Kaonic modes in hyperonic matter and \( p \)-wave kaon condensation

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Abstract

Kaon excitations (kaonic modes) are investigated in hyperonic matter, where hyperons (\( \Lambda, \Sigma^-, \Xi^- \)) are mixed in the ground state of neutron-star matter. \( P \)-wave kaon-baryon interactions as well as the \( s \)-wave interactions are taken into account within chiral effective Lagrangian, and the nonrelativistic effective baryon-baryon interactions are incorporated. When the hyperon \( \Lambda \) is more abundant than the proton at high baryon density, a proton-particle-\( \Lambda \)-hole mode, which has the \( K^+ \) quantum number, appears in addition to other particle-hole modes with the \( K^- \) quantum number. It is shown that the system becomes unstable with respect to a spontaneous creation of a pair of the particle-hole modes with \( K^+ \) and \( K^- \) quantum numbers, stemming from the \( p \)-wave kaon-baryon interaction. The onset density of this \( p \)-wave kaon condensation may be lower than that of the \( s \)-wave \( K^- \) condensation.

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Kaon condensation in high density matter has been investigated extensively from various points of view [1–6]. Its existence makes the equation of state (EOS) of high density matter much softened, which affects static properties of neutron stars [4,7–9] and also dynamical evolution of protoneutron stars [10,11]. A kaon condensate also leads to rapid cooling of neutron stars via enhanced neutrino emissions [7,12–15]. The driving force for kaon condensation is the s-wave kaon-nucleon (KN) interaction which consists of the scalar attraction simulated by the KN sigma term $\Sigma_{KN}$ and the vector interaction corresponding to the Tomozawa Weinberg term. In neutron-star matter, the lowest excitation energy $\omega_{\text{min}}$ of the antikaon decreases with baryon number density $\rho_B$ due to the s-wave $KN$ attraction. At a critical density, $\omega_{\text{min}}$ becomes equal to the kaon chemical potential $\mu_K$, and the $K^-$ appears macroscopically as a Bose-Einstein condensate (BEC).\(^1\)

The critical density $\rho_B^C$ has been estimated as $\rho_B^C = 3 - 4 \rho_0$ with $\rho_0 (=0.16 \text{ fm}^{-3})$ being the nuclear saturation density [1–6]. Recently, some authors examined the possibility of kaon condensation in neutron stars by taking into account many-body effects such as the Pauli-blocking and the nucleon-nucleon correlations [16,17].

Studies of kaon condensation stimulated theoretical investigations of in-medium kaon properties with reference to the kaon-nucleon scattering [18–21], kaonic atoms [22–25], and heavy-ion collisions [26–30]. The momentum dependence of kaon optical potentials in a nuclear medium has also been discussed, which may serve as unified understanding of these phenomena [31–33]. Although there is still a debate concerning the strength of the kaon optical potential, recent experimental results on the subthreshold $K^+K^-$ production in relativistic heavy-ion collisions and proton-nucleus collisions suggest a substantial decrease in the antikaon effective mass [34–36]. Based on the strongly attractive kaonic potential, a possibility of kaonic nuclei has been proposed [37–39].

So far, kaon condensation has been considered mostly in neutron-star matter consisting of neutrons, protons and leptons in chemical equilibrium. However, hyperons ($\Lambda$, $\Xi^-$, $\Sigma^-$, etc.) may appear in neutron-star matter. Since the early suggestion of hyperon-mixing in neutron stars [40,41], the possible existence of hyperonic matter, where hyperons are mixed as well as neutrons, protons, and leptons in the ground state of neutron-star matter, has been discussed by several authors [42–56].

Recent development of hypernuclear experiments enables us to discuss the hyperon-mixing problem in highly dense matter in a realistic situation. It has been shown that hyperons appear at a baryon number density $\rho_B = 2 \sim 3 \rho_0$ based on the relativistic mean-field (RMF) models [45–48], relativistic Hartree-Fock methods [49], the nonrelativistic reaction matrix theory [50,51], the nonrelativistic effective baryon-baryon potential models [52–54], etc. [55,56] The hyperonic matter is also relevant to static and dynamic properties of neutron stars [57,58] and thermal evolution of neutron stars via rapid cooling [59].

The existence region of kaon condensation and hyperons may overlap each other, so that the competition or coexistence problem of these phases has to be clarified. Concerning this problem, it has been pointed out that a critical density of $K^-$ condensation, which

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\(^1\)The kaon chemical potential $\mu_K$ is equal to the electron chemical potential $\mu_e$ in chemical equilibrium. We put $\mu_K = \mu_e = \mu$, and call $\mu$ the charge chemical potential throughout this paper.
may be realized from hyperonic matter, is pushed up to a higher baryon number density
in the presence of the negatively charged hyperons in comparison with the neutron-star
matter consisting of only neutrons, protons, and leptons [46,47]: The negative charge of
the system is carried by the negatively charged hyperons in place of the electrons as the
hyperon-mixing develops, making up with the positive charge and satisfying the charge
neutrality. Accordingly, the number of the electrons $\rho_e$ and the charge chemical potential
$\mu \left[= (3\pi^2 \rho_e)^{1/3} \right]$ get smaller with the increase in the baryon number density, which makes
the onset condition for kaon condensation, $\omega_{\text{min}} = \mu$, difficult to satisfy. In Refs. [46,47],
only the $s$-wave kaon-baryon interaction has been taken into account. However, there
is the $p$-wave kaon-baryon interaction with Yukawa couplings, which may affect kaon
dynamics in dense matter crucially as well as the $s$-wave interaction. In this context,
interrelation between kaons and hyperons has been discussed through the introduction
of a “kaesobar” which is a linear combination of the $K^-$ and $\Sigma^-$-particle-neutron-hole
states produced by the $p$-wave interaction [26,60]. On the other hand, the effects of the
$p$-wave kaon-baryon interaction on kaon condensation have been considered in neutron-
star matter where hyperons are not mixed in the ground state of dense matter [61,62]. In
Ref. [61], the kaon-baryon interaction has been taken into account within chiral effective
Lagrangian. It has been shown that the $s$-wave $K^-$ condensation sets in at a lower density,
and at a higher density, the $p$-wave $K^-$ condensation is realized accompanying hyperon
excitation in the form of quasi-particles which are superimposed states of nucleons and
hyperons. The $K^-$ mode has been shown to be relevant to the onset of $p$-wave $K^-$
condensation. In Ref. [62], pionic intermediate states, kaon fluctuations and residual
interaction have been incorporated in addition to the $s$-wave and $p$-wave kaon-baryon
interactions. It has been shown that the energy of a low-lying $\Lambda$-particle-proton-hole
branch becomes negative beyond some density, leading to a first-order phase transition
to proton matter accompanying kaon condensation.

In this paper, we consider kaon dynamics and mechanisms of kaon condensation in
hyperonic matter by taking into account the $p$-wave kaon-baryon interaction. We discuss
in-medium properties of kaons by obtaining the kaon dispersion relations in hyperonic
matter. Kaon-baryon interactions for both the $s$-wave and $p$-wave type are incorpo-
rated within the effective chiral Lagrangian, while other baryon-baryon interactions are
supplemented by the use of the nonrelativistic effective interactions. We show that a
low-lying proton-particle-$\Lambda$-hole mode with the $K^+$ quantum number develops at high
densities. This mode, together with the other particle-hole modes carrying the $K^-$ quan-
tum number, leads to a new mechanism of kaon condensation stemming from the $p$-wave
kaon-baryon interaction.

The paper is organized as follows. Section II gives the formulation to obtain the
kaonic modes in hyperonic matter. In Sec. III, numerical results and discussion are given.
Summary and concluding remarks are devoted in Sec. IV. In the Appendix, the expression

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2The same situation is applied for the neutral hyperons, e.g., $\Lambda$: As the $\Lambda$ appears, the electron
is absorbed by the process, $p + e^- \rightarrow \Lambda + \nu_e$ (see III B).

3Throughout this paper, the $K^-$ and $K^+$ denote the kaonic modes which reduce to free kaons
in vacuum ($\rho_B \rightarrow 0$), and are distinguished from other particle-hole modes carrying the $K^\pm$
quantum numbers.

4Part of this work has been briefly reported in Ref. [63].
for the potential energy density which is used in the formulation is presented.

II. FORMULATION

A. Kaon-baryon interaction

We start with the effective chiral SU(3)$_L \times$ SU(3)$_R$ Lagrangian used by Kaplan and Nelson [1] for the kaon-baryon interaction.

\[ \mathcal{L} = \frac{1}{4} f^2 \text{Tr} \partial^\mu \Sigma^\dagger \partial_\mu \Sigma + \frac{1}{2} f^2 \Lambda_{\chi SB} (\text{Tr} M (\Sigma - 1) + \text{h.c.)} \]
\[ + \text{Tr} \bar{\Psi} (i \beta - M_B) \Psi + \text{Tr} \bar{\Psi} \gamma^\mu [V_\mu, \Psi] + D \text{Tr} \bar{\Psi} \gamma^\mu \gamma^5 \{ A_\mu, \Psi \} + F \text{Tr} \bar{\Psi} \gamma^\mu \gamma^5 [A_\mu, \Psi] \]
\[ + a_1 \text{Tr} \bar{\Psi} (\xi M^\dagger \xi + \text{h.c.)} \Psi + a_2 \text{Tr} \bar{\Psi} (\xi M^\dagger \xi + \text{h.c.)} + a_3 (\text{Tr} M \Sigma + \text{h.c.)} \text{Tr} \bar{\Psi} \Psi, \quad (1) \]

where \( f (\sim f_\pi = 93 \text{ MeV}) \) is the meson decay constant, \( \Sigma \) is the nonlinear meson field, \( \Sigma \equiv \frac{e^2 \Pi}{f} \), in terms of the octet meson \( \Pi \), which is represented as

\[ \Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & K^0 & -\sqrt{2/3} \eta \end{pmatrix} \quad (2) \]

with the Nambu-Goldstone bosons \( \pi_a \) and the SU(3) generators \( T_a \) (\( a = 1 \sim 8 \)). \( \Lambda_{\chi SB} \) is the chiral symmetry breaking scale, \( \sim 1 \text{ GeV} \), \( M \) the mass matrix which is defined as \( M \equiv \text{diag}(m_u, m_d, m_s) \) with the quark masses \( m_i \), and \( M_B \) is the baryon mass generated as a consequence of spontaneous chiral symmetry breakdown. The baryon fields are given as the spin 1/2 octet \( \Psi \),

\[ \Psi = \begin{pmatrix} \Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & \Sigma^+ \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & p \\ \Xi^- & \Xi^0 & -2\Lambda/\sqrt{6} \end{pmatrix}. \quad (3) \]

Baryons couple with mesons through the mesonic vector current \( V_\mu \), which is defined by \( V_\mu \equiv 1/2 (\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger) \) with \( \xi \equiv \Sigma^{1/2} \), and the axial vector current \( A_\mu \), defined by \( A_\mu \equiv i/2 (\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger) \). In the nonrelativistic limit, the fourth term including \( V_\mu \) in Eq. (1) reduces to the s-wave vector interaction, and the fifth and sixth terms to the p-wave interaction. The last three terms in Eq. (1) represent the s-wave meson-baryon scalar interaction, which explicitly breaks down chiral symmetry. Throughout this paper, we take the same set of parameters as those in Ref. [1]: The axial vector coupling constants \( D \) and \( F \) are determined from the weak decays of the baryon octet states and are chosen as \( D = 0.81 \) and \( F = 0.44 \). The quark masses are taken to be \( m_u = 6 \text{ MeV} \), \( m_d = 12 \text{ MeV} \) and \( m_s = 240 \text{ MeV} \). With these values for \( m_i \), the parameters \( a_1 \) and \( a_2 \) are fixed so as to give the empirical octet baryon mass splittings and given as \( a_1 = -0.28 \), \( a_2 = 0.56 \).

