Noncommutative Gauge Theories on
Fuzzy Sphere and Fuzzy Torus from Matrix Model

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Abstract

We consider a reduced model of four-dimensional Yang-Mills theory with a mass term. This matrix model has two classical solutions, two-dimensional fuzzy sphere and two-dimensional fuzzy torus. These classical solutions are constructed by embedding them into three or four dimensional flat space. They exist for finite size matrices, that is, the number of the quantum on these manifolds is finite. Noncommutative gauge theories on these noncommutative manifolds are derived by expanding the model around these classical solutions and studied by taking two large $N$ limits, a commutative limit and a large radius limit. The behaviors of gauge invariant operators are also discussed.

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1 Introduction

After the discovery of the D-brane, our understanding of string theory has changed drastically. The developments of D-brane physics[1, 2] leads to several proposals to formulate nonperturbative aspects of string theory. One of the notable properties of D-branes is that a system of $N$ coincident D-branes has collective coordinates which are described by $N \times N$ matrices and low energy dynamics of D-branes are described by $U(N)$ supersymmetric gauge theories. This idea leads to several kinds of matrix models which have been proposed for the constructive definition of string theory or M-theory[3, 4]. IIB matrix model[4] is one of these proposals. It is a large $N$ reduced model[5] of ten-dimensional supersymmetric Yang-Mills theory and the action has a matrix regularized form of the Green-Schwarz action of IIB superstring.

In the matrix model, eigenvalues of bosonic variables are interpreted as spacetime coordinates and matter and even spacetime are dynamically emerged out of matrices[6, 7, 11]. Therefore noncommutative geometry appears naturally from the matrix model. The idea of the noncommutative geometry is to modify the microscopic structure of the spacetime. This modification is implemented by replacing fields on the spacetime by matrices. Yang-Mills theories in noncommutative space is first appeared within the framework of toroidal compactification of the matrix model[8]. In string theory, it is pointed out that the world volume theory on D-branes with a constant NS-NS two-form background is described by noncommutative Yang-Mills theory[9]. It was shown[10, 11, 12] that noncommutative Yang-Mills theories in flat backgrounds are obtained by expanding the matrix model around a flat noncommutative background. Fields on the background are appeared as the fluctuation around the background from the matrices. This implies the unification of spacetime and fields. Lattice version of noncommutative gauge theories is formulated in [13, 14].

The IIB matrix model has only flat noncommutative backgrounds as classical solutions. To describe a curved spacetime and a compact spacetime is one of the important problems in matrix models. In our previous paper [15], we treat a two-dimensional fuzzy sphere in a matrix model and show that expanding the model around the fuzzy sphere solution leads to a supersymmetric noncommutative gauge theory on the fuzzy sphere. Although ordinary matrix model does not have a fuzzy sphere as a classical solution, adding a Chern-Simons term to Yang-Mills reduced model enable us to describe a fuzzy sphere as a classical solution and a noncommutative gauge theory on it as discussed in [15]. Owing to the extra term, the matrix model can describe a curved spacetime. In [16], $N$ D-branes action in nontrivial backgrounds is considered and Chern-Simons term appeared from D0-branes action in a constant RR three-form potential. Owing to the Chern-Simons term, D0-branes expand into a noncommutative fuzzy sphere configuration. However, it is not well understood how the extra term is generated from the IIB matrix model.
In this paper, we treat a fuzzy sphere and a fuzzy torus in a matrix model by the same manner as in [10, 11, 15]. We start with a bosonic matrix model which is given by

\[ S = -\frac{1}{g^2} Tr \left( \frac{1}{4} [A_\mu, A_\nu] [A_\mu, A_\nu] + \lambda^2 A_\mu A_\mu \right). \]  

(1)

\( A_\mu \) are \( N \times N \) hermitian matrices. \( \lambda \) is a dimensionful parameter which depends on \( N \). This is a reduced model, which is obtained by reducing the spacetime volume to a single point[5], of a four-dimensional bosonic Yang-Mills theory with a mass term\(^2\). This model possesses \( SO(4) \) symmetry and gauge symmetry expressed by the following unitary transformation,

\[ A_\mu \rightarrow U A_\mu U^\dagger. \]  

(2)

This model does not have translational symmetry of \( A_\mu \) because of the mass term. In spite of this shortcoming, this model has an interesting property. The equation of motion of the action (1) is given by

\[ [A_\nu, [A_\mu, A_\nu]] + 2\lambda^2 A_\mu = 0. \]  

(3)

This equation of motion has two classical solutions with different topology. One is a two-dimensional fuzzy sphere and the other is a two-dimensional fuzzy torus (These solutions are explained in section 2 and section 3 respectively). Both solutions are constructed by embedding them into three or four-dimensional flat spacetime. Flat backgrounds do not exist for finite \( N \), while sphere and torus\(^3\) backgrounds can exist for finite \( N \). In the matrix model picture, \( N \) represents the number of the quantum on the backgrounds (or the number of D-instantons or D-particles). As a flat background with infinite extent has the infinite number of the quantum on it, it does not exist for finite \( N \). On the other hand, since the area of compact backgrounds is finite, the solution of compact backgrounds can be constructed for finite \( N \).

