribution - only terms that are quadratic in $m^2$ - will be found.

Under the above conditions one infers from Eq. (3.4) that on the initial hypersurface
\[
|\Psi_1(R)|b^{3/2} \leq C_1 b^{3/2} \sqrt{g_1(R)} \leq C_1 \sqrt{\frac{b-a}{a}} b^{3/2}\quad (5.1)
\]
and
\[ |\Psi_2(R)| \leq C_2 b^2 \sqrt{b - a} \quad (5.2) \]

are valid. With the same accuracy the inequality (8.3) of the Appendix reads

\[
(\partial_t + \partial_r) \sqrt{H(R, t)} \leq 2m \left( \int_R^{R(b)} dr \eta_r \frac{9 \Psi_1^2}{r^8} \right)^{1/2} + 2m \left( \int_R^{\infty} dr \frac{4 \Psi_2^2}{r^{10}} \right)^{1/2} \leq 6m C_1 \sqrt{\frac{b - a}{a} b^{3/2}} \left( \int_R^{R(b)} dr \frac{1}{r^8} \right)^{1/2} + 4m C_2 b^2 \sqrt{b - a} \left( \int_R^{R(b)} dr \frac{1}{r^{10}} \right)^{1/2}. \quad (5.3)
\]

Herein, the integration extends from \( R \), where \( R \in \tilde{\Gamma}_a \), to \( R(b) \), which is defined by \((R(b), t) \in \tilde{\Gamma}_b \). One has \( R(b) - R = b - a \) up to the term \( m^0 \). The integral \( \int_R^{R(b)} dr \frac{1}{r^8} \) is bounded from above by \((b - a)/R^8 \) and the integral \( \int_R^{R(b)} dr \frac{1}{r^{10}} \) is bounded from above by \((b - a)/R^{10} \), again to lowest order in powers of \( m \).

Thus one arrives at

\[
(\partial_t + \partial_r) \sqrt{H(R, t)} \leq 4m(b - a) b^{3/2} \left( \frac{1.5 C_1}{\sqrt{aR^4}} + \frac{C_2 \sqrt{a}}{R^3} \right). \quad (5.4)
\]

The integration of this inequality along the null geodesic \( \tilde{\Gamma}_a \) yields

\[
\sqrt{H_M} \leq m(b - a) \frac{b^{3/2}}{a^{7/2}} \left( 2C_1 + C_2 \right) + O(m^2). \quad (5.5)
\]

Taking into account the condition that \( b - a < a \), one arrives at

\[
H_M \leq \frac{4m^2}{a^2} \left( \frac{b - a}{a} \right)^2 \left( \frac{b}{a} \right)^3 \left( C_1 + \frac{C_2}{2} \right)^2 + O(m^3/b^3). \quad (5.6)
\]

Since the amount of backscattered energy \( \delta E_a \) is bounded from above by \( 2H_M \), as shown in the preceding section, one finally arrives at the following estimate

\[
\frac{\delta E_a}{E(a, 0)} \leq \frac{8m^2}{a^2} \left( \frac{b - a}{a} \right)^2 \times \left( \frac{b}{a} \right)^3 \left( \frac{3}{2} \sqrt{2} + \sqrt{\frac{2}{3} + \frac{1}{\sqrt{2}}} \right)^2 + O(m^3/a^3) \leq 84 \frac{m^2}{a^2} \left( \frac{b}{a} \right)^3 \left( \frac{b - a}{a} \right)^2 + O(m^3/a^3). \quad (5.7)
\]

If \( (b - a)/a < 0.1 \), then the above formula predicts

\[
\frac{\delta E_a}{E(a, 0)} \leq 0.84 \frac{m^2}{a^2}. \quad (5.8)
\]
It is clear that if the relative width of the initial pulse tends to zero then the effect becomes negligible. This can be translated into the dependence on the wavelength of the radiation [6]: The compression of the support of a function leads to the decrease of its wavelength scale in its Fourier transform.

