EXACT STRING SOLUTIONS IN NONTRIVIAL BACKGROUNDS

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We show how the classical string dynamics in $D$-dimensional gravity background can be reduced to the dynamics of a massless particle constrained on a certain surface whenever there exists at least one Killing vector for the background metric. We obtain a number of sufficient conditions, which ensure the existence of exact solutions to the equations of motion and constraints. These results are extended to include the Kalb-Ramond background. The $D1$-brane dynamics is also analyzed and exact solutions are found. Finally, we illustrate our considerations with several examples in different dimensions. All this also applies to the tensionless strings.

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1 Introduction

The string equations of motion and constraints in curved space-time are highly nonlinear and, in general, non exactly solvable. Different methods have been applied to solve them approximately [1], [2], [3]-[5] or, if possible, exactly [6]-[11]. On the other hand, quite general exact solutions can be found by using an appropriate ansatz, which exploits the symmetries of the underlying curved space-time [9], [12]-[34]. In most cases, such an ansatz effectively decouples the dependence on the spatial world-sheet coordinate $\sigma$ [12]-[26] or the dependence on the temporal world-sheet coordinate $\tau$ [19, 22], [27]-[32]. Then the string equations of motion and constraints reduce to nonlinear coupled ordinary differential equations, which are considerably simpler to handle than the initial ones.

In this article, we obtain some exact solutions of the classical equations of motion and constraints for both tensile and null strings in a $D$-dimensional curved background. This is done by using an ansatz, which reduces the initial dynamical system to the one depending on only one affine parameter. This is possible whenever there exists at least one Killing vector for the background metric. Then we search for sufficient conditions, which ensure the existence of exact solutions to the equations of motion and constraints without fixing particular metric. These results are extended to include the Kalb-Ramond two-form gauge field background. The $D$-string dynamics is also

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analyzed and exact solutions are found. After that, we give several explicit examples in four, five and ten dimensions. Finally, we conclude with a discussion of the derived results.

2 Reduction of the dynamics

To begin with, we write down the bosonic string action in $D$-dimensional curved space-time $\mathcal{M}_D$ with metric tensor $g_{MN}$

$$S = \int d^2 \xi \mathcal{L}, \quad \mathcal{L} = -\frac{T}{2} \sqrt{-\gamma} \gamma^{mn} \partial_m X^M \partial_n X^N g_{MN} (X),$$

where, as usual, $T$ is the string tension and $\gamma$ is the determinant of the auxiliary metric $\gamma^{mn}$.

Here we would like to consider tensile and null (tensionless) strings on equal footing, so we have to rewrite the action (1) in a form in which the limit $T \to 0$ can be taken. To this end, we set

$$\gamma_{\alpha \beta} = \left( -\xi^0, \xi^1 \right) = (\tau, \sigma), \quad m, n = 0, 1, \quad M, N = 0, 1, ..., D - 1,$$

and obtain

$$\mathcal{L} = \frac{1}{4\lambda^0} g_{MN} (X) \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \left( \partial_0 - \lambda^1 \partial_1 \right) X^N - \lambda^0 T^2 g_{MN} (X) \partial_1 X^M \partial_1 X^N.$$

The equations of motion and constraints following from this Lagrangian density are

$$\partial_0 \left[ \frac{1}{2\lambda^0} \left( \partial_0 - \lambda^1 \partial_1 \right) X^K \right] - \partial_1 \left[ \frac{\lambda^1}{2\lambda^0} \left( \partial_0 - \lambda^1 \partial_1 \right) X^K \right] + \frac{1}{2\lambda^0} \Gamma^K_{MN} \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \left( \partial_0 - \lambda^1 \partial_1 \right) X^N = 2\lambda^0 T^2 \partial_1 X^K + \Gamma^K_{MN} \partial_1 X^M \partial_1 X^N + \partial_1 \ln \lambda^0 \partial_1 X^K,$$

$$g_{MN} (X) \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \left( \partial_0 - \lambda^1 \partial_1 \right) X^N + \left( 2\lambda^0 T^2 \right)^2 g_{MN} (X) \partial_1 X^M \partial_1 X^N = 0, \quad (3)$$

$$g_{MN} (X) \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \partial_1 X^N = 0, \quad (4)$$

where

$$\Gamma^K_{MN} = \frac{1}{2} g^{KL} \left( \partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN} \right)$$

is the connection compatible with the metric $g_{MN} (X)$. We will work in the gauge $\lambda^m = \text{constants}$ in which the Euler-Lagrange equations take the form

$$\left( \partial_0 - \lambda^1 \partial_1 \right) \left( \partial_0 - \lambda^1 \partial_1 \right) X^K + \Gamma^K_{MN} \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \left( \partial_0 - \lambda^1 \partial_1 \right) X^N = \left( 2\lambda^0 T^2 \right)^2 \left( \partial_1 X^K + \Gamma^K_{MN} \partial_1 X^M \partial_1 X^N \right).$$
Now we are going to show that by introducing an appropriate ansatz, one can reduce the classical string dynamics to the dynamics of a massless particle constrained on a certain surface whenever there exists at least one Killing vector for the background metric. Indeed, let us split the index \( M = (\mu, a) \), \( \{\mu\} \neq \{\emptyset\} \) and let us suppose that there exist a number of independent Killing vectors \( \eta_\mu \). Then in appropriate coordinates \( \eta_\mu = \frac{\partial}{\partial x^\mu} \) and the metric does not depend on \( X^a \). In other words, from now on we will work with the metric

\[
\mathbb{G}_{MN} = \mathbb{G}_{MN}(X^a) .
\]

On the other hand, we observe that

\[
X^M(\tau, \sigma) = F^M_\pm[w_\pm(\tau, \sigma)], \quad w_\pm(\tau, \sigma) = (\lambda^1 \pm 2\lambda^0 T)\tau + \sigma
\]

are solutions of the equations of motion (5) in arbitrary \( D \)-dimensional background \( \mathbb{G}_{MN}(X^K) \), depending on \( D \) arbitrary functions \( F_\pm^M \) or \( F_-^M \) (see also [6]). Taking all this into account, we propose the ansatz

\[
X^\mu(\tau, \sigma) = C^\mu_\pm w_\pm + y^\mu(\tau), \quad C^\mu_\pm = \text{constants}, \quad C_\pm = \text{constants}, \quad \mu = 0, 1, 2, …, D - 1 .
\]

Inserting (6) into constraints (3) and (4) one obtains (the overdot is used for \( d/d\tau \))

\[
\mathbb{G}_{MN}(y^a)\dot{y}^M \dot{y}^N \pm 2\lambda^0 T C^\mu_\pm \left[ \mathbb{G}_{\mu N}(y^a)\dot{y}^N \pm 2\lambda^0 T C^\nu_\pm \mathbb{G}_{\nu \mu}(y^a) \right] = 0 , \quad C^\mu_\pm \left[ \mathbb{G}_{\mu N}(y^a)\dot{y}^N \pm 2\lambda^0 T C^\nu_\pm \mathbb{G}_{\nu \mu}(y^a) \right] = 0 .
\]

Obviously, this system of two constraints is equivalent to the following one

\[
\mathbb{G}_{MN}(y^a)\dot{y}^M \dot{y}^N = 0 , \quad C^\mu_\pm \left[ \mathbb{G}_{\mu N}(y^a)\dot{y}^N \pm 2\lambda^0 T C^\nu_\pm \mathbb{G}_{\nu \mu}(y^a) \right] = 0 .
\]

Using the ansatz (6) and constraint (8) one can reduce the initial Lagrangian to get

\[
L_{\text{red}}(\tau) \propto \frac{1}{4\lambda^0} \left[ \mathbb{G}_{MN}(y^a)\dot{y}^M \dot{y}^N - 2 \left( 2\lambda^0 T \right)^2 C^\mu_\pm C^\nu_\pm \mathbb{G}_{\mu \nu}(y^a) \right] .
\]

It is easy to check that the constraint (7) can be rewritten as \( \mathbb{G}^M N p_M p_N = 0 \), where \( p_M = \partial L_{\text{red}}(\dot{y}^M) \) is the momentum conjugated to \( y^M \). All this means that we have obtained an effective dynamical system describing a massless point particle moving in a gravity background \( \mathbb{G}_{MN}(y^a) \) and in a potential

\[
U \propto T^2 C^\mu_\pm C^\nu_\pm \mathbb{G}_{\mu \nu}(y^a)
\]

on the constraint surface (8).