The parameter \( a_3 \) is related to the “\( \pi N \) and \( KN \) sigma terms” which simulate the s-wave meson-baryon scalar interactions through the relations, \( \Sigma_{\pi N} = -(a_1 + 2a_3)(m_u + m_d) \).

\footnote{We use the units in which \( \hbar = c = k_B = 1 \) throughout this paper.}
\[ \Sigma_{K_p} = -(a_1 + a_2 + 2a_3)(m_u + m_s), \] and \[ \Sigma_{K_n} = -(a_2 + 2a_3)(m_u + m_s). \] Considering that there is some ambiguity about these values, we take \( a_3 = -0.9 \) and \(-1.1 \), which yields \( \Sigma_{\pi N} = 37 \) MeV, \( \Sigma_{K_p} = 374 \) MeV, \( \Sigma_{K_n} = 305 \) MeV for \( a_3 = -0.9 \), and \( \Sigma_{\pi N} = 45 \) MeV, \( \Sigma_{K_p} = 472 \) MeV, \( \Sigma_{K_n} = 403 \) MeV for \( a_3 = -1.1 \).

In this paper, we mainly concentrate on the kaon dispersion relations in hyperonic matter and onset mechanisms of kaon condensation. Expecting that we can also discuss the EOS of a fully-developed kaon-condensed phase beyond the onset density \([68]\) within the same framework, we give here a formulation to elucidate both issues in a unified way: First, the energy density of the kaon-condensed phase, \( \mathcal{E}_{\text{eff}} \), is obtained, and second, the kaon inverse propagator \( D_K^{-1}(\omega, k; \rho_B) \) is obtained from the expansion of the energy density expression with respect to the classical kaon field \( K^{\text{cl}} \) as
\[ \mathcal{E}_{\text{eff}}(|K^{\text{cl}}|) = \mathcal{E}_{\text{eff}}(|K^{\text{cl}}| = 0) - D_K^{-1}(\omega = \mu, k; \rho_B)|K^{\text{cl}}|^2 + O(|K^{\text{cl}}|^4). \]

The kaon inverse propagator \( D_K^{-1} \) includes the self energy which is related to the forward \( KN \) scattering amplitudes. It is to be noted that there are different ways of off-shell extrapolation from the \( KN \) scattering amplitudes in obtaining the kaon self energy: One is derived from chiral perturbation theory and the other is based on low energy theorems of current algebra and PCAC (partial conservation of axial-vector current). These approaches lead to different off-shell properties of the kaon excitation energy \([64,65,2,4]\). The relations between these approaches have been discussed \([66,67]\). Based on the former way with the chiral effective Lagrangian Eq. (1), the s-wave \( KN \) scattering amplitudes near threshold cannot be reproduced empirically without higher order terms in chiral expansion and a pole contribution from the \( \Lambda (1405) \) (abbreviated to \( \Lambda^* \)) \([65,2,19]\). These corrections enter into the kaon self energy. Nevertheless, it has been shown that part of the higher order terms for \( K^- \), the range term \([\propto \omega(K^-)^2] \), becomes small in high-density matter, since the \( K^- \) excitation energy \( \omega(K^-) \) decreases with density owing to the s-wave \( KN \) attractive interaction. Furthermore, the pole contribution from the \( \Lambda^* \) becomes negligible since the \( K^-N \) threshold lies far below the pole of the \( \Lambda^* \) at high density due to the decrease in \( \omega(K^-) \). Hence these corrections become irrelevant to the s-wave \( K^- \) excitation energy in high-density matter \([2,4,19,65]\), while the \( K^+ \) excitation energy is largely changed due to the repulsive effects from the higher-order corrections.

When hyperons are incorporated, extra terms next to leading order, coming from the p-wave kaon-baryon interaction, contribute to the self energy. These terms are typically proportional to \((k^\mu)^2 = \omega^2 - k^2 \), and are responsible for reproducing the low-energy data on pion- and photon-induced kaon production and elastic and inelastic \( K^-p \) scatterings \([69,70]\). In hyperonic matter, several kaonic modes of particle-hole excitations, \( \Lambda p^{-1}, \Sigma^-n^{-1}, \Xi^-\Lambda^{-1} \), appear in addition to the \( K^\pm \) due to the p-wave interaction, as shown

\[ \text{[6]} \text{Similar methods have been utilized in Refs. [4,61].} \]

\[ \text{[7]} \text{On the other hand, the role of the} \ \Lambda^* \text{in a nuclear medium has been discussed on the assumption that it is a} \ K^-p \text{bound state [18–21]: In Ref. [18], it has been shown that the mass of the} \ \Lambda^* \text{is shifted upwards due to the Pauli blocking of intermediate states, and that the} \ K^- \text{feels an attractive potential by scattering through an intermediate} \ \Lambda^* \text{state. In subsequent works, the mass of the} \ \Lambda^* \text{has been shown to be left as a free space value due to the net effects of the Pauli-blocking and the attraction which comes from the modification of the} \ K^- \text{in the intermediate} \ K^-N \text{states, while the decay width of the} \ \Lambda^* \text{increases with an increase in the baryon number density [20,21].} \]
in Sec. III, and the excitation energies for these modes are small, being of order of the s-wave $K^-$ energy. Thus we expect that the higher order terms coming from the $p$-wave interaction have minor effects on the kaonic modes as far as the kaon momentum is not very large, i.e., $|k| = O(\omega)$ except for the $K^+$. Hence, throughout this paper, we are based on the chiral effective Lagrangian Eq. (1), and don’t take into account higher order terms in chiral expansion with respect to $M/\Lambda_{\chi SB}$ and $\partial/\Lambda_{\chi SB}$.

There are also other $p$-wave subthreshold resonances such as $\Sigma(1385)$ (abbreviated to $\Sigma^*$), which may appear in the dispersion relations of the kaonic modes. However, the excitation energies of the $\Sigma^*-N$ modes are of the order $\sim 450$ MeV so that their branches lie far above the other particle-hole branches, $p\Lambda^{-1}$, $\Sigma^-n^{-1}$, $\Xi^-\Lambda^{-1}$ considered in this paper. Furthermore, their coupling strengths $g_{\Sigma^*N}$ to the $K^-N$ are not so large as compared with $g_{\Lambda p}$, $g_{\Sigma^-n}$, and $g_{\Xi^-\Lambda}$. Hence the inclusion of these resonances would not change the results quantitatively. Quantitative evaluation including the higher order terms in chiral perturbation and effects of the subthreshold resonances remains to be a future investigation.

B. Effective energy density

For simplicity, we take only the $\Lambda$, $\Sigma^-$, and $\Xi^-$ for hyperons in addition to the proton ($p$), neutron ($n$) for nucleons and the ultrarelativistic electron for leptons. The other hyperons, $\Sigma^0$, $\Sigma^+$ and $\Xi^0$, are supposed to be irrelevant because they appear in higher densities due to their heavy baryon masses and due to the fact that the electron chemical potential does not assist to satisfy the threshold condition in contrast to the case of the negatively charged hyperons.

We consider charged kaon condensation in chemically-equilibrated hyperonic matter, and neglect other mesons in the octet meson field $\Sigma$. The classical kaon field is then assumed to be a plane wave type:

$$K^{\pm,cl}(r,t) = \frac{f}{\sqrt{2}} \theta e^{\pm i(\mu_K t - k \cdot r)} ,$$

(4)

where $\theta$ is a chiral angle which means an amplitude of a condensate, $\mu_K$ the kaon chemical potential, and $k$ is the kaon momentum.

The effective Hamiltonian $H_{\text{eff}}$ is derived with the introduction of the charge chemical potential $\mu(=\mu_K=\mu_e)$ by a charge neutrality condition, and is separated into baryon, meson and lepton parts:

$$H_{\text{eff}} = H_{\text{eff}}^B + H_{\text{eff}}^M + H_{\text{eff}}^e ,$$

(5)

where $H_{\text{eff}}^B = H^B + \mu \rho_Q^B$, $H_{\text{eff}}^M = H^M + \mu \rho_Q^M$, and $H_{\text{eff}}^e = H^e + \mu \rho_Q^e$ with the charge density operators $\rho_Q^i$ ($i = B, M, e$).

The axial vector coupling terms in the baryonic part $H_{\text{eff}}^B$ have space-time dependent factors $\exp(\pm ip_K \cdot x)$ with a four momentum $p_K^\mu = [\mu_K, k]$. These factors are eliminated by transformation of the baryon bases $\bar{\psi}^T = (p, \Lambda, \Xi^-, n, \Sigma^-)$ as $\psi^T \rightarrow \bar{\psi}^T =$
\[ (e^{-i\vec{p} \cdot \vec{x}} p, \Lambda, e^{i\vec{p} \cdot \vec{x}} \Xi^-, e^{-i\vec{p} \cdot \vec{x}}/2 n, e^{i\vec{p} \cdot \vec{x}}/2 \Sigma^-). \]

By this transformation, the momentum of each baryon is shifted by \( V_3 \vec{k} \), where \( V_3 \) is the \( V \)-spin defined by \( V_3 = \frac{1}{2}(I_3 + \frac{3}{2}Y) \) with the third component of the isospin \( I_3 \) and the hypercharge \( Y \). After making the Foldy-Wouthuysen-Tani transformation for \( \mathcal{H}_{\text{eff}}^B \) and expanding it up to \( O(1/M_N) \) with \( M_N \) the nucleon mass, one obtains the baryonic part of the Hamiltonian in a nonrelativistic form as \( \mathcal{H}_{\text{eff}}^B = \psi^\dagger \mathcal{H}_{\text{eff}}^B \psi \) with

\[
H_{\text{eff}}^B = \begin{pmatrix}
H_{pp} & H_{p\Lambda} & H_{p\Xi^-} & 0 & 0 \\
H_{p\Lambda} & H_{\Lambda\Lambda} & H_{\Lambda\Xi^-} & 0 & 0 \\
H_{p\Xi^-} & H_{\Xi^-\Lambda} & H_{\Xi^-\Xi^-} & 0 & 0 \\
0 & 0 & 0 & H_{n\Sigma^+} & H_{n\Sigma^-} \\
0 & 0 & 0 & H_{\Sigma^-\Sigma^-} & H_{\Sigma^-\Sigma^-}
\end{pmatrix},
\]