A careful analysis of the higher order terms would show that they give a contribution to (5.7) that also scales like \( \left( \frac{b-a}{a} \right)^2 \). In the case when \( a \approx 2m \), Eq. (5.7) would be of the form

\[
\frac{\delta E_a}{E(a,0)} \leq C(x) \left( \frac{b-a}{2m} \right)^2,
\]

where \( C(x) \) is a large number and \( x \equiv 2m/a \). One can show that \( \lim_{x \to 1} C(x) = \infty \), but on the other hand \( C(x) \) is fixed, when \( 2m/a \) is fixed. Thus (5.9) implies that when \( b \to a \), then the backscatter becomes negligible. Radiation that is dominated by infinitely short wavelengths does not backscatter.

### VI. MORE ESTIMATES ON HIGH FREQUENCY RADIATION

We assume initial data of compact support \([a, b]\). The initial energy \( E(a,0) \) (see the beginning of Sec. III) reads, expressed in terms of functions \( \Psi_0, \Psi_1 \) and \( \Psi_2 \), as follows

\[
E(a,0) = \int_a^b \, dr \rho = \int_a^b \, dr \left[ \frac{6 (r \Psi_0'(r) + \Psi_1(r)) + \Psi_2(r)}{r^6} \right]^2 + \left( \frac{\Psi_0'(r) + r \Psi_1'(r) + \Psi_2'(r)}{r^2} \right)^2 + \left( \frac{-2 \Psi_2(r) - r (\Psi_1(r) + r (\Psi_0'(r) + \Psi_1'(r)) + \Psi_2'(r))}{r^6} \right)^2.
\]

The radiation energy in the wave zone is known to be \( E(a,0) = C \int_a^b \, dr (\Psi_0')^2 \). This can be compatible with (6.1) (modulo a normalization constant, which is not relevant here), if the terms with \( (\Psi_0')^2 \) give a leading contribution.

One notices, that if \( \Psi_\mu(R) \) (\( \mu = 0, 1, 2 \)) are of compact support, then \( |\Psi_\mu(R)| = |\int_a^R \, dr \partial_r \Psi_\mu(r)| \leq \sqrt{(R-a) \int_a^R \, dr \partial_r \Psi_\mu^2(r)} \). Combining this with (2.5) one arrives at
\[ 3|\Psi_0(r)| = |\Psi'_1(r)| \leq 3\sqrt{b-a} \int_a^b \sqrt{\psi^2(r) + 2\int_a^b \sqrt{\psi_0^2(r)}}, \]
\[ |\Psi_1(r)| = |\Psi'_2(r)| \leq 2(b-a)^{3/2} \sqrt{\int_a^b \psi^2(r) + 2\int_a^b \sqrt{\psi_0^2(r)}}, \]
\[ |\Psi_2(r)| \leq 0.8(b-a)^{5/2} \sqrt{\int_a^b \psi^2(r) + 2\int_a^b \sqrt{\psi_0^2(r)}}. \] (6.2)

Taking into account (6.2), one concludes that if

\[ C_1(b-a)/a \ll 1, \] (6.3)

\((C_1 \text{ is a constant of the order of 100})\) then the energy is well approximated by \( E(a,0) = 2 \int_a^b \psi_0^2(r) \).

In such circumstances it is clear that our analysis can be greatly simplified. First of all, the contribution coming from \( \Psi_2 \) to the backscatter is much smaller than that due to \( \Psi_1 \); notice an additional power of \((b-a)/a\) in the relevant estimate of (6.2). Secondly, the estimate (5.1) of \( \Psi_1 \) is now replaced by a stronger result

\[ |\Psi_0(r)| \leq \frac{1}{\sqrt{2}}(b-a)^{3/2} \sqrt{E(a,0)} \]. (6.4)

The repetition of the calculation of Sec. V gives finally (taking into account the above conditions)

\[ \frac{\delta E_a}{E(a,0)} \leq \left( \frac{2m}{a} \right)^2 \left( \frac{b-a}{a} \right)^4. \] (6.5)