Analogous results can be received if one uses the ansatz

\[
X^\mu(\tau, \sigma) = C^\mu_\pm w_\pm + z^\mu(\sigma), \quad X^a(\tau, \sigma) = z^a(\sigma) .
\]
Now putting (9) in (3) and (4) one gets (′ is used for $d/d\sigma$)

$$\left[(2\lambda^0 T)^2 + (\lambda^1)^2\right] g_{MN} z'^M z'^N + 4\lambda^0 T \left[ (2\lambda^0 T + \lambda^1) C^\mu_\pm g_{\mu N} z'^N + 2\lambda^0 T C^\mu_\pm C^\nu_\pm g_{\mu \nu}\right] = 0,$$

$$\lambda^1 g_{MN} z'^M z'^N + \left[ (\lambda^1 \mp 2\lambda^0 T) C^\mu_\pm g_{\mu N} z'^N \mp 2\lambda^0 T C^\mu_\pm C^\nu_\pm g_{\mu \nu}\right] = 0.$$ 

These constraints are equivalent to the following ones

$$g_{MN}(z^a) z'^M z'^N = 0,$$

$$C^\mu_\pm \left[ g_{\mu N}(z^a) z'^N + \frac{2\lambda^0 T}{2\lambda^0 T \mp \lambda^1} C^\nu_\pm g_{\mu \nu}(z^a) \right] = 0. \tag{10}$$

The corresponding reduced Lagrangian obtained with the help of (9) and (10) is

$$L_{\text{red}}(\sigma) \propto \frac{(\lambda^1)^2 - (2\lambda^0 T)^2}{4\lambda^0} \left[ g_{MN}(z^a) z'^M z'^N - 2 \left( \frac{2\lambda^0 T}{2\lambda^0 T \mp \lambda^1} \right)^2 C^\mu_\pm C^\nu_\pm g_{\mu \nu}(z^a) \right]$$

and a similar interpretation can be given as before.

In both cases - the ansatz (6) and the ansatz (9), the reduced Lagrangians do not depend on $y^\mu$ and $z^\mu$ respectively, and their conjugated generalized momenta are conserved.

Let us point out that the main difference between tensile and null strings, from the point of view of the reduced Lagrangians, is the absence of a potential term for the latter.

Because the consequences of (6) and (9) are similar, our further considerations will be based on the ansatz (6).

## 3 Exact solutions in curved background

To obtain the equations which we are going to consider, we use the ansatz (6) and rewrite (5) in the form

$$g_{KL} \ddot{y}^L + \Gamma_{K,MN} \dot{y}^M \dot{y}_N \pm 4\lambda^0 T C^\mu_\pm \Gamma_{K,\mu N} \dot{y}_N = 0. \tag{11}$$

At first, we set $K = \mu$ in the above equality. It turns out that in this case the equations (11) reduce to

$$\frac{d}{d\tau} \left[ g_{\mu \nu} \ddot{y}^\nu + g_{\mu a} \dot{y}^a \pm 2\lambda^0 T C^\nu_\pm g_{\mu \nu}\right] = 0,$$

i.e. we have obtained the following first integrals (constants of the motion)

$$g_{\mu \nu} \ddot{y}^\nu + g_{\mu a} \dot{y}^a \pm 2\lambda^0 T C^\nu_\pm g_{\mu \nu} = A^\pm_\mu = \text{constants} \tag{12}.$$ 

They correspond to the conserved momenta $p_\mu$. From the constraint (8) it follows that the right hand side of (12) must satisfy the condition

$$A^\pm_\mu C^\nu_\pm = 0.$$
Using (12), the equations (11) for $K = a$ and the constraint (7) can be rewritten as

$$
2 \frac{d}{dt} \left( h_{ab} \dot{y}^b \right) - (\partial_a h_{bc}) \dot{y}^b \dot{y}^c + \partial_a V^\pm = 4 \partial_{[a} \left( g_{b] \mu} k^{\mu \nu} A^\pm_{\nu} \right) \dot{y}^b
$$

and

$$
h_{ab} \dot{y}^a \dot{y}^b + V^\pm = 0,
$$

where

$$
h_{ab} \equiv g_{ab} - g_{aq} k^{\mu \nu} g_{qb}, \quad V^\pm \equiv A^{\pm}_{\mu} A^{\pm}_{\nu} k^{\mu \nu} + \left( 2 \lambda^0 T \right)^2 C^{\mu}_{\pm} C^{\nu}_{\pm} g_{\mu \nu},
$$

and $k^{\mu \nu}$ is by definition the inverse of $g^{\mu \nu}$: $k^{\mu \lambda} k_{\lambda \nu} = \delta^{\mu \nu}$. For example, when $g_{MN}$ does not depend on the coordinate $y^q$

$$
h_{ab} = g_{ab} - \frac{g_{aq} g_{qb}}{g_{qq}},
$$

when $g_{MN}$ does not depend on two of the coordinates (say $y^q$ and $y^s$)

$$
h_{ab} = g_{ab} - \frac{g_{aq} g_{qs} g_{qb} - 2 g_{aq} g_{qs} g_{sb} + g_{as} g_{qq} g_{sb}}{g_{qq} g_{ss} - g_{qs}^2},
$$

and so on.

At this stage, we restrict the metric $h_{ab}$ to be a diagonal one, i.e.

$$
g_{ab} = g_{aq} k^{\mu \nu} g_{qb}, \quad \text{for} \quad a \neq b.
$$

This allows us to transform further equations (13) and obtain (there is no summation over $a$)

$$
\frac{d}{dt} (h_{aa} \dot{y}^a)^2 + \dot{y}^a \partial_a \left( h_{aa} V^\pm \right) + \dot{y}^a \sum_{b \neq a} \left[ \partial_a \left( \frac{h_{aa}}{h_{bb}} \right) \left( h_{bb} \dot{y}^b \right)^2 - 4 \partial_{[a} A^\pm_{b]} h_{aa} \dot{y}^b \right] = 0,
$$

where we have introduced the notation

$$
A^\pm_{a} \equiv g_{aq} k^{\mu \nu} A^\pm_{\nu}.
$$

In receiving (16), the constraint (14) is also used after taking into account the restriction (15).

To reduce the order of the differential equations (16) by one, we first split the index $a$ in such a way that $y^r$ is one of the coordinates $y^a$, and $y^a$ are the others. Then we impose the conditions

$$
\partial_\alpha \left( \frac{h_{aa}}{h_{aa}} \right) = 0, \quad \partial_\alpha (h_{aa} \dot{y}^a)^2 = 0, \quad \partial_r (h_{aa} \dot{y}^a)^2 = 0, \quad A^\pm_{a} = \partial_\alpha f^\pm.
$$

The result of integrations, compatible with (14) and (15), is the following

$$
(h_{aa} \dot{y}^a)^2 = D_\alpha (y^a \neq y^a) + h_{aa} \left[ 2 \left( A^\pm_r - \partial_r f^\pm \right) \dot{y}^r - V^\pm \right] = E_a \left( y^a \right),
$$

$$
(h_{rr} \dot{y}^r)^2 = h_{rr} \left\{ \left( \sum_{a} -1 \right) V^\pm - \sum_{a} \frac{D_\alpha}{h_{aa}} \right\} + \left[ \sum_{a} \left( A^\pm_r - \partial_r f^\pm \right) \right]^2 = E_r \left( y^r \right),
$$

for $a \neq b$. For $a = b$, the result is

$$
(h_{aa} \dot{y}^a)^2 = D_\alpha (y^a \neq y^a) + h_{aa} \left[ 2 \left( A^\pm_r - \partial_r f^\pm \right) \dot{y}^r - V^\pm \right] = E_a \left( y^a \right).
$$
where $D_{\alpha}, E_{\alpha}, E_{r}$ are arbitrary functions of their arguments, and

$$
\dot{z}^r \equiv \dot{y}^r + \frac{\sum_{\alpha} \alpha}{\dot{h}_{rr}} \left( A^+_r - \partial_r f^+_r \right).
$$

To find solutions of the above equations without choosing particular metric, we have to fix all coordinates $y^a$ except one. If we denote it by $y^A$, then the exact solutions of the equations of motion and constraints for a string in the considered curved background are given by

$$
X^\mu \left( X^A, \sigma \right) = X^\mu_0 + C^\mu_\pm \left( \lambda^1 \tau + \sigma \right) - \int_{X^A_0}^{X^A} h^{\mu \nu}_0 \left[ \frac{g^0_{\nu A} \mp A^\pm_\nu}{V^\pm_0} \right]^{1/2} du, \tag{20}
$$

$$
X^a = X^a_0 = \text{constants} \quad \text{for} \quad a \neq A, \quad \tau \left( X^A \right) = \tau_0 \pm \int_{X^A_0}^{X^A} \left( - \frac{h^{0}_{AA}}{V^\pm_0} \right)^{1/2} du,
$$

where $X^\mu_0, X^A_0$ and $\tau_0$ are arbitrary constants. In these expressions

$$
h^{0}_{AA} = h^{0}_{AA} \left( X^A \right) = h_{AA} \left( X^A, X^a_{0 \neq A} \right)
$$

and analogously for $V^\pm_0, k^{\mu \nu}_0$ and $g^{0}_{\nu A}$.