where the nonvanishing matrix elements are given by

\[
\begin{align*}
H_{pp} &= \frac{1}{2M_N} \left\{ (p - k \cos \theta)^2 + \left( \frac{1}{2}g_{\Lambda p} \mu \sin \theta \right)^2 \right\} + \mu \cos \theta - \Sigma_{Kp}(1 - \cos \theta) \\
H_{p\Lambda} &= \frac{i}{2}g_{\Lambda p} \sin \theta \left\{ k - \frac{\mu}{2M_N}(2p - k \cos \theta) \right\} \cdot \sigma = H_{p\Lambda}^* \\
H_{p\Xi^-} &= \frac{1}{8M_N} g_{\Xi^- p} g_{\Xi^- \Xi^-}(\mu \sin \theta)^2 = H_{p\Xi^-} \\
H_{\Lambda\Lambda} &= \frac{1}{2M_N} \left[p^2 + \left\{ \left( \frac{g_{\Lambda p}}{2} \right)^2 + \left( \frac{g_{\Xi^- \Xi^-}}{2} \right)^2 \right\}(\mu \sin \theta)^2 \right] + \delta M_{\Lambda N} - \Sigma_{K\Lambda}(1 - \cos \theta) \\
H_{\Lambda\Xi^-} &= \frac{i}{2}g_{\Xi^- \Xi^-} \sin \theta \left\{ k - \frac{\mu}{2M_N}(2p + k \cos \theta) \right\} \cdot \sigma = H_{\Xi^-\Lambda}^* \\
H_{\Xi^-\Xi^-} &= \frac{1}{2M_N} \left\{ (p + k \cos \theta)^2 + \left( \frac{1}{2}g_{\Xi^- \Xi^-} \mu \sin \theta \right)^2 \right\} + \delta M_{\Xi^- \Xi^-} - \mu \cos \theta - \Sigma_{K\Xi^-}(1 - \cos \theta) \\
H_{n\Sigma^+} &= \frac{1}{2M_N} \left\{ (p - \frac{1}{2}k \cos \theta)^2 + \left( \frac{1}{2}g_{\Sigma^+ \Sigma^+} \mu \sin \theta \right)^2 \right\} - \mu \sin^2 \frac{\theta}{2} - \Sigma_{K\Sigma^+}(1 - \cos \theta) \\
H_{n\Sigma^-} &= \frac{i}{2}g_{\Sigma^- \Sigma^-} \sin \theta \left\{ k - \frac{\mu}{M_N}p \right\} \cdot \sigma = H_{\Sigma^-\Sigma^-}^* \\
H_{\Sigma^-\Sigma^-} &= \frac{1}{2M_N} \left\{ (p + \frac{1}{2}k \cos \theta)^2 + \left( \frac{1}{2}g_{\Sigma^- \Sigma^-} \mu \sin \theta \right)^2 \right\} + \delta M_{\Sigma^- \Sigma^-} - \mu \cos^2 \frac{\theta}{2} - \Sigma_{K\Sigma^-}(1 - \cos \theta).
\end{align*}
\]

In Eq. (7), \( \Sigma_{K\Lambda} \equiv \Sigma_{K\Lambda} \equiv \Sigma_{K\Xi^-} = \Sigma_{K\Sigma^-} \), and \( \Sigma_{K\Sigma^-} = \Sigma_{Kp} \cdot \delta M_{YN} \) are the hyperon-nucleon mass difference. The p-wave axial vector coupling strengths are defined by \( g_{pN} = (D + 3F)/\sqrt{6} (=0.87) \), \( g_{pN} = D - F (=0.37) \), and \( g_{p\Xi^-} = (-D + 3F)/\sqrt{6} (=0.21) \). For simplicity, the nucleon mass is taken to be the free neutron mass, \( M_N = M_p = M_n = 939.57 \) MeV, and the mass differences \( \delta M_{YN} \) are taken from the free baryon masses: \( M_N = 1115.7 \) MeV, \( M_{\Sigma^-} = 1197.4 \) MeV, and \( M_{\Xi^-} = 1321.3 \) MeV. The p-wave kaon-baryon Yukawa couplings are brought about through the terms proportional to \( \mathbf{k} \cdot \sigma \) with \( \sigma \) the spin matrix in the off-diagonal matrix elements in Eq. (7). There appear terms of order \( O(1/M_N) \) coming from meson-baryon recoils, \( \mathbf{p} \cdot \mathbf{k}/M_N, \mathbf{p} \cdot \mathbf{k}^2/M_N, \mu \mathbf{p} \cdot \sigma/M_N, \mu^2 \sin^2 \theta/M_N, \) etc. in Eq. (7). The magnitude of the baryon momentum \( |\mathbf{p}| \) is typically the Fermi momentum \( p_F \), which is in the range \( p_F \lesssim 4m_\pi \) with \( m_\pi \) being the pion mass. Thus these terms are considered to be small as compared with the kinetic energy \( \mathbf{p}^2/(2M_N) \) of each baryon so long as \( \mu = O(m_\pi) \) and \( |\mathbf{k}| \lesssim 3m_\pi \). Numerically, these terms
are not necessarily small. Nevertheless, we would like to elucidate mechanisms of kaon condensation within the basic $p$-wave kaon-baryon interaction in hyperonic matter, at the cost of detailed corrections to the effective Hamiltonian. Hence, for simplicity, we neglect the terms of order $O(1/M_N)$ except for the kinetic energy term.

After diagonalization of $H_B^{\text{eff}}$, the baryonic eigenstates are represented as “quasiparticles” which consist of superposition of the baryons, e.g., $|\tilde{p}\rangle = \alpha|p\rangle + \beta|\Lambda\rangle + \gamma|\Xi^-\rangle$, $|\tilde{n}\rangle = \delta|n\rangle + \epsilon|\Sigma^+\rangle$, etc., where $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ are coefficients. The baryon contribution to the effective energy density, $\mathcal{E}_B^{\text{eff}}$, is obtained from occupation of the quasiparticles over each Fermi sea [68].

$$\mathcal{E}_B^{\text{eff}} = \sum_i \sum_{|p| \leq |p_F(i)|} E_s^{(i)}(p),$$

(8)

where $p_F(i)$ $(i = \tilde{p}, \tilde{\Lambda}, \tilde{\Xi}^-, \tilde{n}, \tilde{\Sigma}^-)$ are the Fermi momenta, and the subscript ‘$s$’ stands for the spin states for the quasiparticles.

The mesonic contribution to the effective energy density is given by the substitution of the classical kaon field (4) into $H_M^{\text{eff}}$ as

$$\epsilon_M^{\text{eff}} = -\frac{1}{2} f^2 (\mu^2 - k^2) \sin^2 \theta + f^2 m_K^2 (1 - \cos \theta),$$

(9)

where $m_K \equiv [\Lambda_{\chi SB}(m_u + m_s)]^{1/2}$, which is identified with the free kaon mass, and is replaced by the experimental value, 493.7 MeV. The leptonic contribution in Eq. (5) reduces to the effective energy density

$$\epsilon_e^{\text{eff}} = \frac{\mu^4}{4\pi^2} - \mu \frac{\mu^3}{3\pi^2} = -\frac{\mu^4}{12\pi^2}$$

(10)

for the ultra-relativistic electrons. Thus one obtains the total effective energy density as $\mathcal{E}_{\text{eff}} = \mathcal{E}_B^{\text{eff}} + \mathcal{E}_M^{\text{eff}} + \mathcal{E}_e^{\text{eff}}$.

C. Potential contribution

Here we introduce baryon-baryon interactions beyond the framework of chiral symmetry. The potential energy density $\mathcal{E}_{\text{pot}}$ produced from baryon-baryon interactions is crucial to obtaining the EOS and matter composition of the ground state for not only the normal (noncondensed) phase but also the condensed phase. We suppose that $\mathcal{E}_{\text{pot}}$ depends on the extent of kaon condensation only implicitly through the change of each number density $\rho_i$ $(i = p, \Lambda, \Xi^-, n, \Sigma^-)$ due to condensation. Then the form of $\mathcal{E}_{\text{pot}}$ is assumed to be $\mathcal{E}_{\text{pot}}(\rho_p, \rho_\Lambda, \rho_{\Xi^-}, \rho_n, \rho_{\Sigma^-})$. Following a procedure for incorporating nuclear interactions in case of pion condensation [71,72], we define the potential for the baryon $V_i$ $(i=p, \Lambda, \Xi^-, n, \Sigma^-)$ in hyperonic matter as

$$V_i = \partial \mathcal{E}_{\text{pot}} / \partial \rho_i,$$

(11)

In general, the $\Sigma^0$ state is also superposed to the quasiparticle states $|\tilde{p}\rangle$, $|\tilde{\Lambda}\rangle$, and $|\tilde{\Xi}^-\rangle$ through the $p$-wave couplings, but we simply neglect the effects of the $\Sigma^0$ on these quasiparticle states.
which corresponds to a potential contribution to each baryon chemical potential, i.e.,
estimated at the Fermi momentum of each baryon. Thus momentum dependence of the
potential is neglected, which is reasonable for the $\Sigma^-$ and $\Lambda$ potentials because of their
weak momentum dependence, as suggested from recent microscopic calculations [50,51].
It should be reminded, however, that it is not a good approximation for the nuclear
potentials where the potential depths at zero momentum are deeper than the value at the
Fermi momentum.

The potential terms $V_i$ are added to each diagonal matrix element of the baryonic part
of the effective Hamiltonian $H^B_{\text{eff}}$ [Eq. (7)] as $H_i \rightarrow H'_i = H_i + V_i$. Then the baryonic part of the
modified effective energy density $E'_{\text{eff}}$ is given by

$$E'_{\text{eff}} = \sum_i \sum_{|p| \leq |p_F(i)|} \sum_{s=\pm 1/2} E_s^{(i)}(p) + E_{\text{pot}} - \sum_{i=p,\Lambda,\Sigma^-,n,\Sigma^-} \rho_i V_i,$$

where $E_s^{(i)}(p)$ are eigenvalues diagonalized with inclusion of the $V_i$ in the diagonal parts
of $H^B_{\text{eff}}$. The last term on the r. h. s. in Eq. (12) is introduced in order to subtract the
double counting of the baryon interaction energies in the first sum over the quasiparticle
energies $E_s^{(i)}(p)$.

For a practical use of the potential energy density $E_{\text{pot}}$ in hyperonic matter, we adopt
the nonrelativistic expression by Balberg and Gal [52], which includes hyperon-hyperon
interactions as well as hyperon-nucleon ones, and higher order terms in $\rho_i$ simulating the
many-body effects. The expression for the $E_{\text{pot}}$ is given in Appendix A. We take the
exponents $\delta$ and $\gamma$ in the density-dependent terms in Eq. (A1) to be $\delta = \gamma = 5/3$, which
gives the moderate stiffness of the EOS among the three cases in Ref. [52]. Some of
the parameters in $E_{\text{pot}}$ are refitted so as to be consistent with recent empirical data on
nuclear and hypernuclear experiments and the saturation properties of symmetric nuclear
matter: We take the saturation density $\rho_0=0.16$ fm$^{-3}$, the binding energy 16 MeV, and the
incompressibility 210 MeV at $\rho_0$ in symmetric nuclear matter, from which $a_{NN}$ and $c_{NN}$
are fixed for the isovector terms in the nucleon-nucleon $(NN)$ part of the potential energy
[Eq. (A1)]. From the symmetry energy $\sim$ 30 MeV at $\rho_B = \rho_0$, one obtains $b_{NN}$ for the
isospin-dependent term for the $NN$ part.10 For the $\Lambda N$ part, $a_{\Lambda N}$ and $c_{\Lambda N}$ are taken to be
the same as in Ref. [52]. The depth of the $\Lambda$ potential in nuclear matter is then equal to the
empirical value, i.e., $V_\Lambda (\rho_p = \rho_n = \rho_0/2, \rho_\Lambda = \rho_{\Sigma^-} = \rho_{\Sigma^-} = 0) = a_{\Lambda \Lambda} \rho_0 + c_{\Lambda \Lambda} \rho_0^2 = -27$
MeV [75]. For the $\Xi^- N$ part, $a_{\Xi N}$ and $c_{\Xi N}$ are related with each other by the use of the