**VII. CONCLUSIONS**

In our paper we derived upper bounds for the backscattering of gravitational quadrupole waves propagating outward from a central compact object. The calculations were restricted to situations where the initial configuration was either an arbitrarily shaped wave with support outside some radius \( a \) or the wave was a sharp pulse, i. e. its extension was small compared to its initial location \( a \). The obtained upper bounds show that, for a given central object, the backscattering is the weaker the more outside from the central object the waves
start propagating, and that is also the weaker the more compact the pulses are, i.e. the higher the involved frequencies are. The both results do confirm previous completely different calculations by Price, Pullin and Kundu [10]. Backscattering should thus be strongest for pulses which start propagating outward close to the horizon of a black hole. This claim, however, needs further investigation for the following reasons. First, we gave bounds from above and not from below for the amount of backscattered energy. Second, the linear approximation may not be accurate enough very close to the horizon.

Results of Flanagan and Hughes [11] and Buonanno and Damour [12] have shown that the merger part of the gravitational wave signal could be a significant part of the total energy emitted. The wave pulse during the merger phase can be inside 3m. For a very compact pulse located in this region the inequality (5.9) of Sec. V can still yield a nontrivial bound, but in the general case our estimates fail. The main reason why we are loosing much in the accuracy is that we are forced to use several times - for the sake of generality - the Schwarz inequality. The present bounds can be significantly improved if initial data are explicitly known, since in this case they can be numerically bounded by an exact expression involving the initial energy and the Schwarz inequality would be used only once. On the other hand, it has been discovered that the backscattering can be quite strong when a signal propagates from within the photon sphere [13].

In a forthcoming paper we shall discuss, and compare with the results of our present paper, several aspects of the backscattering of gravitational waves where the sources of the gravitational waves are taken into account.

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In order to show the estimate (4.7) one begins with the second inequality of (4.6). Notice that $H(t = 0) = 0$, since $\delta(R, t = 0) = \partial_t \delta(R, t)|_{t=0}$. Thence the integration of (4.6) along $\tilde{\Gamma}_{a,(R,t)}$ yields

$$H_M \leq \int_a^\infty \frac{dr}{\eta_r} |RHS(r)|, \quad (8.1)$$

where $RHS(r)$ stands for the right hand side of Eq. (4.6). Our task consists in estimating the line integral of $|RHS(r)|$.

In order to do this one uses the estimates of (3.4). The calculation is quite long and we will describe only the main points. In the first step one uses the Schwarz inequality in the right hand side of (4.6), in order to obtain an expression of the type

$$4m \left( \int_R^\infty dr \frac{(\partial_t \delta)^2}{\eta_r} \right)^{1/2} \times \left( \int_R^\infty dr \eta_r \frac{f^2(r)}{r^8} \right)^{1/2}, \quad (8.2)$$

where $f(r)$ denotes $(-3\Psi_1)^2$, $(2\Psi_2/r)^2$, or the squares of the terms that are proportional to $63m$. The first integral can be bounded by $\sqrt{H(R)}$, therefore (4.6) and (8.2) yield

$$(\partial_t + \partial_r) \sqrt{H(R,t)} \leq 2m \left( \int_R^\infty dr \eta_r \frac{9\Psi_1^2}{r^8} \right)^{1/2} + 2m \left( \int_R^\infty dr \eta_r \frac{4\Psi_2^2}{r^{10}} \right)^{1/2} +$$

$$2m \left( \int_R^\infty dr \frac{63m(1 + \frac{m}{r})^2}{4(1 + \frac{3m}{2r})^2} \frac{2\Psi_0^2}{r^8} \right)^{1/2} + 2m \left( \int_R^\infty dr \frac{63m(1 + \frac{m}{r})^2}{4(1 + \frac{3m}{2r})^2} \frac{2\Psi_1^2}{r^{10}} \right)^{1/2} +$$

$$2m \left( \int_R^\infty dr \frac{63m(1 + \frac{m}{r})^2}{4(1 + \frac{3m}{2r})^2} \frac{2\Psi_2^2}{r^{12}} \right)^{1/2}. \quad (8.3)$$