## 4 Turning on the B field

Here we are going to obtain exact string solutions when the background also includes the Kalb-Ramond antisymmetric gauge field $B_{MN}(X)$. To this end, we start with the bosonic part of the Green-Schwarz superstring action

$$
S_1 = -\frac{T}{2} \int d^2 \xi \left[ \sqrt{-\gamma} \gamma^{mn} \partial_m X^M \partial_n X^N g_{MN}(X) - \epsilon^{mn} \partial_m X^M \partial_n X^N B_{MN}(X) \right]. \tag{21}
$$

Varying (21) with respect to $X^M$ and $\gamma_{mn}$, we obtain the equations of motion

$$
- g_{LK} \left[ \partial_m \left( \sqrt{-\gamma} \gamma^{mn} \partial_n X^K \right) + \sqrt{-\gamma} \gamma^{mn} \Gamma^K_{MN} \partial_m X^M \partial_n X^N \right] = \frac{1}{2} H_{LMN} \epsilon^{mn} \partial_m X^M \partial_n X^N, \quad H = dB,
$$

and the constraints

$$
\left( \gamma^{kl} \gamma^{mn} - 2 \gamma^{km} \gamma^{ln} \right) \partial_m X^M \partial_n X^N g_{MN} \left( X \right) = 0.
$$

In the gauge $\gamma^{mn} = \text{constants}$ and using (2), the Euler-Lagrange equations can be rewritten as

$$
g_{LM} \left[ \left( \partial_0 - \lambda^1 \partial_1 \right)^2 - (2\lambda_0^0 T)^2 \partial_1^2 \right] X^M + \Gamma_{LMN} \left[ \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \left( \partial_0 - \lambda^1 \partial_1 \right) X^N - (2\lambda_0^0 T)^2 \partial_1 X^M \partial_1 X^N \right] = 2\lambda_0^0 T H_{LMN} \partial_0 X^M \partial_1 X^N, \tag{22}
$$
and the independent constraints take the same form (3), (4) as before. Putting the ansatz (6) into (22) one obtains
\begin{align}
    g_{KL} \ddot{y}^L + \Gamma_{MN} \dot{y}^M \dot{y}^N + 2\lambda^0 T C^\mu_\pm (H_{K\mu N} \pm 2\Gamma_{K,\mu N}) \dot{y}^N = 0. \tag{23}
\end{align}

Now we suppose that the tensor field $B_{MN}$ has the same symmetry as the background metric $g_{MN}$, i.e. $\partial_\mu g_{MN} = \partial_\mu B_{MN} = 0$. Then it follows that
\begin{align}
    g_{\mu N} \dot{y}^N + 2\lambda^0 T C^\nu_\pm (B_{\mu \nu} \pm g_{\mu \nu}) = B^\pm_\mu = \text{constants} \tag{24}
\end{align}

are first integrals of the equations of motion (23). These conserved quantities are compatible with the constraint (8) when $B^\pm_\mu C^\mu_\pm = 0$. Using (24), the equations of motion for $y^a$ and the other constraint (7) can be transformed into
\begin{align}
    &2 \frac{d}{d\tau} \left( h_{ab} \dot{y}^b \right) - (\partial_a h_{bc}) \dot{y}^b \dot{y}^c + \partial_a V^\pm_B = 4\partial_a B^\pm_\alpha \dot{y}^\alpha, \tag{25} \\
    &h_{ab} \dot{y}^a \dot{y}^b + V^+_B = 0, \tag{26}
\end{align}

where now
\begin{align}
    V^\pm_B &= \left( 2\lambda^0 T \right)^2 C^\mu_\pm C^\nu_\pm g_{\mu \nu} + \left( B^\pm_\nu - 2\lambda^0 T B_\nu C^\lambda_\pm \right) k^{\mu \nu} \left( B^\pm_\nu - 2\lambda^0 T B_\nu C^\rho_\pm \right), \\
    B^\pm_a &= g_{a \mu} k^{\mu \nu} B^\pm_\nu + 2\lambda^0 T (B_\alpha - g_{a \mu} k^{\mu \nu} B_\nu \lambda) C^\lambda_\pm.
\end{align}

The comparison of (25), (26) with (13), (14) shows that the former can be obtained from the latter by the replacements
\begin{align}
    V^\pm \mapsto V^\pm_B, \quad A^\pm_a \mapsto B^\pm_a,
\end{align}

i.e these equalities are form-invariant. Therefore, the first integrals (19) will have the same form as before under the same conditions on the metric $h_{ab}$ and with $B^\pm_a = \partial_a f^\pm_B$.

The corresponding generalization of the exact solutions (20) for $B_{MN}(X) \neq 0$ will be
\begin{align}
    X^\mu \left( X^A, \tau \right) &= X^\mu_0 + C^\mu_\pm \left( \lambda^3 \tau + \sigma \right) \\
    &- \int_{X^A_0}^{X^A} k^{\mu \nu}_0 \left[ g^0_{\nu A} \mp \left( B^\pm_\nu - 2\lambda^0 T B_\nu C^\lambda_\pm \right) \left( -\frac{h^0_{AA}}{V^\pm_B} \right)^{1/2} \right] du, \\
    X^a &= X^a_0 = \text{constants} \quad \text{for} \quad a \neq A, \quad \tau \left( X^A \right) = \tau_0 \pm \int_{X^A_0}^{X^A} \left( -\frac{h^0_{AA}}{V^\pm_B} \right)^{1/2} du.
\end{align}

We note that till now we have used the string frame ($\sigma$ - model) metric. It would be useful to have the obtained result written in Einstein frame metric, i.e. the frame in which the $D$-dimensional Einstein-Hilbert action is free from dilatonic scalar factor. In particular, we will need it in one of the following sections. In $D$ dimensions, the connection between these two types of metrics is [35]
\begin{align}
    g_{MN} \equiv g_{MN}^{\text{string}} = \exp \left( \frac{a(D)}{2} \phi \right) g_{MN}^E, \tag{28}
\end{align}

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where for the string

\[ a(D) = \sqrt{\frac{8}{D - 2}} \]

and \( \phi(X) \) is the dilaton field. With the help of (28), one can check that (27) will give the right formulas for the exact solution in the Einstein frame, if one takes all metric coefficients in this frame and replaces \( V_\pm^0 \) with

\[ \exp \left( \frac{a(D)}{2} \phi^0 \right) V_\pm^0 \]

in the expression for \( X^\mu \), and with

\[ \exp \left( -\frac{a(D)}{2} \phi^0 \right) V_B^\pm \]

in the expression for \( \tau \), where \( \phi^0 = \phi^0 \left( X^A \right) \), and

\[ V_\pm^B \equiv \left( B_\mu^\pm - 2\lambda^0 T B_{\mu\lambda} C_\pm^\lambda \right) k_{E\mu} \left( B_\nu^\pm - 2\lambda^0 T B_{\nu\rho} C_\pm^\rho \right) \exp \left( -\frac{a}{2} \phi \right) + \left( 2\lambda^0 T \right)^2 C_\pm^\mu C_\pm^\nu g_{\mu\nu} \exp \left( \frac{a}{2} \phi \right). \]