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10It is to be noted that the potential contribution $V_{\text{sym}}(\rho_B)$ to the symmetry energy is read
from the isovector term for the $NN$ interactions in Eq. (A1) as $V_{\text{sym}}(\rho_B) = b_{NN} \rho_B^2/2$, which
mimics the density dependence of the $V_{\text{sym}}(\rho_B)$ in the RMF models. In normal neutron-star
matter, this linear density dependence leads to a large proton fraction $\rho_p/\rho_B \gtrsim 0.1$ at high
density and the large electron chemical potential $\mu_e$ through the relation $\mu_e = (3\pi^2 \rho_e)^{1/3}$ and
the charge neutrality $\rho_e = \rho_p$. The large proton fraction concerns with a possibility of the
direct Urca process in neutron stars [73,74], and the large electron chemical potential assists
the onset of the negatively charged hyperons, e.g., the $\Sigma^-$, through the chemical equilibrium
condition, $\mu_n + \mu_e = \mu_{\Sigma^-}$ [54]. However, there is an ambiguity about the density dependence of
the symmetry energy. E. g., the nonrelativistic potential models show its moderate increase in
density in comparison with the RMF case [15,74].
depth of the $\Xi^-$ potential in nuclear matter, which is deduced from the recent ($K^-, K^+$) experimental data as $-14 - 20$ MeV [76,77]. Here we put $V_\Xi^- (\rho_p = \rho_n = \rho_0/2$, $\rho_\Lambda = \rho_{\Xi^-} = \rho_{\Sigma^-} = 0) = a_{\Xi N} \rho_0 + c_{\Xi N} \rho_0^2 = -16$ MeV. Further, the crossover density $\rho_{\xi 0}$, where $V_\Xi^- (\rho_B = \rho_{\omega}) = 0$, is taken to be equal to that for the $\Lambda$ [52]. As for the $V_{\Sigma^-}$, recent ($K^-, \pi^+$) experiment at BNL [79] suggests a strong isospin dependence in the $\Sigma^-$-nucleus potential. It has been shown that the experimental data is compatible with the analysis by the Nijmegen model F for the baryon-baryon interaction, which gives a repulsive $\Sigma^-$ potential in nuclear matter, which is deduced from the recent ($K^-, K^+$) experiment at BNL [79] suggests a strong isospin dependence in the $\Sigma^-$-nucleus potential. The analysis of $\Sigma^-$ atom data also shows the repulsive $\Sigma^-$ potential [80]. Theoretically, a repulsive $\Sigma$ potential in nuclear matter has been obtained from the $G$ matrix calculation based on the SU(6) quark model baryon-baryon interaction [81]. Referring to these results, we first take the potential depth of the $\Sigma^-$ in nuclear matter to be repulsive as follows: In Ref. [78], the $\Sigma^-$ potential has been parametrized as $V_{\Sigma^-} (k_{\Sigma^-}) = V_0 (k_{\Sigma^-}) - \frac{1}{2} V_1 (k_{\Sigma^-}) \cdot \frac{2Z - A}{A}$, where $k_{\Sigma^-}$ is the $\Sigma^-$ momentum, and $A, Z$ are the mass number and the atomic number, respectively. The calculated values in Ref. [78] for $V_0$ and $V_1$ at $k_{\Sigma^-}=0$ in nuclear matter based on the Nijmegen model F are identified with the corresponding terms in our model such that $V_{\Sigma^-} (\rho_p = \rho_n = \rho_0/2, \rho_\Lambda = \rho_{\Xi^-} = \rho_{\Sigma^-} = 0) = a_{\Sigma N} \rho_0 + c_{\Sigma N} \rho_0^2 = V_0 = 23.5$ MeV, and $b_{\Sigma N} \rho_0 = V_1/2 = 40.2$ MeV. We call this parametrization for the repulsive $\Sigma^-$ potential Case I. For the second case, the parameters $a_{\Sigma N}, b_{\Sigma N}$, and $c_{\Sigma N}$ are taken to be the same as in Ref. [52]. The depth of the $V_{\Sigma^-}$ is then equal to that of the $\Lambda$ as an extreme case for the attractive $\Sigma^-$ potential, i.e., $a_{\Sigma N} \rho_0 + c_{\Sigma N} \rho_0^2 = -27$ MeV. We call this parametrization for the attractive $\Sigma^-$ potential Case II.

The remaining parameters in the potential energy density $\mathcal{E}_{\text{pot}}$, relevant to the hyperon-hyperon interactions, are taken to be the same as those in Ref. [52]. The numerical values for the parameters are summarized in Tables I and II in Appendix A.

In general, there are additional off-diagonal matrix elements in $\mathcal{H}_{\text{eff}}^B$ in the presence of condensation. These off-diagonal matrix elements include the densities $\bar{\rho}_{p\Lambda} \equiv i \langle p^\dagger \sigma \cdot \bar{k} \Lambda \rangle$, $\bar{\rho}_{p\Xi^-} \equiv i \langle \Lambda p^\dagger \sigma \cdot \bar{k} \Xi^- \rangle$, $\bar{\rho}_{p\Sigma^-} \equiv i \langle n^\dagger \sigma \cdot \bar{k} \Sigma^- \rangle$, where $\langle \cdots \rangle$ means the ground state expectation value with $\bar{k} \equiv k/|k|$. The strengths in these off-diagonal matrix elements are approximately related to the Landau-Migdal parameters, which simulate the short-range correlations between baryons, as is the case of pion condensation [71,72]. However, the values of the Landau-Migdal parameters for the particle-holes including hyperons are hardly known both theoretically and experimentally. Hence, in this paper, we don’t take into account these extra off-diagonal matrix elements, by putting emphasis on a brief discussion about onset mechanisms of the $p$-wave kaon condensation within a simple model.

**D. Kaon propagation in hyperonic matter**

After expanding the total effective energy density $\mathcal{E}_{\text{eff}}' (\equiv \mathcal{E}_{\text{eff}}'^B + \mathcal{E}_{\text{eff}}'^M + \mathcal{E}_{\text{eff}}'^e)$ with respect to $\theta$ around $\theta = 0$ as $\mathcal{E}_{\text{eff}}' = \mathcal{E}_{\text{eff}}'(\theta = 0) - \frac{f^2}{2} D_{K}^{-1}(\mu, k; \rho_B) \theta^2 + O(\theta^4)$, one obtains the kaon inverse propagator:

$$D_{K}^{-1}(\omega, k; \rho_B) = \omega^2 - k^2 - m_K^2 - \Pi_K(\omega, k; \rho_B)$$ (13)
with the kaon self energy \( \Pi_K(\omega, k; \rho_B) = \Pi_K^s(\omega, k; \rho_B) + \Pi_K^p(\omega, k; \rho_B) \), where

\[
\Pi_K^s(\omega, k; \rho_B) = -\frac{1}{f^2} \sum_i \rho_i \Sigma_i k_i - \frac{1}{f^2} (\rho_p + \frac{1}{2} \rho_n - \frac{1}{2} \rho_{\Sigma^-} - \rho_{\Xi^-}) \omega 
\]

\[
\Pi_K^p(\omega, k; \rho_B) = -\frac{1}{2f^2} \left[ \frac{(\rho_p - \rho_\Lambda)(g_{\Lambda p} k)^2}{\delta M_{\Lambda p} - \omega + V_\Lambda - V_p} + \frac{(\rho_n - \rho_{\Sigma^-})(g_{\Sigma^- n} k)^2}{\delta M_{\Sigma^- n} - \omega + V_{\Sigma^-} - V_n} \right. 
\]

\[
\left. + \frac{(\rho_\Lambda - \rho_{\Xi^-})(g_{\Xi^- \Lambda} k)^2}{\delta M_{\Xi^- \Lambda} - \omega + V_{\Xi^-} - V_\Lambda} \right].
\]

The first term on the r.h.s. of Eq. (14a) gives the s-wave scalar attraction with \( \rho_i \) the number densities for baryons \( i \) (\( i = p, \Lambda, \Xi^-, n, \Sigma^- \)). The second term on the r.h.s. of Eq. (14a) gives the s-wave vector interaction, where the coefficients in front of the number densities come from the \( V \)-spin \((V_3)\). This term gives attraction for proton and neutron, while repulsion for \( \Sigma^- \) and \( \Xi^- \). The \( p \)-wave part \( \Pi_K^p [\text{Eq. (14b)}] \) consists of the pole contributions from the \( p \)-wave kaon-baryon interaction. The diagrams corresponding to each term in Eq. (14b) are depicted in Fig. 1.

The excitation energies for kaonic modes are obtained as zero points of the kaon inverse propagator, \( D_K^{-1}(\omega, k; \rho_B) \), which depends on the composition of the ground state of noncondensed hyperonic matter, i.e., the number densities for the particles, \( \rho_i \) (\( i = p, \Lambda, \Xi^-, n, \Sigma^- \)). The particle number densities are determined from charge neutrality condition, \( \rho_p = \rho_{\Xi^-} + \rho_{\Sigma^-} + \rho_e \), baryon number conservation, \( \rho_p + \rho_\Lambda + \rho_{\Xi^-} + \rho_n + \rho_{\Sigma^-} = \rho_B \), and chemical equilibrium conditions between \( p, \Lambda, \Xi^-, n, \Sigma^- \) and \( e^- \),

\[
\mu_n = \mu_p + \mu_e \quad \text{for} \quad n \leftrightarrow pe^- \bar{\nu}_e, 
\]

\[
\mu_\Lambda = \mu_p + \mu_e \quad \text{for} \quad pe^- \leftrightarrow \Lambda \nu_e, 
\]

\[
\mu_{\Xi^-} = \mu_\Lambda + \mu_e \quad \text{for} \quad \Lambda e^- \leftrightarrow \Xi^- \nu_e, 
\]

\[
\mu_{\Sigma^-} = \mu_\Lambda + \mu_e \quad \text{for} \quad ne^- \leftrightarrow \Sigma^- \nu_e, 
\]

where the chemical potentials for baryons \( \mu_i \) are given by

\[
\mu_i = (3\pi^2 \rho_i)^{2/3}/(2M_N) + \delta M_{iN} + \nu_i \quad (\delta M_{iN} = 0 \text{ for } i = p, n).
\]

### III. NUMERICAL RESULTS AND DISCUSSION

#### A. \( K^- \) optical potential

We take a preliminary view of the in-medium kaon properties within our model by estimating a \( K^- \) optical potential \( V_{\text{opt}}(k; \rho_B) \) at \( \rho_B = \rho_0 \) in symmetric nuclear matter. The \( K^- \) optical potential \( V_{\text{opt}}(k; \rho_B) \) is defined in terms of the kaon self energy \( \Pi_K [\text{Eq. (14)}] \) as

\[
V_{\text{opt}}(k; \rho_B) = \Pi_K(\omega(k, \rho_B), k; \rho_B)/2\omega(k, \rho_B),
\]

where \( \omega(k, \rho_B) \) is the \( K^- \) excitation energy obtained from the dispersion equation, \( D_K^{-1}(\omega, k; \rho_B) = 0 \) at given \( \rho_B \) and \( k \).
In Fig. 2, we show $V_{\text{opt}}(k; \rho_0)$ as a function of the kaon momentum $|k|$. The solid lines represent the total values coming from both the $s$ and $p$-wave interactions [see Eq. (14)], and the dashed lines represent the contribution from the $p$-wave interaction. The bold lines are for $a_3 = -0.9$ ($\Sigma K_n=305$ MeV), and the thin lines for $a_3 = -0.28$ ($\Sigma K_n=0$). Since the numerical result depends little on the choice of the $\Sigma^-$ potential (Case I or Case II), only Case I is shown in the figure. The total attractive potential energy decreases monotonically with $k$. The potential depth at zero momentum is $\sim -120$ MeV for $a_3 = -0.9$ and $\sim -55$ MeV for $a_3 = -0.28$. It is to be noted that the total potential energy is attractive even if the $s$-wave scalar interaction is almost absent (thin solid line) due to the existence of the $s$-wave $K^-N$ vector attraction (the Tomozawa-Weinberg term). Since the excitation energy for $K^-$ is larger than those at the $\Sigma^-$ and $\Lambda$ poles, i.e., $\delta M_{\Sigma^-=n} - \omega + V_{\Sigma^-} - V_n < 0$ and $\delta M_{\Lambda p} - \omega + V_\Lambda - V_p < 0$, the $p$-wave part of the optical potential is repulsive for $K^-$ at this density, as seen from Eq. (14), and has a minor contribution in comparison with the $s$-wave attraction. The $K^-$-nucleus potential has been obtained phenomenologically from kaonic atom data in several works. In Ref. [22], a strongly attractive potential inside a nucleus, Re $V_{\text{opt}} = -200 \pm 20$ MeV, has been obtained with a nonlinear density dependent term. On the other hand, the energy shifts and widths have been well reproduced in Ref. [24] with much reduced attraction $\sim -45$ MeV in a local density approximation.