The goal of this paper is to consider a curved spacetime or a compact spacetime (a noncommutative sphere and a noncommutative torus) in matrix models. We show that expanding the model around classical solutions leads to noncommutative gauge theories on the solutions.

This paper is organized as follows. Section 2 and 3 are devoted to the analysis of the noncommutative gauge theories on the fuzzy sphere and the fuzzy torus respectively. Section 2 is based on our previous paper [15]. The relation between the model considered in the previous paper [15] and the model used in this paper is also discussed. Construction of a noncommutative torus in terms of unitary matrices is well known. As the eigenvalue of

\(^2\)We now comment on the mass term. This mass term corresponds to a negative mass. Therefore we cannot avoid an unstable mode which is originated from the negative mass term.

\(^3\)Another construction of torus is to impose a periodic boundary condition in a flat background[2, 17]. This construction make us introduce infinite copies of the original matrices. Therefore we cannot construct for finite \( N \).
unitary matrices are distributed over $S^1$, compactness is naturally described in terms of unitary matrices. In section 3, we construct a fuzzy torus in terms of hermitian matrices by dividing a unitary matrix into two hermitian matrices. To investigate the noncommutative gauge theories on these two noncommutative manifolds, we consider two large $N$ limits. One is a commutative limit and another is a large radius limit. The first limit gives ordinary gauge theories on a commutative sphere and a commutative torus. On the other hand, the second limit gives a noncommutative gauge theory on a noncommutative flat space. Although these two gauge theories are same from the matrix model point of view, two large $N$ limits distinguish these gauge theories. We observe the difference of the symmetry ($SO(3)$ versus $SO(2) \times SO(2)$) by taking a commutative limit. In section 4, the behavior of gauge invariant operators on a noncommutative sphere and a noncommutative torus is investigated. Section 5 is devoted to summary and discussions.

2 Noncommutative gauge theory on noncommutative sphere

In this section, we treat a noncommutative sphere. The fuzzy sphere [18, 19, 20, 21, 22, 23, 24, 25, 26, 27] can be constructed by introducing a cut off parameter $N - 1$ for angular momentum of the spherical harmonics. The number of independent functions is $\sum_{l=0}^{N-1}(2l + 1) = N^2$. Therefore, we can replace the functions by $N \times N$ hermitian matrices on the fuzzy sphere. Thus, the algebra on the fuzzy sphere becomes noncommutative.

A noncommutative gauge theory on a noncommutative sphere is considered in our previous paper[15]. Chern-Simons term is added in this case. Since the construction of the noncommutative gauge theory is parallel between Chern-Simons term case and mass term case, we mainly follow our previous paper[15]. A noncommutative sphere is constructed by embedding it into $\mathbb{R}^3$. It is represented by the following algebra,

$$[\hat{x}_i, \hat{x}_j] = i\alpha\epsilon_{ijk}\hat{x}_k \quad (i, j, k = 1, 2, 3),$$
$$\hat{x}_4 = 0.$$  

(4)

$A_\mu = \hat{x}_\mu$ satisfy (3) if we set $\lambda^2 = \alpha^2$. Indices $i, j, k$ are used for $1, 2, 3$ and $\mu, \nu, \tau$ for $1, 2, 3, 4$. We impose the following condition for $\hat{x}_i$,

$$\hat{x}_1\hat{x}_1 + \hat{x}_2\hat{x}_2 + \hat{x}_3\hat{x}_3 = \rho^2.$$  

(5)

This solution preserves $SO(3)$ symmetry. In the $\alpha \to 0$ limit, $\hat{x}_i$ becomes commutative coordinates $x_i$:

$$x_1 = \rho \sin \theta \cos \phi$$

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\[ x_2 = \rho \sin \theta \sin \phi \]
\[ x_3 = \rho \cos \theta, \]

where \( \rho \) denotes the radius of the sphere. The metric tensor of a commutative sphere is

\[ ds^2 = \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 \equiv \rho^2 g_{ab} d\sigma^a d\sigma^b. \]

Matrices \( \hat{x}_i \) can be constructed from the generators of the \( N \)-dimensional irreducible representation of \( SU(2) \) as

\[ \hat{x}_i = \alpha \hat{L}_i \]

with

\[ [\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk} \hat{L}_k. \]

The radius of the sphere is given by the quadratic Casimir of \( SU(2) \) as

\[ \rho^2 = \frac{N^2 - 1}{4} \alpha^2. \]

The Plank constant on the fuzzy sphere, which represents the area occupied by the unit quantum on the fuzzy sphere, is given by

\[ \frac{4\pi \rho^2}{N} = \frac{N^2 - 1}{N} \pi \alpha^2. \]

Now we show that an expansion of the model around the classical background (4) by the similar procedure as in [10, 11] leads to a noncommutative Yang-Mills on a fuzzy sphere. We first consider \( U(1) \) noncommutative gauge theory on the fuzzy sphere. We expand the bosonic matrices around the classical solution (4) as

\[ A_i = \hat{x}_i + \alpha \rho \hat{a}_i = \alpha \rho (\frac{\hat{L}_i}{\rho} + \hat{a}_i) \]
\[ A_4 = \alpha \rho \hat{\phi} \]

\( \hat{a}_i \) and \( \hat{\phi} \) are fields which propagate on the fuzzy sphere. A notable point is that the background \( \hat{x}_i \) and the fields \( \hat{a}_i \) and \( \hat{\phi} \) are dynamically generated from matrix \( A_i \) and they are treated on the same footing.