Let us finally give the induced metric \( G_{mn} \) which arises on the string worldsheet in our parameterization of the auxiliary metric \( \gamma^{mn} \) given by (2) and after taking into account the ansatz (6). It is

\[ G_{00} = \left[ \left( \lambda^1 \right)^2 - \left( 2\lambda^0 T \right)^2 \right] G_{11}, \quad G_{01} = \lambda^1 G_{11}, \quad G_{11} = C_\pm^\mu C_\pm^\nu g_{\mu\nu} \left( X^a \right). \]

5 Exact D-string solutions

In this section our aim is to consider the D-string dynamics in nontrivial backgrounds. We will use the action

\[ S_D = -\frac{T_D}{2} \int d^2 \xi \exp (-a\phi) \sqrt{-\mathcal{K}^{mn}} \left( G_{mn} + B_{mn} + (g_sT_D)^{-1} F_{mn} \right), \quad (29) \]

introduced in [36], which is classically equivalent to the Dirac-Born-Infeld action

\[ S_{DBI} = -T_D \int d^2 \xi \exp (-a\phi) \sqrt{-\det \left( G_{mn} + B_{mn} + (g_sT_D)^{-1} F_{mn} \right)}. \]

The notations used in (29) are as follows. \( T_D = T/g_s \) is the D-string tension, where \( T = (2\pi\alpha')^{-1} \) is the (fundamental) string tension and \( g_s = \exp(\langle \phi \rangle) \) is the string coupling expressed by the dilaton vacuum expectation value \( \langle \phi \rangle \). \( G_{mn}(X) = \partial_m X^M \partial_n X^N g_{MN}(X) \), \( B_{mn}(X) = \partial_m X^M \partial_n X^N B_{MN}(X) \) and \( \phi(X) \) are the pullbacks of the background metric, antisymmetric tensor and dilaton to the D-string worldsheet, while \( F_{mn}(\xi) \) is the field
strength of the worldsheet $U(1)$ gauge field $A_m(\xi)$. $K$ is the determinant of the matrix $K_{mn}$, $K^{mn}$ is its inverse, and these matrices have symmetric as well as antisymmetric part

$$K^{mn} = K^{(mn)} + K^{[mn] \equiv \gamma^{mn}_D + \omega^{mn},}$$

where the symmetric part $\gamma^{mn}_D$ is the analogue of the auxiliary metric $\gamma^{mn}$ in the string actions (1) and (21).

To proceed further, we set in (29)

$$\gamma^{mn}_D = \left( -\frac{1}{\lambda^1} - (\lambda^1 + \lambda^2) \left( \lambda^1 - \lambda^2 \right) + (2\lambda^0 T_D)^2 \right), \quad \omega^{mn} = -\lambda^2 \varepsilon^{mn},$$

and obtain the Lagrangian density

$$L_D = \exp\left( -a\phi \right) \left\{ g_{MN} \left( \partial_0 - \lambda^1 \partial_1 \right) X^M \left( \partial_0 - \lambda^1 \partial_1 \right) X^N ight. - \left[ (\lambda^2)^2 + \left( 2\lambda^0 T_D \right)^2 \right] g_{MN} \partial_1 X^M \partial_1 X^N 
+ 2\lambda^2 \left[ B_{MN} \partial_0 X^M \partial_1 X^N + (g_s T_D)^{-1} F_{01} \right] \right\}.$$
The following step is to apply our ansatz (6). However, in the $D$-string case it does not work properly - a little modification is needed. Actually, now the background independent solutions of the equations of motion (31) are

$$X^M(\tau, \sigma) = F^M_\pm [(\lambda^1 + A_\pm) \tau + \sigma], \quad A_\pm = \pm \sqrt{(2\lambda^0 T_D)^2 + (\lambda^2)^2},$$

where again $F^M_\pm$ are arbitrary functions of their arguments. Therefore, the appropriate ansatz is

$$X^\mu(\tau, \sigma) = C^\mu_\pm [(\lambda^1 + A_\pm) \tau + \sigma] + y^\mu(\tau), \quad X^a(\tau, \sigma) = y^a(\tau).$$

(32)

The insertion of (32) into the equations (31) reduce them to the following ones

$$g_{KL} \ddot{y}^L + \Gamma_{KMN} \ddot{y}^M \dot{y}^N + C_\pm^\mu \left( \lambda^2 H_{K\mu N} + 2A_\pm \Gamma_{K,\mu N} \right) \dot{y}^N = 0,$$

which possess the first integrals

$$g_{\mu N} \dot{y}^N + C_\pm^\mu \left( \lambda^2 B_{\mu \nu} + A_\pm g_{\mu \nu} \right) = B^\pm_\mu = \text{constants}, \quad B^\pm_\mu C_\pm = 0.$$

In full analogy with the previously considered case, the equations of motion for $y^a$ can be transformed into the form (25), where now instead of $V^\pm_B$ and $B_a^\pm$ we have

$$V^\pm_B \equiv A_\pm^2 C_\pm^\mu C_\pm^\nu g_{\mu \nu} + \left( B^\pm_\nu - \lambda^2 B_{\nu \lambda} C_\pm^\lambda \right) k^{\mu \nu} \left( B^\pm_\nu - \lambda^2 B_{\nu \rho} C_\pm^\rho \right),$$

$$B^\pm_a \equiv g_{ab} k^{\mu \nu} B^\pm_\nu + \lambda^2 \left( B_{a \lambda} - g_{ab} k^{\mu \nu} B_{\nu \lambda} \right) C_\pm^\lambda.$$

The corresponding exact solution is

$$X^\mu \left( X^A, \sigma \right) = X^\mu_0 + C_\pm^\mu \left( \lambda^1 \tau + \sigma \right)$$

$$- \int_{X^0_0}^{X^A} k^{\mu \nu}_0 \left[ g^0_{\nu \lambda} \mp \left( B^\pm_\nu - \lambda^2 B^0_{\nu \lambda} C^\lambda_\pm \right) \left( -\frac{h^{\nu \lambda}}{V^\pm} \right)^{1/2} \right] du,$$

$$X^a = X^a_0 = \text{constants} \quad \text{for} \quad a \neq A, \quad \tau \left( X^A \right) = \tau_0 \pm \int_{X^0_0}^{X^A} \left( -\frac{h^{\nu \lambda}}{V^\pm} \right)^{1/2} du,$$

$$\frac{F_{01}}{g_s T_D} = \lambda^2 C_\pm^\mu g^0_{\mu \nu} C_\pm^\nu - \left( B^\pm_\rho - \lambda^2 B^0_{\rho \lambda} C^\lambda_\pm \right) \left( -\frac{h^{\rho \mu}}{V^\pm} \right)^{1/2}$$

$$\mp \left( B^0_{\lambda \nu} - g_{ab} k^{\rho \mu} B^0_{\rho \nu} \right) C_\pm^\nu \left( -\frac{h^{\rho \mu}}{V^\pm} \right)^{-1/2}, \quad \phi = \phi_0 = \text{constant}.$$

6 Examples

Let us first give an explicit example of exact solution for a string moving in four dimensional cosmological Kasner type background. Namely, the line element is ($x^0 \equiv t$)

$$ds^2 = g_{MN} dx^M dx^N = -(dt)^2 + \sum_{\mu=1}^{3} t^{2q_\mu} (dx^\mu)^2,$$

$$\sum_{\mu=1}^{3} q_\mu = 1, \quad \sum_{\mu=1}^{3} q_\mu^2 = 1.$$
For definiteness, we choose $q_\mu = (2/3, 2/3, -1/3)$. The metric (34) depends on only one coordinate $t$, which we identify with $X^a = y^a(\tau)$ according to our ansatz (6). Correspondingly, the last two terms in (16) vanish and there is no need to impose the conditions (18). Moreover, the metric (34) is a diagonal one, so we have $h_{aa} = g_{aa} = -1$. Taking this into account, we obtain the exact solution of the equations of motion and constraints (20) in the considered particular metric expressed as follows

$$X^\mu (t, \sigma) = X^\mu_0 + C^\mu_\pm \left( \lambda^1 \tau + \sigma \right) \pm A^\mu_\pm I^\mu (t), \quad \tau(t) = \tau_0 \pm I^0 (t), \quad (36)$$

$$I^M (t) \equiv \int_0^t du u^{-2q_M} \left( V^\pm \right)^{-1/2} , \quad q_M = (0, 2/3, 2/3, -1/3).$$

Although we have chosen relatively simple background metric, the expressions for $I^M$ are too complicated. Because of that, we shall write down here only the formulas for the two limiting cases $T = 0$ and $T \to \infty$ for $t > t_0 \geq 0$. The former corresponds to considering null strings (high energy string limit).