Several authors have elaborated the momentum-dependent $K^-$ optical potential in nuclear matter. With a coupled channel approach based on chiral models [32,21], or a $G$-matrix method with the Jülich $K N$ interaction [33] taking into account the in-medium modification of the kaon, they have obtained more moderate momentum dependence for Re $V_{\text{opt}}$ than ours over the relevant momentum region. On the other hand, our result for $a_3 = -0.9$ is similar to the result by a dispersion relation approach in Ref. [31]. At present, there is a controversy about the magnitude of the $K^-$ potential at finite momentum as well as at zero momentum.

### B. Particle fractions in hyperonic matter

Before considering the behaviors of kaonic modes, we make a survey of the ground state properties of hyperonic matter by obtaining the matter composition $\rho_i$ ($i = p, n, \Lambda, \Sigma^-, \Xi^-, e^-$), which enters into the kaon self energy Eq. (14), being responsible for kaon dynamics. The particle fractions $\rho_i/\rho_B$ as functions of the baryon number density $\rho_B$ are shown in Fig. 3 (a) for Case I (the repulsive $V_{\Sigma^-}$) and Fig. 3 (b) for Case II (the attractive $V_{\Sigma^-}$).

In Case I, the $\Lambda$ appears at $\rho_B \sim 0.37$ fm$^{-3}$, and its fraction rapidly increases with density, exceeding the proton fraction ($\rho_\Lambda > \rho_p$) at $\rho_B \sim 0.40$ fm$^{-3}$. Soon after the appearance of $\Lambda$, the $\Xi^-$ are mixed at $\rho_B \sim 0.42$ fm$^{-3}$, and it increases with density. On the other hand, the electron fraction rapidly decreases after the appearance of the $\Lambda$ and $\Xi^-$. The $\Sigma^-$ does not appear over the relevant densities because of its repulsive potential.

In Case II, the $\Sigma^-$ first appears at $\rho_B \sim 0.31$ fm$^{-3}$, and the $\Lambda$ appears at $\rho_B \sim 0.40$ fm$^{-3}$ after the onset of $\Sigma^-$. Both fractions rapidly increase with density. In particular, the fraction of $\Lambda$ exceeds the proton fraction at $\rho_B \sim 0.47$ fm$^{-3}$. The electron fraction decreases due to the appearance of the negatively charged hyperon $\Sigma^-$, which is qualitatively the same feature as for Case I. The electron chemical potential $\mu_e = [(3\pi^2 \rho_e)^{1/3}]$ also
becomes small with increase in density, which is unfavorable to matching the chemical equilibrium condition, $\mu_{\Xi^-} = \mu_{\Lambda} + \mu_n$. Thus the existence region of the $\Xi^-$ is pushed up to very high densities as compared with that in Case I.

These results for matter composition in Case I and II qualitatively reproduce the results of Figs. 4 and 3 in Ref. [52], respectively.

C. Kaonic modes in hyperonic matter

1. Case I (the repulsive $V_{\Xi^-}$)

(i) The weaker $s$-wave scalar interaction ($a_3 = -0.9$)

Here we discuss kaonic excitations in hyperonic matter in Case I. In Fig. 4 (a), we show the excitation energies for kaonic modes as functions of the kaon momentum $|k|$ for $a_3 = -0.9$ ($\Sigma_K = 305$ MeV) and $\rho_B = 0.38$ fm$^{-3}$ in Case I. Around this density, the $\Lambda$ begins to appear in a neutron-star matter [Fig. 3 (a)], where $\rho_\Lambda < \rho_p$. In addition to the $K^-$ and $K^+$ branches, there are three particle-hole branches: $\Lambda p^{-1}$, $\Xi^- \Lambda^{-1}$, and $\Sigma^{-n-1}$. The excitation modes are discriminated by a sign of the residue ($\partial D_K^{-1}/\partial \omega$)$^{-1}$ at their pole of the Green’s function $D_K$, as is the case with pionic modes [82–84]: If $\partial D_K^{-1}(\omega, k; \rho_B)/\partial \omega > 0$, the mode has a $K^-$ quantum number, while if $\partial D_K^{-1}(\omega, k; \rho_B)/\partial \omega < 0$, the mode has a $K^+$ quantum number. In Fig. 4 (b), the value of the inverse kaon propagator $D_K^{-1}$ is shown as a function of the excitation energy $\omega$ at $|k| = 500$ MeV for the same $a_3$ and density as Fig. 4 (a). The intersection with the $\omega$ axis denotes an excitation energy for each mode. One finds that the three particle-hole modes have the $K^-$ quantum numbers at this density.

As is seen from Fig. 4 (a), the energies of the particle-hole branches depends little on the momentum $|k|$, since the energy for each particle-hole mode is essentially determined from the location of the pole in the limit $|k| \rightarrow 0$ in the $p$-wave part of the self energy (Eq. 14b), so long as the $p$-wave kaon-baryon coupling strength is not very strong. On the other hand, the energies of the $K^+$ and $K^-$ branches are sensitive to $|k|$.

Now we look into the behavior of the excitation modes at the higher density where the $\Lambda$ is fully mixed and satisfies $\rho_\Lambda > \rho_p$. In Fig. 5 (a), we show the excitation energies for kaonic modes as functions of $|k|$ for $a_3 = -0.9$ and $\rho_B = 0.50$ fm$^{-3}$ (dashed lines) and $\rho_B = 0.57$ fm$^{-3}$ (solid lines). The inset in Fig. 5 (a) shows the magnified part of the $p\Lambda^{-1}$ and $\Xi^- \Lambda^{-1}$ branches.

In Fig. 5 (b), the value of $D_K^{-1}(\omega, k; \rho_B)$ as a function of $\omega$ at certain momentum $|k| = k^C$ (=984 MeV) is shown for the same $a_3$ and $\rho_B$ as in Fig. 5 (a). For these high densities, the $p\Lambda^{-1}$ branch which has a quantum number of the $K^+$ appears instead of the $\Lambda p^{-1}$ branch: E.g., for $\rho_B = 0.50$ fm$^{-3}$, $\partial D_K^{-1}/\partial \omega < 0$ at the pole of the $p\Lambda^{-1}$ mode [Fig. 5 (b)]. In order to go into details about the condition for the appearance of the $p\Lambda^{-1}$ mode, one obtains from Eqs.(13) and (14),

$$
\partial D_K^{-1}/\partial \omega = 2\omega + \frac{1}{f^2} \left( (\rho_p + \frac{1}{2} \rho_n - \frac{1}{2} \rho_{\Xi^-} - \rho_{\Xi^+}) + \frac{1}{2} (\rho_p - \rho_\Lambda) \left( \frac{g_{\Lambda p} k}{\delta M_{\Lambda p} - \omega + V_{\Lambda} - V_p} \right)^2 
+ \frac{1}{2} (\rho_\Lambda - \rho_{\Xi^-}) \left( \frac{g_{\Xi^- \Lambda} k}{\delta M_{\Xi^- \Lambda} - \omega + V_{\Xi^-} - V_\Lambda} \right)^2 \right)
$$
\[
\frac{1}{2} (\rho_n - \rho_{\Sigma^-}) \left( \frac{g_{\Sigma^- n k}}{\delta M_{\Sigma^- n} - \omega + V_{\Sigma^-} - V_n} \right)^2 .
\]

(18)

Since the excitation energy \( \omega(p\Lambda^{-1}) \) for the \( p\Lambda^{-1} \) mode roughly satisfies \( \delta M_{\Lambda p} - \omega(p\Lambda^{-1}) + V_{\Lambda} - V_p \sim 0 \), the second term in the bracket on the r.h.s. of Eq. (18) coming from the \( K^- p\Lambda \) interaction is dominant, and a sum of the remaining terms in (18) is positive. Hence, in order to satisfy \( \partial D_K^{-1}/\partial \omega < 0 \), the \( \Lambda \) have to be more abundant than the proton \( (\rho_\Lambda > \rho_p) \), which is the necessary (but not sufficient) condition for the existence of the \( p\Lambda^{-1} \) mode. In Case I, the \( p\Lambda^{-1} \) mode appears at \( \rho_B \sim 0.40 \text{ fm}^{-3} \), where \( \rho_p \simeq \rho_\Lambda \) [see Fig. 3].

As the baryon number density increases, the locations of the particle-hole branches become lower owing to the \( p \)-wave interactions, as shown in Fig. 5 (a). In particular, the \( \Xi^- \Lambda^{-1} \) and \( p\Lambda^{-1} \) branches get close to each other, and they merge at certain density \( (\rho_B \simeq 0.57 \text{ fm}^{-3}) \) with a critical momentum \( k^C \) (\( =984 \text{ MeV} \)). At \( |k| = k^C \), these two excitation modes merge at the \( \omega \) axis in the \( D_K^{-1} - \omega \) plane [Fig. 5 (b)], where the double-pole condition,

\[
D_K^{-1}(\omega, k; \rho_B) = 0 ,
\]

(19a)

\[
\partial D_K^{-1}(\omega, k; \rho_B)/\partial \omega = 0 ,
\]

(19b)

and the extremum condition with respect to \( |k| \),

\[
\partial D_K^{-1}(\omega, k; \rho_B)/\partial |k| = 0 ,
\]

(20)

are satisfied. It means that a pair of the two modes, \( \Xi^- \Lambda^{-1} \) and \( p\Lambda^{-1} \), are created spontaneously with no cost of energy because the energy of \( p\Lambda^{-1} \) mode with the quantum number \( K^+ \) is to be reversed in sign. Hence the system is unstable with respect to a pair creation of \( [\Xi^- \Lambda^{-1}] \) and \( [p\Lambda^{-1}] \) modes. This instability originates from the \( p \)-wave kaon-baryon interaction, and we call this instability \( p \)-wave kaon condensation. The onset mechanism of the \( p \)-wave kaon condensation is similar to that of pion condensation, where a driving force is given by the \( p \)-wave \( \pi N \) interaction [82–85]. From Eqns. (19a), (19b), (20), and by putting \( \omega = \mu \) (the charge chemical potential), one obtains the baryon number density \( \rho^C_B \), the charge chemical potential \( \mu^C \), and the kaon momentum \( k^C \) at the onset of condensation. It is to be noted that the numerical value of \( k^C \) obtained for \( a_3 = -0.9 \) in Case I may be very large. As a realistic effect, form factors at the \( p \)-wave kaon-baryon vertices might reduce the large \( p \)-wave attraction, resulting in a moderate value of \( k^C \), while pushing a critical density \( \rho^C_B \) to a higher baryon number density.