The integrands of (8.3) are taken at a time $t$ and $(R,t) \in \tilde{\Gamma}_a$; the integration extends over the part $r \geq R$ of the Cauchy hypersurface $\Sigma_t$. At this place one inserts the bounds on $\Psi_0$, etc.
\( \Psi_1 \) and \( \Psi_2 \). That requires some care; the estimates hold true on the initial hypersurface \( \Sigma_0 \), while here one needs estimates on \( \Sigma_t \). This point is clarified later. It is useful to introduce dimensionless variables \( x = R/a \) and \( \tilde{m} = m/a \).

i) First we shall consider the contribution that is due to \( 3\Psi_1 \). Let \( r_0 \) be defined by \( (r, t) \in \tilde{\Gamma}_r \). The insertion of the bound given in (3.4) bounds \( 2m \left( \int_R^\infty dr \eta r^{9/2} \right)^{1/2} \) by \( 6mC_1 \left( \int_R^\infty dr \eta r^{4} g_1(r) \right)^{1/2} \). Notice that \( g_1(r) \) is an increasing function, therefore if one replaces \( g_1(r_0) \) by \( g_1(r) \), then the integral that appears here cannot be smaller. In this way one utilizes the initial information (the energy inequality (3.4)) in order to control the evolution. The integral in question can be performed explicitly, with the result

\[
6mC_1 \left( \int_R^\infty dr \eta r^{4} g_1(r) \right)^{1/2} = \frac{6mC_1}{a^2} \sqrt{G_1(x)}. \tag{8.4}
\]

Here, it holds

\[
-G_1(x) = \frac{\tilde{m}^4(137 - 770\tilde{m} + 1880\tilde{m}^2 - 2160\tilde{m}^3 + 960\tilde{m}^4)}{30(-1 + 2\tilde{m})^5x^8} - \frac{\tilde{m}^3(991 - 5110m + 10840\tilde{m}^2 - 8880\tilde{m}^3 - 720\tilde{m}^4 + 3360\tilde{m}^5)}{105(-1 + 2\tilde{m})^5x^7} + \frac{\tilde{m}^2(2981 - 13010m + 18440\tilde{m}^2 + 7920\tilde{m}^3 - 41520\tilde{m}^4 + 27360\tilde{m}^5)}{420(-1 + 2\tilde{m})^5x^6} - \frac{\tilde{m}(4497 - 11370m - 21720\tilde{m}^2 + 133040\tilde{m}^3 - 200240\tilde{m}^4 + 101600\tilde{m}^5)}{2100(-1 + 2\tilde{m})^5x^5} + \frac{375 + 4650\tilde{m} - 35400\tilde{m}^2 + 93200\tilde{m}^3 - 110000\tilde{m}^4 + 49376\tilde{m}^5}{3360(-1 + 2\tilde{m})^5x^4} - \frac{11}{336\tilde{m}x^3} - \frac{11}{448\tilde{m}^2x^2} - \frac{11}{448\tilde{m}^3x} + \frac{11\ln(\frac{x}{-2\tilde{m} + x})}{896\tilde{m}^4} - \frac{\ln(\frac{-2\tilde{m} + x}{1-2\tilde{m}})}{140x^8}). \tag{8.5}
\]

This rather long expression is quite well approximated by \( G_1 = (1+4\ln x)/(16x^4) \) if \( m/a \ll 1 \). The integration of (8.4) along a null cone \( C_a \) is done as follows. The integral \( \int_1^\infty \sqrt{G_1(x)} \) is bounded from above,

\[
\int_1^\infty dx \eta x^{-1} \sqrt{G_1(x)} \leq \left( \int_1^\infty dx x^{2} G_1(x) \right)^{1/2} \left( \int_1^\infty dx x^{-2} \eta x^{-2} \right)^{1/2}. \tag{8.6}
\]