When $T = 0$, $I^M$ reads

$$I^M = \frac{1}{2} \left[ \left( A^+ \right)^2 + \left( A^- \right)^2 \right] \left[ \frac{t^{2/3-2q_M}}{(q_M - 1/3)} A F \left( 1/2, q_M - 1/3; q_M + 2/3; - \frac{1}{A^2 t^2} \right) \right]$$

$$+ \frac{t^{5/3-2q_M}}{q_M - 5/6} F \left( 1/2, 5/6 - q_M; 11/6 - q_M; -A^2 t_0^2 \right) + \frac{\Gamma (q_M - 1/3) \Gamma (5/6 - q_M)}{\sqrt{\pi} A^{5/3-2q_M}},$$

where

$$A^2 \equiv \frac{\left( A^+ \right)^2}{\left( A^+ \right)^2 + \left( A^- \right)^2},$$

$F (a, b; c; z)$ is the Gauss’ hypergeometric function and $\Gamma (z)$ is the Euler’s $\Gamma$-function.

When $T \to \infty$, $I^M$ is given by the equalities

$$I^0 = \pm \frac{1}{4 \lambda^0 T C^3_\pm} \left[ \frac{6}{C} t^{1/3} F \left( 1/2, -1/6; 5/6; - \frac{1}{C^2 t^2} \right) \right.$$

$$- \frac{3}{2} \lambda^0 t^{4/3} \left. F \left( 1/2, 2/3; 5/3; -C^2 t_0^2 \right) + \frac{\Gamma (-1/6) \Gamma (2/3)}{\sqrt{\pi} C^{4/3}} \right],$$

$$I^{1,2} = \pm \frac{1}{4 \lambda^0 T C^3_\pm} \left[ \ln \left( \frac{1 + C^2 t^2}{} \right)^{1/2} - 1 \right] - \ln \left( \frac{1 + C^2 t^2}{} \right)^{1/2} 1 \right],$$

$$I^3 = \pm \frac{1}{2 \lambda^0 T C^3_\pm C^2} \left[ \left( 1 + C^2 t^2 \right)^{1/2} - \left( 1 + C^2 t_0^2 \right)^{1/2} \right],$$

where

$$C^2 \equiv \frac{\left( C^+_\pm \right)^2 + \left( C^-_\pm \right)}{\left( C^+_\pm \right)^2}. $$
The solutions for $t < t_0$ may be obtained from the above ones by the exchange $t \leftrightarrow t_0$.

Let us consider the asymptotic behaviour of the solutions. The tensionless string solution has the following asymptotics as a function of the cosmic time $t$

\[
|\tau(t) - \tau_0| \xrightarrow{t \to 0} \frac{3}{5 \left[ (A^+_{1})^2 + (A^+_{2})^2 \right]^{1/2}} t^{5/3},
\]

\[
X^{1,2}(t, \sigma) \xrightarrow{t \to 0} X_{0}^{1,2} + C_{\pm}^{1,2} \mp \frac{3A_{1,2}^\pm}{\left[ (A^+_{1})^2 + (A^+_{2})^2 \right]^{1/2}} t^{1/3},
\]

\[
X^{3}(t, \sigma) \xrightarrow{t \to 0} X^0_3 + C^3_{\pm} \left\{ \sigma \mp \frac{3\lambda^1}{5 \left[ (A^+_{1})^2 + (A^+_{2})^2 \right]^{1/2}} t^{5/3} \right\},
\]

\[
|\tau(t) - \tau_0| \xrightarrow{t \to \infty} \frac{3}{2 |A^3_{\pm}|} t^{2/3},
\]

\[
X^{1,2}(t, \sigma) \xrightarrow{t \to \infty} X_{0}^{1,2} + C_{\pm}^{1,2} \left\{ \sigma \mp \frac{3\lambda^1}{2A^3_{\pm}} t^{2/3} \right\},
\]

\[
X^{3}(t, \sigma) \xrightarrow{t \to \infty} X^0_3 + C^3_{\pm} \sigma \mp \frac{3}{4} t^{4/3}.
\]

The asymptotic behaviour of the same solution as a function of the worldsheet time parameter $\tau$ is given by

\[
t(\tau) \equiv X^0(\tau) \xrightarrow{\tau \to 0} \frac{5}{3} \left[ (A^+_{1})^2 + (A^+_{2})^2 \right]^{3/10} |\tau - \tau_0|^{3/5},
\]

\[
X^{1,2}(\tau, \sigma) \xrightarrow{\tau \to 0} X_{0}^{1,2} + C_{\pm}^{1,2} \mp A^+_{1,2} \left\{ \frac{3^{15} \left| \tau - \tau_0 \right|}{\left[ (A^+_{1})^2 + (A^+_{2})^2 \right]^{2}} \right\}^{1/5},
\]

\[
X^{3}(\tau, \sigma) \xrightarrow{\tau \to 0} X^0_3 + C^3_{\pm} \left\{ \sigma \mp \lambda^1 \left| \tau - \tau_0 \right| \right\};
\]

\[
X^0(\tau) \xrightarrow{\tau \to \infty} \left( \frac{2}{3} |A^3_{\pm}| \right)^{3/2},
\]

\[
X^{1,2}(\tau, \sigma) \xrightarrow{\tau \to \infty} X_{0}^{1,2} + C_{\pm}^{1,2} (\sigma + \lambda^1 \tau),
\]

\[
X^{3}(\tau, \sigma) \xrightarrow{\tau \to \infty} X^0_3 + C^3_{\pm} \mp \frac{(A^3_{\pm})^2}{3} \tau^2.
\]

In the limit $T \to \infty$, the tensile string solution has the following behaviour for early times

\[
\tau(t) - \tau_0 \xrightarrow{t \to 0} - \frac{3}{8\lambda^0 T C^3_{\pm}} t^{4/3},
\]
For late times, the asymptotic behavior is:
\[
\tau(t) - \tau_0 \xrightarrow{t \to \infty} \frac{3}{2\lambda^0 TC^3_\pm} t^{1/3}, \\
X^{1,2}(t, \sigma) \xrightarrow{t \to \infty} X^{1,2}_0 + C^{1,2}_\pm \left( \sigma + \frac{3\lambda^1}{2\lambda^0 TC^3_\pm} t^{1/3} \right), \\
X^3(t, \sigma) \xrightarrow{t \to \infty} X^3_0 + C^3_\pm \sigma + \frac{A^{1,2}_\pm}{2\lambda^0 TC^3_\pm} t.
\]

The asymptotic behaviour of this solution as a function of $\tau$ is given by
\[
X^0(\tau) \xrightarrow{\tau \to \tau_0} \left[ \frac{8\lambda^0 TC^3_\pm}{3} (\tau_0 - \tau) \right]^{3/4}, \\
X^{1,2}(\tau, \sigma) \xrightarrow{\tau \to \tau_0} X^{1,2}_0 + C^{1,2}_\pm \left[ \sigma + \lambda^1 (\tau - \tau_0) \right] + \frac{A^{1,2}_\pm}{4\lambda^0 TC^3_\pm} \Gamma(0), \\
X^3(\tau, \sigma) \xrightarrow{\tau \to \tau_0} X^3_0 + C^3_\pm \left[ \sigma + \lambda^1 (\tau - \tau_0) \right] ;
\]
\[
X^0(\tau) \xrightarrow{\tau \to \infty} \left( \frac{2\lambda^0 TC^3_\pm}{3} \tau \right)^3, \\
X^{1,2}(\tau, \sigma) \xrightarrow{\tau \to \infty} X^{1,2}_0 + C^{1,2}_\pm (\sigma + \lambda^1 \tau), \\
X^3(\tau, \sigma) \xrightarrow{\tau \to \infty} X^3_0 + C^3_\pm + \frac{A^3_\pm (2\lambda^0 TC^3_\pm)^2}{27} \tau^3.
\]

The proper string size is
\[
L_s \propto \sqrt{\frac{C^\mu C^\nu g_{\mu \nu}(X^a)}{1 - (\lambda^1/2\lambda^0 T)^2}}.
\]

In the considered Kasner space-time, it grows like $t^{-1/3}$ for $t \to 0$, and like $t^{2/3}$ for $t \to \infty$. Recall that the space volume depends on the cosmic time $t$ linearly.