The population of the modes can be seen from the spectral density which is defined as

\[
A(\omega, k; \rho_B) = -2 \text{Im} D_R(\omega, k; \rho_B) ,
\]

where \( D_R(\omega, k; \rho_B) \) is the retarded Green’s function for kaons. In our formulation, the particle-hole continuum states are not included, so that the spectral density has a form

\[
A(\omega, k; \rho_B) = 2\pi \sum_i Z_i \delta(\omega - \omega_i) ,
\]

(21)

where \( \omega_i \)s are the solutions of the dispersion equation, \( D_K^{-1}(\omega, k; \rho_B) = 0 \), and \( Z_i \)s are the residues of \( D_K(\omega, k; \rho_B) \) at \( \omega = \omega_i \). The spectral density has a sum rule,
\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega A(\omega; k; \rho_B) = \sum_i \omega_i Z_i = 1 , \]

which follows from the canonical commutation relation for the charged kaon field. In Fig. 6, we show occupation factors \([62]\) which are defined as \(\Gamma(i) \equiv \omega_i Z_i\) for the kaonic modes \((i = K^-, p\Lambda^{-1}, \Sigma^{-n-1}, \Xi^{-\Lambda^{-1}}, K^+)\) as functions of \(|k|\) for \(a_3 = -0.9\) and \(\rho_B = 0.57\) fm\(^{-3}\) in Case I. The \(p\Lambda^{-1}\) mode has a negative contribution to the occupation factor. For \(|k| < k_C\), the occupation factors for both the \(\Xi^{-\Lambda^{-1}}\) and \(p\Lambda^{-1}\) are small in comparison with those for the other modes. Near the critical momentum \(k_C\), however, they become large and diverge at \(k_C\), which implies the instability of the system.

The effects of the \(p\)-wave kaon-baryon interaction on the kaon dynamics near the onset density of condensation are evaluated from the self energy \(\Pi_K\). For each kaonic mode, we show, in Fig. 7, the kaon self energy for the \(s\)-wave part \(\Pi^s_K(\omega; k; \rho_B)\) \([\text{Eq. (14a)}]\) and the \(p\)-wave part \(\Pi^p_K(\omega; k; \rho_B)\) \([\text{Eq. (14b)}]\) by the dashed lines and the solid lines, respectively, as a function of \(|k|\) for \(a_3 = -0.9\) and \(\rho_B = \rho_B^C = 0.57\) fm\(^{-3}\). For the \(p\Lambda^{-1}\), \(\Sigma^{-n-1}\), and \(\Xi^{-\Lambda^{-1}}\), the attractive \(p\)-wave part \(\Pi^p_K\) gets large almost proportionally to \(|k|^2\), and the magnitude becomes comparable to that of the \(s\)-wave part \(\Pi^s_K\) at \(|k| \sim 500\) MeV.

On the other hand, the \(p\)-wave part \(\Pi^p_K\) for the \(K^-\) works repulsively at small \(|k|\), and it decreases monotonically with \(|k|\). At a high momentum, the excitation energy for the \(K^-\) mode is so large that the \(K^-\) mode is located far beyond the poles for the other particle-hole modes, which yields \(\delta M_{\Lambda^p} - \omega(K^-) + V_{\Lambda} - V_{\rho} \ll 0\), \(\delta M_{\Sigma^{-n-1}} - \omega(K^-) + V_{\Sigma} - V_{\pi} \ll 0\), and \(\delta M_{\Xi^{-\Lambda^{-1}}} - \omega(K^+) + V_{\Xi} - V_{\Lambda} \ll 0\) in the \(\Pi^p_K\). Hence the magnitude of the \(p\)-wave part \(\Pi^p_K\) for the \(K^-\) is tiny, and the \(s\)-wave part \(\Pi^s_K\) is dominant in the self energy.

(ii) The stronger \(s\)-wave scalar interaction \((a_3 = -1.1)\)

Next we consider a case for the stronger \(s\)-wave scalar attraction. In Fig. 8 (a), we show the excitation energies of kaonic modes as functions of \(|k|\) for \(a_3 = -1.1\) \((\Sigma_{Kn}=403\) MeV\) and \(\rho_B = 0.48\) fm\(^{-3}\) just beyond the onset of condensation. In Fig. 8 (b), the value of the kaon inverse propagator \(D^{-1}_K(\omega; k; \rho_B)\) is shown as a function of \(\omega\) at \(|k| = k_C = 118\) MeV for the same values of \(a_3\) and \(\rho_B\) as those in Fig. 8 (a). As is the case with \(a_3 = -0.9\), the \(p\)-wave condensation of the \(\Sigma^{-n-1}\) and \(p\Lambda^{-1}\) pairs occurs but at a smaller density \(\rho_B = 0.48\) fm\(^{-3}\) and a smaller momentum \(|k|^C = 118\) MeV than those for \(a_3 = -0.9\). The difference of the critical density and the critical momentum between the two cases is attributed to the difference of the microscopic structures of the kaonic modes. Figure 8 (a) shows that there are level crossings between the \(K^-\), \(\Sigma^{-n-1}\) and \(\Xi^{-\Lambda^{-1}}\) branches, whereas there is a level crossing only between the \(K^-\) and \(\Sigma^{-n-1}\) branches for \(a_3 = -0.9\) \([\text{Fig. 5 (a)}]\). The difference of the structures for the kaonic modes between the \(a_3 = -0.9\) and \(-1.1\) cases can also be seen from the dependence of the kaonic modes on the baryon number density. In Fig. 9, we show the dependence of the excitation energies of kaonic modes on the baryon number density except for the \(K^+\). (a) is for \(a_3 = -0.9\) and \(|k|=500\) MeV which is smaller than \(k_C\) (=984 MeV), and (b) is for \(a_3 = -1.1\) and \(|k|=100\) MeV, which is smaller than but near \(k_C\) (=118 MeV). For \(a_3 = -0.9\), the \(K^-\) is repelled far from the remaining particle-hole modes over the relevant densities, and there is no level crossing. The \(\Xi^{-\Lambda^{-1}}\) and \(p\Lambda^{-1}\) modes merge at \(\rho_B \simeq 0.60\) fm\(^{-3}\), which corresponds to the instability with respect to \(p\)-wave condensation. The qualitative feature is also applied to the case at the critical momentum \(k_C\) which satisfies the conditions Eqs. (19) and (20).
The particle-hole branches such as the $\Xi^{-}\Lambda^{-1}$ and $p\Lambda^{-1}$ do not depend much on the value of $|k|$, nor does the critical point for the $p$-wave condensation.

For $a_3 = -1.1$, on the other hand, the excitation energy of the $K^{-}$ is small as compared with the $a_3 = -0.9$ at a given density due to the larger $s$-wave scalar attraction and the smaller momentum $|k|$, and one can see in Fig. 9 (b) that there are energy gaps between $K^{-}$ and $\Sigma^{-}n^{-1}$ branches and the $\Sigma^{-}n^{-1}$ and $\Xi^{-}\Lambda^{-1}$ branches owing to the level crossings.\footnote{The appearance of the collective modes was also pointed out in relation to the level crossing in Ref. [64].} As a result of the level crossings, the $\Xi^{-}\Lambda^{-1}$ branch takes over the characteristics of the $K^{-}$, the excitation energy of which changes appreciably depending on the magnitude of the $s$-wave scalar interaction simulated by $a_3$.\footnote{This mode corresponds to the kaesobar [26,60].} Thus, when the level crossing occurs, the behavior of the $\Xi^{-}\Lambda^{-1}$ branch is sensitive to the value of $a_3$, and the critical point for the $p$-wave condensation, which is given by the merge point of the $\Xi^{-}\Lambda^{-1}$ and $p\Lambda^{-1}$ branches, also depends on $a_3$. This mechanism of $p$-wave condensation is similar to that of $p$-wave pion condensation, where the $\pi^{-}$ mode, which reduces to a free $\pi^{-}$ in vacuum, and the spin-isospin zero sound (called $\pi_{s}^{+}$) are spontaneously created in pairs [82–85].

The larger mixing of the $\Lambda$ than the proton in hyperonic matter is a necessary condition for the appearance of the low-lying particle-hole mode ($p\Lambda^{-1}$) having a $K^{+}$ quantum number, and this condition is crucial to the realization of $p$-wave kaon condensation considered here. The result should be compared with a mechanism of $p$-wave kaon condensation realized from the conventional neutron-star matter where only the nucleons $n$, $p$ are present as baryons [61]. In the latter case, there is no low-lying collective mode having the $K^{+}$ quantum number in the ground state of the neutron-star matter, so that condensation of pair modes with the $K^{+}$ and $K^{-}$ quantum numbers cannot be expected. Instead, it is a single mode having the $K^{-}$ quantum number that is relevant to kaon condensation in the (nucleonic) neutron-star matter [61,62]: In particular, it has been shown in Ref. [61] that a Bose-Einstein condensation of the $s$-wave $K^{-}$ mode occurs at some density as a result of the $s$-wave $K^{-}N$ attraction, where hyperons are still absent in matter. It has also been shown that only at higher densities, the classical $K^{-}$ field simply acquires a momentum, increasing the attractive energy through the $p$-wave kaon-baryon interaction in addition to the $s$-wave one, accompanying hyperon excitation [61].

As seen in Fig. 9, the $\Delta p^{-1}$ branch crosses the charge chemical potential, i. e., $\omega(\Delta p^{-1}) = \mu$ at $\rho_B \sim 0.38$ fm$^{-3}$. It apparently suggests an onset of another type of Bose-Einstein condensation of $\Delta p^{-1}$ mode. In Fig. 10, we show the occupation factors $\Gamma(i)$ as functions of $|k|$ for $a_3 = -0.9$ and $\rho_B=0.38$ fm$^{-3}$ in Case I. The value of $\Gamma$ for the $\Delta p^{-1}$ mode as well as the $\Xi^{-}\Lambda^{-1}$ is negligible over the relevant kaon momentum $|k|$. This is explained as follows: The excitation energy for the $\Delta p^{-1}$ mode is determined from $\delta M_{\Delta p} - \omega(\Delta p^{-1}) + V_{\Lambda} - V_{p} \sim 0$, and it depends little on the momentum $k$ [ see Fig. 4 (a) and the discussion at the beginning of Sec. III C 1-(i)]. In this case, the second term in the bracket on the r.h.s. of Eq. (18) for the $\partial D_{K^{-1}}^{1}/\partial \omega$ becomes very large at $\omega = \omega(\Delta p^{-1})$, so that $\Gamma(\Delta p^{-1}) \left[ \propto 1/(\partial D_{K^{-1}}^{1}/\partial \omega)_{\omega=\omega(\Delta p^{-1})} \right]$ becomes very small. The small population of the $\Xi^{-}\Lambda^{-1}$ is also explained in a similar way. The main population of kaonic modes is thus exchanged between the $K^{-}$ and $\Sigma^{-}n^{-1}$ modes before and after an avoided level-
crossing point $|k| \sim 200$ MeV [see Fig. 4 (a)]. On the other hand, the number of $K^-$ with $k$ is given as $n(k) = \int \frac{d\omega}{2\pi} \omega f_K(\omega) A(\omega, k; \rho_B) = \sum_i f_K(\omega_i) \Gamma(i)$, where $f_K(\omega) (=1/\exp\{(\omega - \mu)/T\} - 1)$ is the Bose-Einstein distribution function. Since the $f_K(\omega)$ becomes divergent at $\omega = \omega(\Lambda p^{-1})$, the $\Lambda p^{-1}$ mode brings about a singular behavior of $n(k)$, however small the population $\Gamma(\Lambda p^{-1})$ is. In this respect, the physical significance of this instability should be considered carefully. In this paper, however, we don’t go into details about this possible instability, and only concentrate on the characteristic features of the pair-mode condensation.