Numerical integration yields
\[ 6C_1 \frac{m}{a} \sqrt{\int_1^\infty \frac{(1 + 4 \ln x)}{16x^2}} + O((m/a)^2) \approx 8.24 \frac{m\sqrt{E(a,0)}}{a} + O((m/a)^2). \]  

(8.7)

One can check that the neglected terms can give a contribution comparable to the leading term only at distances smaller than 6.6m.

ii) The calculation concerning the contribution of the \( \Psi_2 \) function is similar. The leading (proportional to \( m^0 \)) term is \( \int_R^\infty dr \frac{\Psi_2^2}{r^2} \). \( |\Psi_2| \) is bounded in terms of \( g_2(x) \). \( g_2(x) \) is an increasing function, and a reasoning similar to what was made when discussing \( g_1(x) \), leads to the conclusion that one can again use the initial energy inequality given by (3.4). One finds that \( \int_R^\infty dr \frac{\Psi_2^2}{r^2} \) is bounded from above by

\[
- \frac{1}{a^4} G_2(x) := - \frac{1}{a^4} \int_x^\infty dy \frac{g_2(y)}{y^6} (1 - 2\bar{m}/y)^2 = \\
\frac{4\bar{m}^2(-3 + 44\bar{m} - 120\bar{m}^2 + 96\bar{m}^3)}{21(-1 + 2\bar{m})^3x^7} - \\
\frac{2\bar{m}(-21 + 236\bar{m} - 408\bar{m}^2 - 192m^3 + 576\bar{m}^4)}{63(-1 + 2\bar{m})^3x^6} + \\
\frac{-21 + 88\bar{m} + 480\bar{m}^2 - 1968\bar{m}^3 + 1760\bar{m}^4}{105(-1 + 2\bar{m})^3x^5} - \\
\frac{34}{105x^4} - \frac{630\bar{m}x^3}{840\bar{m}^2x^2} - \\
\frac{31}{31 \ln(\frac{x}{\bar{m}})} - \\
\frac{31}{840\bar{m}^3x} + \frac{1680\bar{m}^4}{8\bar{m}(60\bar{m}^2 - 70\bar{m}x + 21x^2) \ln(\frac{2\bar{m}+x}{1-2\bar{m}})} \frac{2\bar{m}+x}{105x^7}. 
\]  

(8.8)

In the limit of \( m \to 0 \) the function \( G_2(x) \) coincides with \((−4 + 5x)/(20x^5)\). Similarly as before, in order to get a term bounding \( \sqrt{H_M} \), one should integrate \( \sqrt{G_2(x)}/(1 − 2\bar{m}/x) \) along a null cone \( C_a \). That gives 0.15, up to terms \( O(m) \), after manipulations similar to those done earlier. The \( O(m) \) correction becomes dominant when \( 2m/a > 0.3 \). After a reasoning similar to that applied above in the case of \( \Psi_1 \) one finds that the total contribution due to the bound on the \( \Psi_2 \) function is equal to

\[
4C_2 \sqrt{\int_1^\infty dx \frac{(-4 + 5x)}{20x^3}} \frac{m\sqrt{E(a,0)}}{a} = 2.19 \frac{m\sqrt{E(a,0)}}{a}. 
\]  

(8.9)
In summary, one obtains

\[ \sqrt{H_M} \leq 10.43 \frac{m \sqrt{E(a,0)}}{a} + O(m^2). \] (8.10)

In the above analysis, in (8.3), we neglected the terms proportional to 63m. They give corrections of the order \( O(m^2) \) to the right hand side of (8.10). We checked that their contribution is small in the region \( a > 6.6m \). Our final result (4.10) tells us that \( a > \sqrt{218} m \approx 15m \) is valid for a nontrivial estimate. Therefore, all higher order terms in (8.10) can be safely neglected.