Our choice of the scale factors $\left( t^{2/3}, t^{2/3}, t^{-1/3} \right)$ was dictated only by the simplicity of the solution. However, this is a very special case of a Kasner type metric. Actually, this is one of the two solutions of the constraints (35) (up to renaming of the coordinates $x^\mu$) for which two of the exponents $q_\mu$ are equal. The other such solution is $q_\mu = (0, 0, 1)$ and it corresponds to flat space-time. Now, we will write down the exact null string solution ($T = 0$) for a gravity background with arbitrary, but different $q_\mu$. It is given
by (36), where

\begin{align}
I^M(t) &= \text{constant} - \frac{x}{A_2} \sum_{k=0}^{\infty} \frac{(A_3^+/A_2^+)^{2k}}{k! \Gamma (1/2 - k)} t^P \times \\
F \left( 1/2 + k, \frac{P}{2(q_2 - q_1)}; 2(q_2 - q_1)k + 3q_2 - 2q_1 + 1 - 2q_M; \right) - \left( \frac{A_3^+/A_2^+}{2} \right)^2 t^{2(q_2 - q_1)}, \ \text{for } q_1 > q_2,
\end{align}

\begin{align}
\mathbb{P} \equiv 2(q_2 - q_3)k + q_2 + 1 - 2q_M, \quad q_M = (0, q_1, q_2, q_3),
\end{align}

and

\begin{align}
I^M(t) &= \text{constant} + \frac{x}{A_1} \sum_{k=0}^{\infty} \frac{(A_3^+/A_1^+)^{2k}}{k! \Gamma (1/2 - k)} t^Q \times \\
F \left( 1/2 + k, \frac{Q}{2(q_1 - q_2)}; 2(q_1 - q_2)k + 3q_1 - 2q_2 + 1 - 2q_M; \right) - \left( \frac{A_3^+/A_1^+}{2} \right)^2 t^{2(q_1 - q_2)}, \ \text{for } q_1 < q_2.
\end{align}

Because there are no restrictions on \( q_\mu \), except \( q_1 \neq q_2 \neq q_3 \), the above probe string solution is also valid in generalized Kasner type backgrounds arising in superstring cosmology [37]. In string frame, the effective Kasner constraints for the four dimensional dilaton-moduli-vacuum solution are

\begin{align}
\sum_{\mu=1}^{3} q_{\mu} = 1 + K, \quad \sum_{\mu=1}^{3} q_{\mu}^2 = 1 - B^2, \\
-1 - \sqrt{3 (1 - B^2)} \leq K \leq -1 + \sqrt{3 (1 - B^2)}, \quad B^2 \in [0, 1].
\end{align}

In Einstein frame, the metric has the same form, but in new, rescaled coordinates and with new powers \( \tilde{q}_\mu \) of the scale factors. The generalized Kasner constraints are also modified as follows

\begin{align}
\sum_{\mu=1}^{3} \tilde{q}_{\mu} = 1, \quad \sum_{\mu=1}^{3} \tilde{q}_{\mu}^2 = 1 - \tilde{B}^2 - \frac{1}{2} \tilde{K}^2, \quad \tilde{B}^2 + \frac{1}{2} \tilde{K}^2 \in [0, 1].
\end{align}

Actually, the obtained tensionless string solution is also relevant to considerations within a pre-big bang context, because there exist a class of models for pre-big bang cosmology, which is a particular case of the given generalized Kasner backgrounds [37].

Our next example is for a string moving in the following ten dimensional supergravity background given in Einstein frame [35]

\begin{align}
ds^2 = g_{MN}^E dx^M dx^N &= \exp(2A) \eta_{mn} dx^m dx^n + \exp(2B) \left( dr^2 + r^2 d\Omega_7^2 \right) , \\
\exp \left[ -2(\phi - \phi_0) \right] &= 1 + \frac{k}{r^6}, \quad \phi_0, k = \text{constants}, \\
A &= \frac{3}{4}(\phi - \phi_0), \quad B = -\frac{1}{4}(\phi - \phi_0), \quad B_0 = -\exp \left[ -2 \left( \phi - \frac{3}{4} \phi_0 \right) \right].
\end{align}
All other components of $B_{MN}$ as well as all components of the gravitino $\psi_M$ and dilatino $\lambda$ are zero. If we parameterize the sphere $S^7$ so that

$$g_{10-j,10-j}^E = \exp(2B)r^2 \prod_{l=1}^{j-1} \sin^2 x^{10-l}, \quad j = 2, 3, \ldots, 7, \quad g_{99}^E = \exp(2B)r^2,$$

the metric $g_{MN}^E$ does not depend on $x^0$, $x^1$ and $x^3$, i.e. $\mu = 0, 1, 3$. Then we set $y^\alpha = y_0^\alpha = \text{constants}$ for $\alpha = 4, \ldots, 9$ and obtain a solution of the equations of motion and constraints as a function of the radial coordinate $r$:

$$X^\mu (r, \sigma) = X_0^\mu + C_+^\mu \left( \lambda^1 \tau + \sigma \right) \pm I^\mu (r), \quad \tau (r) = \tau_0 \pm I(r),$$

$$I^m (r) = \eta^{mn} \int_{r_0}^r du \left[ B_{n-1}^+ + 2\lambda^0 T \varepsilon_{nk} C_+^k \exp (-\phi_0/2) \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) W_{-1/2} \right],$$

$$I^3 (r) = \frac{B_3^+}{s^2} \int_{r_0}^r du W_{-1/2}, \quad s \equiv \prod_{l=1}^6 \sin y_{0}^{10-l}, \quad I(r) = \exp (\phi_0/2) \int_{r_0}^r du W_{-1/2},$$

where

$$W_\pm = \left\{ \left[ B_0^+ + 2\lambda^0 TC_+^1 \exp (-\phi_0/2) \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) \right] \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) - \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) - \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) \right\} \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) - \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) \right\} \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) - \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) \right\} \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) - \left( \begin{pmatrix} 1 + \frac{k}{u^6} \end{pmatrix} \right) \right\}$$

This is the solution also in the string frame, because we have one and the same metric in the action expressed in two different ways.

The above solution extremely simplifies in the tensionless limit $T \to 0$. Let us give the manifest expressions for this case. For $r_0 < r$, they are:

$$\lim_{T \to 0} I^m (r) = \eta^{mn} B_m^+ \left( J^0 + k J^6 \right), \quad \lim_{T \to 0} I^3 (r) = \frac{B_3^+}{s^2} J^2, \quad \lim_{T \to 0} I (r) = J^0,$$

where

$$J^3 (r) = -\pi \sqrt{\left( B_0^+ \right)^2 - \left( B_1^+ \right)^2} \times \left\{ \frac{1}{r^\beta-1} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{6n+\beta-5}{4} \right) \left( k/r^6 \right)^n}{(6n + \beta - 1) \Gamma \left( \frac{1-2n}{2} \right) \Gamma \left( \frac{1-2n+\beta-5}{4} \right)} P_n \left( \frac{-6n+\beta-1}{4} \right) - n-1 \right\} \left( 1 - 2 \frac{\delta}{k} r^4 \right),$$

$$-\frac{1}{r_0^\beta-1} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{2n+\beta+3}{4} \right) \left( -\delta/r_0^2 \right)^n}{(2n + \beta - 1) \Gamma \left( \frac{1-2n}{2} \right) \Gamma \left( \frac{6n+\beta+3}{4} \right)} P_n \left( \frac{2n+\beta-1}{4} \right) - n-1 \right\} \left( 1 - 2 \frac{\delta}{k} r_0^4 \right),$$

$$\delta \equiv \left( \frac{B_3^+/s}{B_0^+} \right)^2 \left( \frac{B_1^+/s}{B_1^+} \right)^2.$$
and \( P_n^{(\alpha,\beta)}(z) \) are the Jacobi polynomials. To obtain the solution for \( r_0 > r \), one has to exchange \( r \) and \( r_0 \) in the expression for \( J^3 \).