2. Case II (the attractive $V_{\Sigma^-}$)

Next we discuss behaviors of kaonic modes in Case II (the attractive $V_{\Sigma^-}$). In Fig. 11, we show the dependence of the excitation energies of kaonic modes on the baryon number density in Case II. (a) is for $a_3 = -0.9$ ($\Sigma K n = 305$ MeV) and $|k| = 500$ MeV, and (b) is for $a_3 = -1.1$ ($\Sigma K n = 403$ MeV) and $|k| = 30$ MeV. Due to the strong attraction of $V_{\Sigma^-}$, the $\Sigma^- n^{-1}$ branch is softer than those of the $\Xi^- \Lambda^{-1}$ and $K^-$ branches for $\rho_B \gtrsim 0.40$ fm$^{-3}$, and the $\Sigma^- n^{-1}$ merges first with the $p \Lambda^{-1}$ branch instead of the $\Xi^- \Lambda^{-1}$. Hence, in Case II, $p$-wave condensation is brought about by a spontaneous creation of the $\Sigma^- n^{-1}$ and $p \Lambda^{-1}$ pair. For the larger $a_3$ [Fig. 11 (b)], there are level crossings between the $K^-$, $\Sigma^- n^{-1}$ and $\Xi^- \Lambda^{-1}$. The $\Sigma^- n^{-1}$ branch takes over the characteristics of the $K^-$ around the crossing point with the $\Xi^- \Lambda^{-1}$ branch [$\rho_B \sim 0.53$ fm$^{-3}$], so that the critical point for the $p$-wave condensation is sensitive to the value of $a_3$, which is similar to the stronger $s$-wave attraction case ($a_3 = -1.1$) in Case I [see Fig. 9 (b)].

The critical density and the corresponding momentum are given from Eq. (20) in addition to the double-pole condition Eq. (19) as $\rho_B^2 = 0.64$ fm$^{-3}$, $k^C = 978$ MeV for $a_3 = -0.9$, and $\rho_B^C = 0.53$ fm$^{-3}$, $k^C = 39$ MeV for $a_3 = -1.1$. Quantitatively, the critical density in Case II is a little larger than that in Case I. As seen from Fig. 9 and Fig. 11, the strongly attractive $V_{\Sigma^-}$ in Case II modifies the density dependence of the relevant kaonic modes, especially $\Sigma^- n^{-1}$ and $p \Lambda^{-1}$ ($\Lambda p^{-1}$), from that in Case I (the repulsive $V_{\Sigma^-}$) through changing chemical composition of highly dense matter. The quantitative estimation of the density at which the two modes merge is subtle depending on the specific density dependence of these modes.

3. Comparison of the critical densities for $p$-wave and $s$-wave condensations

Here we compare the critical density for the $p$-wave condensation discussed in the preceding subsections with that for the $s$-wave $K^-$ condensation which is obtained within the present framework. In Fig. 12, we show the dependence of the minimum excitation energy of the $s$-wave $K^-$ on the baryon number density for $|k| = 0$. (a) is for Case I, and (b) is for Case II. The solid line is for $a_3 = -0.9$ and the dashed line is for $a_3 = -1.1$. The $K^-$ energy decreases with density due to the $s$-wave kaon-baryon interaction in $\Pi_K^s$ [Eq. (14a)]. However, due to the substantial decrease of the charge chemical
potential $\mu$ with density in the presence of hyperons, the onset condition for the s-wave $K^-$ condensation, $\omega_{\text{min}}(K^-) = \mu$, is met at a much larger density (filled circles) than that in the conventional neutron-star matter which consists of only the nucleons $n$, $p$ and $e^-$. On the other hand, the critical density of the $p$-wave condensation is indicated by the arrows in Fig. 12 (the solid arrow for $a_3 = -0.9$ and the dashed arrow for $a_3 = -1.1$). One finds that the $p$-wave condensation precedes the s-wave $K^-$ condensation.

In Fig. 13, we summarize the dependence of the critical density of the $p$-wave kaon condensation $\rho_B^C(p)$ (solid and dotted lines) on the kaon-neutron sigma term $\Sigma_{Kn}$ [$(a_2+2a_3)/(m_n+m_a)$]. For comparison, the critical density of the $s$-wave $K^-$ condensation $\rho_B^C(s)$ (the dashed line) is also shown. (a) is for Case I, and (b) is for Case II. For the strong $s$-wave scalar attraction such that $\Sigma_{Kn} \gtrsim 340$ MeV ($a_3 \lesssim -0.98$), there are level crossings between the $K^-$ and the particle-hole branches. As a result, the critical density $\rho_B^C(p)$ is sensitive to the magnitude of $|a_3|$. $\rho_B^C(p)$ is slightly smaller than $\rho_B^C(s)$ in both Cases I and II, but the difference is small, as shown by the dotted lines and dashed lines. For the weaker $s$-wave attraction such that $\Sigma_{Kn} \lesssim 340$ MeV ($a_3 \gtrsim -0.98$), the critical density of the $p$-wave condensation $\rho_B^C(p)$ depends little on the magnitude of $a_3$ (the solid lines). This is because the $p\Lambda^{-1}$, $\Xi^-\Lambda^{-1}$. and $\Sigma^{-n^{-1}}$ branches hardly depend on the magnitude of $a_3$ as long as no level crossing with the $K^-$ branch occurs before the onset of instability. The critical density $\rho_B^C(p)$ changes little over the range from $\Sigma_{Kn} = 0$ ($a_3 = -0.28$) to $\Sigma_{Kn} \sim 340$ MeV ($a_3 \sim -0.98$) in both Cases I and II. On the other hand, the critical density for the $s$-wave $K^-$ condensation $\rho_B^C(s)$ becomes large as $\Sigma_{Kn}$ (or $|a_3|$) becomes small, due to the reduced contribution from the $s$-wave scalar attraction to the $K^-$ excitation energy $\omega(K^-)$. In conclusion, the critical density of the $p$-wave condensation is always smaller than that of the $s$-wave $K^-$ condensation as long as $|a_3|$ is not too large, and the difference between these critical densities gets remarkable with the decrease in the magnitude of the $s$-wave scalar attraction.

**D. Outline of the condensed phase**

We address qualitative features of the $p$-wave kaon condensation discussed in this paper. The details will be given elsewhere [68].

At a critical density $\rho_B^C$, pairs of the kaonic modes, $[p\Lambda^{-1}] - [\Xi^-\Lambda^{-1}]$ in Case I or $[p\Lambda^{-1}] - [\Sigma^{-n^{-1}}]$ in Case II, are spontaneously created via the reaction, $\Lambda\Lambda \rightarrow \Xi^-p$ in Case I or $\Lambda n \rightarrow \Sigma^-p$ in Case II. Thus, formation of a condensate in hyperonic matter proceeds essentially through the strong reactions, which should be compared with the $s$-wave $K^-$ condensation: In the latter case, a condensate is formed through the weak reactions such as $nn \rightarrow npK^-$, $e^- \rightarrow K^-\nu_e$. In the $p$-wave condensed phase, baryonic system consists of the Fermi seas of the quasiparticles $\tilde{p}$, $\tilde{\Lambda}$, $\tilde{\Xi}^-$, $\tilde{n}$, $\tilde{\Sigma}^-$ (see Sec. II B). The absolute value of total negative strangeness of the system increases as the baryon number density increases by virtue of the weak reactions. The relevant strangeness-changing weak processes are given by $\bar{Y}(K^-) \rightarrow \bar{Y} e^-\nu_e$, $\bar{Y} e^- \rightarrow \bar{Y} \langle K^- \rangle \nu_e$ (for $\bar{Y} = \tilde{p}$, $\tilde{\Lambda}$, $\tilde{\Xi}^-$, $\tilde{n}$, $\tilde{\Sigma}^-$), where the classical kaon field $\langle K^- \rangle$ given by Eq. (4) supplies the system with the energy and momentum to satisfy the kinematical condition for these reactions. These weak processes may also be relevant to enhanced cooling of neutron stars via neutrino emissions just like a case of pion condensed phase [82,83,85].

18
The EOS of the $p$-wave kaon-condensed phase would be further softened as compared with that of noncondensed hyperonic matter owing to the $p$-wave kaon-baryon attractive interaction in addition to the $s$-wave one. The significant softening will make phase transition of a first order after Maxwell’s construction, leading to a drastic change of the internal structure of neutron stars. In particular, the first-order phase transition may imply a mixed phase where droplets of a kaon condensate are immersed in the normal phase [9]. The first-order phase transition has also important effects on the dynamical evolution of newly-born neutron stars accompanying a delayed collapse, which has already been discussed by several authors in case of the $s$-wave kaon condensation [10,11].

IV. SUMMARY AND CONCLUDING REMARKS

We have discussed in-medium properties of kaonic modes in hyperonic matter by taking into account the $p$-wave kaon-baryon interaction as well as the $s$-wave one on the basis of chiral symmetry. Nonrelativistic effective baryon-baryon interactions, which are parametrized by the use of the recent hypernuclear experimental data, have been used. It has been shown that a collective $p\Lambda^{-1}$ mode with the $K^+$ quantum number appears over the densities where the $\Lambda$ is more abundant than the proton. The system becomes unstable with respect to a creation of $[\Xi^-\Lambda^{-1}]$ and $[p\Lambda^{-1}]$ pair or $[\Sigma^-n^{-1}]$ and $[p\Lambda^{-1}]$ pair ($p$-wave kaon condensation), which stems from the $p$-wave kaon-baryon interaction. The onset density of this instability is lower than that of the $s$-wave $K^-$ condensation for a standard value of the parameter $a_3$ simulating the magnitude of the $s$-wave kaon-baryon scalar interaction, and it hardly depends on the value of $a_3$ as long as $|a_3|$ is not too large.

The possibility of the $p$-wave kaon condensation depends on composition of baryons in hyperonic matter. In particular, large mixing of $\Lambda$ as compared with that of the proton is needed for the appearance of the $p\Lambda^{-1}$ mode. The details about the onset densities of hyperons and their fractions at high densities differ between specific models for the baryonic potentials. One of the important ingredients which control matter composition is three-body forces for baryons [50,54]. It has been shown that phenomenological inclusion of three-body forces for only nucleons makes hyperon-mixing favorable [50]. However, it has been pointed out that inclusion of three-body forces for hyperons on the same footing as the nucleons may considerably change the results on the matter composition and the resultant EOS of the hyperonic matter [54]. Thus one has to be careful for the parametrization of the effective baryon-baryon interactions used in this paper, keeping consistency with these other model calculations.

Our model used for the $p$-wave kaon-baryon interaction is based on the leading order expansion in the chiral perturbation theory. Higher order terms in chiral expansion which are relevant to the $p$-wave meson-baryon scatterings have been estimated with reference to the experimental results such as pion and photon-induced reactions [69] or elastic and inelastic $K^-p$ scatterings [70]. It needs more consideration whether these higher order terms are quantitatively important to kaon dynamics in highly dense matter.

It has to be elucidated whether the instability of the system with respect to the $p$-wave condensation leads to a fully condensed phase beyond the critical density. In this context, the EOS of the $p$-wave condensed phase and the characteristic features of the system have to be examined [68]. Mixing of hyperons only already leads to appreciable
softening of the EOS\cite{45,49–58}. Hence, further development of kaon condensates in hyperonic matter would make the EOS too soft to obtain the observed neutron star masses \(\sim 1.4M_\odot\)\cite{86,87} or even much larger masses\(\sim 2.0M_\odot\) if the recent analyses from the observations of the quasi-periodic oscillations (QPO)\cite{88} are confirmed\cite{51}. Relativistic effects may help weaken the softness of the EOS, since it has been shown that the energy gain of kaon condensation coming from the \(s\)-wave scalar attraction is suppressed by relativistic effects\cite{8,4}: As a kaon condensate develops, the effective nucleon mass \(M^*\) decreases due to the scalar attraction, which leads to suppression of the scalar density, \(\rho_s = \int \frac{d^3p}{(2\pi)^3} M^*/\sqrt{p^2 + M^2}\). Thus the energy gain of the condensed phase from the scalar attraction \((\propto \rho_s \Sigma_{KN}\) with the \(KN\) sigma term \(\Sigma_{KN}\)) and the growth of a condensate are suppressed.