Now let us turn to the case of a \( D \)-string living in five dimensional \textit{anti de Sitter} space-time. The corresponding metric may be written as

\[
g_{00} = -\left(1 + \frac{r^2}{R^2}\right), \quad g_{11} = \left(1 + \frac{r^2}{R^2}\right)^{-1},
\]

\[
g_{22} = r^2 \sin^2 x^3 \sin^2 x^4, \quad g_{33} = r^2 \sin^2 x^4, \quad g_{44} = r^2,
\]

where \( K = -1/R^2 \) is the constant curvature. Now \( g_{MN} \) does not depend on \( x^0, x^2 \) and \( B_{MN} = 0 \). If we fix the coordinates \( x^3, x^4 \) and use the generic formula (33), the exact solution as a function of \( r \equiv x^1 \) will be

\[
X^0 (r, \sigma) = X^0_0 + C^0_+ \left( \lambda^1 \tau + \sigma \right) \mp B^\pm_0 \int_{r_0}^{r} du \left(1 + \frac{u^2}{R^2}\right)^{-1} \left(g_{00} V^\pm_D\right)^{-1/2},
\]

\[
X^2 (r, \sigma) = X^2_0 + C^2_+ \left( \lambda^1 \tau + \sigma \right) \pm \frac{B^\pm_2}{c^2} \int_{r_0}^{r} \frac{du}{u^2} \left(g_{00} V^\pm_D\right)^{-1/2},
\]

\[
\tau (r) = \tau_0 \pm \int_{r_0}^{r} du \left(g_{00} V^\pm_D\right)^{-1/2}, \quad c \equiv \sin x^0_0 \sin x^4_0,
\]

\[
F_{01}(r) = -g_{sT_D} \lambda^2 \left\{ \left(\frac{C^0_+}{R}\right)^2 - \left(cC^2_+\right)^2 \right\} r^2 + \left(C^0_+\right)^2,
\]

where

\[
g_{00} V^\pm_D = \left[(B^\pm_0)^2 - \left(\frac{B^\pm_2}{cR}\right)^2 + \left(A^0_\pm\right)^2\right] - \left(\frac{B^\pm_2}{c}\right)^2 \frac{1}{u^2}
\]

\[
+ A^2_\pm \left[2 \left(\frac{C^0_+}{R}\right)^2 - \left(cC^2_+\right)^2\right] u^2 + \left(\frac{A^\pm_2}{R}\right)^2 \left[\left(\frac{C^0_+}{R}\right)^2 - \left(cC^2_+\right)^2\right] u^4.
\]

This solution describes a \( D \)-string evolving in the subspace \((x^0, x^1, x^2)\).

Alternatively, we could fix the coordinates \( r = r_0, \quad x^4 = x^4_0 \equiv \psi_0 \) and obtain a solution as a function of the coordinate \( x^3 \equiv \theta \). In this case, the result is the following

\[
X^0 (\theta, \sigma) = X^0_0 + C^0_+ \left( \lambda^1 \tau + \sigma \right) \mp \frac{B^\pm_0 \theta}{g_{00}} \int_{\theta_0}^{\theta} \frac{du}{u} \left(-V^\pm_D\right)^{-1/2},
\]

\[
X^2 (\theta, \sigma) = X^2_0 + C^2_+ \left( \lambda^1 \tau + \sigma \right) \pm \frac{B^\pm_2}{\varrho} \int_{\theta_0}^{\theta} \frac{du}{\sin^2 u} \left(-V^\pm_D\right)^{-1/2},
\]

\[
\tau (\theta) = \tau_0 \pm \frac{\varrho}{\varrho} \int_{\theta_0}^{\theta} du \left(-V^\pm_D\right)^{-1/2}, \quad \varrho \equiv r_0 \sin \psi_0,
\]

\[
F_{01}(\theta) = g_{sT_D} \lambda^2 \left[\left(\frac{C^0_+}{R}\right)^2 g_{00}^0 + \left(\varrho C^2_+\right)^2 \sin^2 \theta\right],
\]

where

\[
-V^\pm_D = \left[(B^\pm_0)^2 g_{11}^0 - \left(A^\pm_0 C^2_+\right)^2 g_{00}^0\right] - \left(A^\pm_0 \varrho\right)^2 \sin^2 u - \left(\frac{B^\pm_2 \varrho}{\sin^2 u}\right)^2.
\]
This is a solution for $D$-string placed in the subspace described by the coordinates $(x^0, x^2, x^3)$.

Another possibility is when the coordinates $r$ and $\theta$ are kept fixed, while the coordinate $\psi$ is allowed to vary. However, it is easy to show that in this case the dependence of the solution on $\psi$ will be the same as on $\theta$, with some constants changed.

Finally, we will give an example of exact solution for a $D$-string moving in a non-diagonal metric. To this end, let us consider the ten dimensional black hole solution of [38]. In string frame metric, it can be written as [39]

$$ds^2 = \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-1/2} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{-1/2} \times \left\{dt^2 + (dx^9)^2 + \frac{r_0^2}{r^2} \left(\cosh \chi dt + \sinh \chi dx^9\right)^2 + \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right) \left[(dx^5)^2 + (dx^6)^2 + (dx^7)^2 + (dx^8)^2\right]\right\} \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{1/2} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{1/2} \left[1 - \frac{r_0^2}{r^2}\right]^{-1} dr^2 + r^2 d\Omega_3^2\right\},$$

$$\exp \left[-2(\phi - \phi_\infty)\right] = \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-1} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right). \tag{37}$$

The equalities (37) define a solution of type IIB string theory, which low energy action in Einstein frame contains the terms

$$\int d^{10}x \sqrt{-g} \left[R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{12} \exp(\phi) H'^2\right], \tag{38}$$

where $H'$ is the Ramond-Ramond three-form field strength. The Neveu-Schwarz 3-form field strength, the selfdual 5-form field strength and the second scalar are set to zero. Therefore, in the solution (33), we also have to set $B_{MN} = 0$. Besides, our solution corresponds to a constant dilaton field $\phi = \phi_0$. If we identify $\phi_0 \equiv \phi_\infty$, this leads to $\alpha = \gamma$ in (37). On the other hand, we have not included in the $D$-string action (29) the Ramond-Ramond 2-form gauge field, so we have to set $H' = 0$. It follows from here that $\alpha = \gamma = 0$ [39]. All this results in a simplification of the metric (37), and in this simplified metric the exact $D$-string solution as a function of the radial coordinate $r$ reads

$$X^\mu(r, \sigma) = X_0^\mu + C_\pm^\mu \left(\lambda^1 \tau + \sigma\right) \pm I^\mu(r), \quad \mu = 0, 2, 5, 6, 7, 8, 9,$n
$$X^{3,4} = X^{3,4}_0 = \text{constants}, \quad \tau(r) = \tau_0 \pm I(r),$$

where

$$I^0 = -\int_{r_0}^r du \left[B_0^\pm g_{09} - B_9^\pm g_{09}\right] g_{11} W^{-1/2},$$

$$I^9 = \int_{r_0}^r du \left[B_0^\pm g_{09} - B_9^\pm g_{09}\right] g_{11} W^{-1/2},$$
\[ I^l = B^l \int_{r_0}^r \frac{du}{g_{00}} W^{-1/2}, \quad l = 2, 5, 6, 7, 8, \quad I = \int_{r_0}^r du W^{-1/2}, \]

\[ F_{01} = g_s T_D \lambda^2\sqrt{\mu} C^\mu_\pm C^\nu_\pm g_{\mu\nu}(r), \]

\[ W = \left[ \left( B^\pm_9 \right)^2 - \frac{(A^\pm_9)^2}{g_{11}} \right] g_{00} + \left[ \left( B^\pm_9 \right)^2 - \frac{(A^\pm_9)^2}{g_{11}} \right] g_{99} \]

\[ -2 \left( B^\pm_9 B^\pm_9 + \frac{A^\pm_9 C^\pm_9}{g_{11}} \right) g_{99} \]

\[ -\frac{1}{g_{11}} \left\{ \sum_{l=5}^8 \left[ (A^l_\pm C^l_\pm)^2 + (B^l_\pm)^2 \right] + \left( \frac{B_2^2}{c} \right)^2 \frac{1}{u^2} + (A^2_\pm C^2_\pm)^2 u^2 \right\}. \]

Actually, after fixing the dilaton to constant and \( H' \) to zero, we have obtained a solution for a \( D \)-string moving in ten dimensional Einstein gravity background, as is seen from (38). That is why, now we are going to consider the case \( \phi = \phi_0, H' \neq 0 \). To this aim, we have to add a Wess-Zumino term to the action (29), describing the coupling of the \( D \)-string to the Ramond-Ramond two-form gauge potential \( B'_{MN} \) \( (H' = dB') \).

It can be shown that this leads to the following change in the equations of motion (31)

\[ \lambda^2 H_{LMN} \longrightarrow \lambda^2 H_{LMN} + 2\lambda^0 \mu \exp(a \phi_0) H'_{LMN}, \]

where \( \mu \) is the \( D \)-string charge \( (\mu = \pm T_D \) from the requirements of space-time supersymmetry and worldsheet \( \kappa \)-invariance of the super \( D \)-string action). Then one proceeds as before to obtain an exact solution of the equations of motion and constraints. This solution is given by (33), where the replacement

\[ \lambda^2 B_{MN} \longrightarrow \lambda^2 B_{MN} + 2\lambda^0 \mu \exp(a \phi_0) B'_{MN}, \]

must be done. Now we can put there the background (37) with \( \alpha = \gamma \neq 0 \) to obtain an explicit probe \( D \)-string solution.

7 Discussion

In this article we performed some investigation on the classical string dynamics in \( D \)-dimensional (super)gravity background. In Section 2 we begin with rewriting the string action for curved background in a form in which the limit \( T \to 0 \) could be taken to include also the null string case, known to be a good approximation for string dynamics in strong gravitational fields. Then we propose an ansatz, which reduces the initial dynamical system depending on two worldsheet parameters \( (\tau, \sigma) \) to the one depending only on \( \tau \), whenever the background metric does not depend at least on one coordinate. An alternative ansatz is also given, which leads to a system depending only on \( \sigma \).

Let us note that the usually used conformal gauge for the auxiliary worldsheet metric \( \gamma^{mn} \) corresponds to \( \lambda^1 = 0, 2\lambda^0 T = 1 \). However, the latter does not allow for unified description of both tensile and tensionless strings.
The ansatz (6) and the ansatz (9) generalize the ones used in [12]-[32] for finding exact string solutions in curved backgrounds. It is also worth noting that (6) and (9) are based on the obtained string solutions \( F_M^\pm( w^\pm) \), which do not depend on the background metric. In conformal gauge, they reduce to the solutions for left- or right-movers, known to be the only background independent non-perturbative solutions for an arbitrary static metric, which are stable and have a conserved topological charge being therefore topological solitons [6].

In Section 3, using the existence of an abelian isometry group \( G \) generated by the Killing vectors \( \partial/\partial x^\mu \), the problem of solving the equations of motion and two constraints in \( D \)-dimensional curved space-time \( \mathcal{M}_D \) with metric \( g_{MN} \) is reduced to considering equations of motion and one constraint in the coset \( \mathcal{M}_D/\mathcal{G} \) with metric \( h_{ab} \). As might be expected, an interaction with an effective gauge field appears in the Euler-Lagrange equations. In this connection, let us note that if we write down \( A^\pm_a \), introduced in (17), as
\[
A^\pm_a = A^\nu_A^\pm A^\nu_a ,
\]
this establishes a correspondence with the usual Kaluza-Klein type notations and
\[
g_{MN}dy^Mdy^N = h_{ab}dy^ady^b + g_{\mu\nu}(dy^\mu + A^\mu_a dy^a) (dy^\nu + A^\nu_b dy^b) .
\]

In the remaining part of Section 3, we impose a number of conditions on the background metric, sufficient to obtain exact solutions of the equations of motion and constraints. These conditions are such that the metric is general enough to include in itself many interesting cases of curved backgrounds in different dimensions.

In Section 4, the previous results are generalized to include a nontrivial Kalb-Ramond background gauge field \( B_{MN} \), which arises in the supergravity theories - the low energy limits of superstring theories. The \( B_{MN} \) is restricted to depend on these coordinates on which the background metric does. There are also indirect restrictions on \( g_{MN} \) and \( B_{MN} \). They follow from the condition on the effective one-form potential \( B^\pm_a \) in the equations of motion (25) to be oriented along one of the coordinate axes, all other components being pure gauges. It is explained how the derived probe string solution looks like also in Einstein frame metric - the one used when searching for brane solutions of the string-theory effective field equations. At the end of the section, the induced worldsheet metric is given and it depends only on the \( g_{\mu\nu} \) part of the background metric, which corresponds to its Killing vectors \( \partial/\partial x^\mu \).

In section 5, we consider the \( D \)-string dynamics. In an appropriate parameterization of the auxiliary worldsheet field \( \mathcal{K}^{mn} \) and with the help of a modified ansatz, we succeeded to reduce the task of finding exact solutions to the application of methods used for this purpose in the previous section. In that case, the limit \( T_D \to 0 \) also could be taken and this will give a solution of the equations of motion and constraints for a tensionless \( D \)-string. However, the tensionless \( D \)-string is not a null string, i.e. the induced worldsheet metric is not degenerate. Really, the induced metric is
\[
G_{00} = \left[ (\lambda^1)^2 - (\lambda^2)^2 - (2\lambda^0 T_D)^2 \right] G_{11} , \\
G_{01} = \lambda^1 G_{11} , \\
G_{11} = C_\pm^a C_\pm^\nu g_{\mu\nu}(X^a) ,
\]

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\[ \text{det} \, G_{mn} = - \left[ (\lambda^2)^2 + (2\lambda^0 T_D)^2 \right] G^2_{11}. \]

Consequently, when \( T_D = 0 \), \( \text{det} \, G_{mn} \) is still different from zero. Moreover, the tensile fundamental string can be viewed as a particular case of the tensionless \( D \)-string [40]. Indeed, it is not difficult to show by using (2) and (30) that the fundamental string action (21) can be obtained from the \( D \)-string action (29) by setting \( A_m = 0 \), \( \phi = 0 \), \( \lambda^2 = 2\lambda^0 T \) and taking \( T_D \to 0 \).

The next section is devoted to four examples of exact string solutions in different backgrounds in four, five and ten dimensions. As an example of solution in cosmological type background, we consider four dimensional Kasner space-time. The next example is for a string moving in ten dimensional supergravity background. The \( D \)-string solutions are illustrated with two examples - in five dimensional anti de Sitter and in ten dimensional black hole backgrounds. It is evident from the solution (27) and from the first two examples that the (exact) null string dynamics in curved backgrounds, [41] - [43], [20], [44] - [48], is simpler than the tensile string one. Moreover, the tensionless string does not interact with the \( B_{MN} \) background, according to the action (21). This is a consequence of the condition for \( \kappa \)-invariance of the Green-Schwarz superstring action, which bosonic part is (21). Actually, the null super \( p \)-brane actions are \( \kappa \)-invariant in flat space-time without Wess-Zumino terms [49, 50]. The \( D \)-string case is different. The tensionless \( D \)-string does interact with the Kalb-Ramond background.

Let us finally note some specific features of the received solutions. The parameterization of the worldsheet fields \( \gamma^{mn} \) and \( K^{mn} \) is such that permit for a unified description of the tensile and tensionless string solutions. The ansatz (6) and the ansatz (32) contain the terms \( \pm 2\lambda^0 T C_{\pm m} \) and \( A_{\pm n} C_{\pm n} \) respectively, which disappear in the expressions (27) and (33) for the exact \( (D) \)-string solutions as a function of one of the coordinates. There, they are replaced by the corresponding integrals. The other part of the background independent solution, \( C_{\pm}^n (\lambda^1 \tau + \sigma) \), is a particular case of such solution for the null \( p \)-branes [51, 25]. Another distinguishing feature of our exact solutions is that we do not restrict ourselves to the usually used static gauge \( X^m (\xi) = \xi^m \), when probe brane dynamics is investigated. It is possible to impose this gauge on the solutions and this will result in additional conditions on them.

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References