In addition, in view of making the EOS consistent with observations, some realistic effects which reduce the \(p\)-wave kaon-baryon attraction should be taken into account: (1) vertex renormalization at the \(p\)-wave kaon-baryon vertices in terms of form factors. (2) short-range correlations between baryons. For the \(p\)-wave part, the off-diagonal matrix elements in the baryonic part of the effective Hamiltonian [Eq. (6)] are to be added by the particle-hole densities, the strengths of which are related with the Landau-Migdal parameters in the relevant channel in the same way as pion condensation\cite{71,82–85}.

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APPENDIX A: POTENTIAL ENERGY DENSITY IN HYPERONIC MATTER

We show the expression for the potential energy density \(\mathcal{E}_{\text{pot}}\) based on the nonrelativistic baryon-baryon interactions by Balberg and Gal [Eq. (6) in\cite{52}]. Since only \(p\), \(\Lambda\), \(\Xi^-\), \(n\), and \(\Sigma^-\) are incorporated for the baryons in this paper, the other terms relevant to \(\Sigma^0\), \(\Sigma^+\), and \(\Xi^0\) are omitted.

\[
\mathcal{E}_{\text{pot}} = \frac{1}{2} \left[ a_{NN}(\rho_p + \rho_n)^2 + b_{NN}(\rho_p - \rho_n)^2 + c_{NN}(\rho_p + \rho_n)^{6+1} \right] \\
+ a_{\Lambda N}(\rho_p + \rho_n)\rho_\Lambda + c_{\Lambda N} \left[ \frac{(\rho_p + \rho_n)^{\gamma+1}}{\rho_p + \rho_n + \rho_\Lambda} + \frac{\rho_\Lambda^{\gamma+1}}{\rho_p + \rho_n + \rho_\Lambda}(\rho_p + \rho_n) \right] \\
+ \frac{1}{2} \left[ a_{YY}\rho_\Lambda^2 + c_{YY}\rho_\Lambda^{\gamma+1} + (a_{YY} + b_{\Xi\Xi})\rho_{\Xi^-}^{-2} + c_{YY}\rho_{\Xi^-}^{-\gamma+1} \right] \\
+ a_{\Xi N}(\rho_p + \rho_n)\rho_{\Xi^-} + b_{\Xi N}(\rho_n - \rho_p)\rho_{\Xi^-} \\
+ c_{\Xi N} \left[ \frac{(\rho_p + \rho_n)^{\gamma+1}}{\rho_p + \rho_n + \rho_{\Xi^-}} + \frac{\rho_{\Xi^-}^{\gamma+1}}{\rho_p + \rho_n + \rho_{\Xi^-}}(\rho_p + \rho_n) \right]
\]
\[\begin{align*}
&+ a_{YY} \rho_{\Xi^-} \rho_{\Lambda} + c_{YY} \left[ \frac{\rho_{\Lambda \gamma + 1}}{\rho_{\Xi^-}^\uparrow + \rho_{\Lambda}} \rho_{\Xi^-} + \frac{\rho_{\Xi^- \gamma + 1}}{\rho_{\Xi^-}^\uparrow + \rho_{\Lambda}} \rho_{\Lambda} \right] \\
&+ a_{\Sigma N} (\rho_p + \rho_n) \rho_{\Sigma^-} + b_{\Sigma N} (\rho_n - \rho_p) \rho_{\Sigma^-} \\
&+ c_{\Sigma N} \left[ \frac{(\rho_p + \rho_n)^{\gamma + 1}}{\rho_p + \rho_n + \rho_{\Sigma^-}} \rho_{\Sigma^-} + \frac{\rho_{\Sigma^- \gamma + 1}}{\rho_p + \rho_n + \rho_{\Sigma^-}} (\rho_p + \rho_n) \right] \\
&+ a_{YY} \rho_{\Sigma^-} \rho_{\Lambda} + c_{YY} \left[ \frac{\rho_{\Sigma^- \gamma + 1}}{\rho_{\Sigma^-}^\uparrow + \rho_{\Lambda}} \rho_{\Lambda} + \frac{\rho_{\Lambda \gamma + 1}}{\rho_{\Sigma^-}^\uparrow + \rho_{\Lambda}} \rho_{\Sigma^-} \right] \\
&+ a_{YY} \rho_{\Sigma^-} \rho_{\Xi^-} + b_{\Sigma \Xi} \rho_{\Xi^-} \rho_{\Sigma^-} + c_{YY} \left[ \frac{\rho_{\Xi^- \gamma + 1}}{\rho_{\Xi^-}^\uparrow + \rho_{\Sigma^-}} \rho_{\Sigma^-} + \frac{\rho_{\Sigma^- \gamma + 1}}{\rho_{\Xi^-}^\uparrow + \rho_{\Sigma^-}} \rho_{\Xi^-} \right] \\
&+ \frac{1}{2} \left[ (a_{YY} + b_{\Sigma \Sigma}) \rho_{\Sigma^-}^2 + c_{YY} \rho_{\Sigma^- \gamma + 1} \right] \quad \text{(A1)}
\end{align*}\]

The parameters relevant to the $NN$ and $YN$ parts in Eq. (A1) have been refitted in reference to the recent empirical data on the nuclear and hypernuclear properties. Numerical values of the parameters are listed in Tables I and II.
REFERENCES


22


FIG. 1. Pole contributions to the $K^-$ self energy from the $p$-wave kaon-baryon interactions: (a) $\Lambda$-particle-proton-hole and proton-particle-$\Lambda$-hole states, (b) $\Sigma^-$-particle-neutron-hole and neutron-particle-$\Sigma^-$-hole states, and (c) $\Xi^-$-particle-$\Lambda$-hole and $\Lambda$-particle-$\Xi^-$-hole states.

FIG. 2. The $K^-$ optical potential at $\rho_B = \rho_0$ in symmetric nuclear matter.
FIG. 3. Particle fractions $\rho_i/\rho_B$ as functions of the baryon number density $\rho_B$ for (a) Case I (the repulsive $V_{\Sigma^-}$) and (b) Case II (the attractive $V_{\Sigma^-}$).

FIG. 4. (a) The excitation energies for kaonic modes as functions of the kaon momentum $|k|$ for $a_3 = -0.9$ ($\Sigma K_n = 305$ MeV) and the baryon number density $\rho_B =$ 0.38 fm$^{-3}$. (b) The value of the inverse kaon propagator $D_K^{-1}$ as a function of the excitation energy $\omega$ at $|k| = 500$ MeV for the same $a_3$ and density as Fig. 4 (a).
FIG. 5. (a) The excitation energies for kaonic modes as functions of |k| for $a_3 = -0.9$ and $\rho_B=0.50$ fm$^{-3}$ (dashed line) and $\rho_B=0.57$ fm$^{-3}$ (solid line). (b) The inverse kaon propagator $D_K^{-1}(\omega, k; \rho_B)$ as a function of $\omega$ at $|k|=k^C$ (=984 MeV) for the same $a_3$ and $\rho_B$ as in Fig. 5 (a).

FIG. 6. Occupation factors $\Gamma(i) \equiv \omega(\partial D_K^{-1}/\partial \omega)^{-1}|_{\omega=\omega_i}$ for the kaonic modes $(i = K^-, p\Lambda^{-1}, \Sigma^-n^{-1}, \Xi^-\Lambda^{-1}, K^+)$ as functions of |k| for $a_3 = -0.9$ and $\rho_B=0.57$ fm$^{-3}$ in Case I.
FIG. 7. Contributions to the kaon self energy $\Pi_K(\omega, k; \rho)$ from the $p$-wave (solid lines) and $s$-wave (dashed lines) kaon-baryon interactions for the kaonic modes as a function of $|k|$ for $a_3 = -0.9$ and $\rho_B = 0.57 \text{ fm}^{-3}$ in Case I.

FIG. 8. (a) The excitation energies of kaonic modes as functions of $|k|$ for $a_3 = -1.1$ ($\Sigma_K n = 403 \text{ MeV}$) and $\rho_B = 0.48 \text{ fm}^{-3}$ just beyond the onset of condensation in Case I. (b) The value of the kaon inverse propagator $D_K^{-1}(\omega, k; \rho_B)$ as a function of $\omega$ at $|k| = k_C = 118 \text{ MeV}$. The values of $a_3$ and $\rho_B$ are the same as those in Fig. 8 (a).
FIG. 9. (a) The dependence of the excitation energies of kaonic modes on the baryon number density for $a_3 = -0.9$ ($\Sigma_{Kn}=305$ MeV) and $|k|=500$ MeV in Case I. The charge chemical potential $\mu$ is also shown as a function of $\rho_B$ by a dotted line. (b) The same as Fig. 9 (a), but for $a_3 = -1.1$ ($\Sigma_{Kn}=403$ MeV) and $|k|=100$ MeV. The charge chemical potential $\mu$ (the dotted line) is identical to that in Fig. 9 (a).

FIG. 10. Occupation factors $\Gamma(i) \equiv \omega(\partial D_{K^{-}}/\partial \omega)^{-1}_{\omega=\omega_i}$ for the kaonic modes ($i = K^{-}, \Lambda p^{-1}, \Sigma^{-} n^{-1}, \Xi^{-} \Lambda^{-1}$) as functions of $|k|$ for $a_3 = -0.9$ and $\rho_B=0.38$ fm$^{-3}$ in Case I.
FIG. 11. (a) The dependence of the excitation energies of kaonic modes on the baryon number density for $a_3 = -0.9$ ($\Sigma_{Kn}=305$ MeV) and $|k|=500$ MeV in Case II. The charge chemical potential $\mu$ is also shown as a function of $\rho_B$ by a dotted line. (b) The same as Fig. 11 (a), but for $a_3 = -1.1$ ($\Sigma_{Kn}=403$ MeV) and $|k|=30$ MeV. The charge chemical potential $\mu$ (the dotted line) is identical to that in Fig. 11 (a).

FIG. 12. (a) The dependence of the minimum excitation energy of the $s$-wave $K^-$ on the baryon number density for $|k| = 0$ in Case I. The solid line is for $a_3 = -0.9$ and the dashed line is for $a_3 = -1.1$. The charge chemical potential $\mu$ (the dotted line) is identical to that in Fig. 9. For reference, the charge chemical potential in neutron-star matter consisting of only the nucleons $n$, $p$ and $e^-$ is shown by the dash-dotted line. (b) The same as Fig. 12 (a), but for Case II. The charge chemical potential $\mu$ (the dotted line) is identical to that in Fig. 11.
FIG. 13. (a) The critical density of the $p$-wave kaon condensation (the solid and dotted lines) as a function of the kaon-neutron sigma term $\Sigma_{Kn}$ in Case I. For comparison, the critical density for the $s$-wave $K^-$ condensation is shown by the dashed line. (b) The same as (a), but for Case II. See the text for the details.
### TABLE I. Parameters in the potential energy density. (\(^{a}\text{MeV} \cdot \text{fm}^3\), \(^{b}\text{MeV} \cdot \text{fm}^3 \cdot \gamma\))

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<td>(a_{YY}^{a})</td>
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<td>(c_{NN}^{b})</td>
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<td>(c_{YY}^{b})</td>
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### TABLE II. Parameters for the \(\Sigma^- N\) part in the potential energy density. (\(^{a}\text{MeV} \cdot \text{fm}^3\), \(^{b}\text{MeV} \cdot \text{fm}^3 \cdot \gamma\))

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<td>(b_{\Sigma N}^{a})</td>
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<td>(c_{\Sigma N}^{b})</td>
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