We first study a correspondence between matrices and functions on a sphere. Ordinary functions on the sphere can be expanded by the spherical harmonics,

\[ a(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega), \]
where
\[ Y_{lm} = \rho^{-l} \sum_a f_{a_1,a_2,\ldots,a_l}^{(m)} x^{a_1} \cdots x^{a_l} \] (14)
is a spherical harmonics and \( f_{a_1,a_2,\ldots,a_l}^{(m)} \) is a traceless and symmetric tensor. The traceless condition comes from \( x_i x_i = \rho^2 \). The normalization of the spherical harmonics is fixed by
\[ \int d\Omega Y_{lm}^* Y_{l'm} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta Y_{lm}^* Y_{l'm} = \delta_{ll'} \delta_{m'm}. \] (15)
Matrices on the fuzzy sphere, on the other hand, can be expanded by the noncommutative spherical harmonics \( \hat{Y}_{lm} \) as
\[ \hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}. \] (16)
\( \hat{Y}_{lm} \) is a \( N \times N \) matrix and defined by
\[ \hat{Y}_{lm} = \rho^{-l} \sum_a f_{a_1,a_2,\ldots,a_l}^{lm} \hat{x}^{a_1} \cdots \hat{x}^{a_l}, \] (17)
where the same coefficients as (14) are used. Angular momentum \( l \) is bounded at \( l = N - 1 \) and these \( \hat{Y}_{lm} \)'s form a complete basis of \( N \times N \) hermitian matrices. From the symmetry of the indices, the ordering of \( \hat{x} \) corresponds to the Weyl type ordering 4. A hermiticity condition requires that \( a_{lm}^* = a_{l-m} \). Normalization of the noncommutative spherical harmonics is given by
\[ \frac{1}{N} Tr(\hat{Y}_{l'm}^* \hat{Y}_{lm}) = \delta_{ll'} \delta_{m'm}. \] (18)
Let us consider a map from matrices to functions:
\[ \hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \rightarrow a(\Omega) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega). \] (19)
This map is formally given as
\[ a(\Omega) = \frac{1}{N} \sum_{lm} Tr(\hat{Y}_{lm}^* \hat{a}) Y_{lm}(\Omega), \] (20)
and correspondingly a product of matrices is mapped to the star product on the fuzzy sphere:
\[ a \star b(\Omega) = \frac{1}{N} \sum_{lm} Tr(\hat{Y}_{lm}^* \hat{a} \hat{b}) Y_{lm}(\Omega). \] (21)

\(^4\)In [15], normal ordered basis is investigated by a stereographic projection from a sphere to a complex plane and Berezin type star product is obtained.
This product is noncommutative corresponding to the noncommutativity of matrices. The explicit form of this product is calculated in [15, 27]. Let us consider the product of the two spherical harmonics, $\hat{Y}_{lm}$ and $\hat{Y}_{l'm'}$. We have required that the maximal maximal value of $l$ is $N - 1$. This product is expanded by the spherical harmonics and it contains $\hat{Y}_{l+l'}$. We assume that $N$ is large such that $l + l'$ does not exceed $N - 1$. This assumption guarantees that the map (20) is one to one.

We next study derivative operators corresponding to the adjoint action of $\hat{L}_i$. Action of $\text{Ad}(\hat{L}_3)$ is calculated as

$$\text{Ad}(\hat{L}_3)\hat{a} = \sum_{lm} a_{lm}[\hat{L}_3, \hat{Y}_{lm}] = \sum_{lm} a_{lm}m\hat{Y}_{lm}. \quad (22)$$

This property and $SO(3)$ symmetry gives the following correspondence:

$$\text{Ad}(\hat{L}_i) \rightarrow L_i \equiv \frac{1}{i} \epsilon_{ijk} x_j \partial_k. \quad (23)$$

The Laplacian on the fuzzy sphere is given by

$$\frac{1}{\rho^2} \text{Ad}(\hat{L}_i)^2 \hat{a} = \frac{1}{\rho^2} \sum_{lm} a_{lm}[\hat{L}_i, [\hat{L}_i, \hat{Y}_{lm}]] = \sum_{lm} \frac{l(l+1)}{\rho^2} a_{lm} \hat{Y}_{lm}. \quad (24)$$

We can rewrite $L_i$ in terms of Killing vectors on the sphere as

$$L_i = -i K^a_i \partial_a. \quad (25)$$

The metric tensor is also given in terms of Killing vectors as

$$g^{ab} = K^a_i K^b_i. \quad (26)$$

The explicit forms of the Killing vectors are shown in the appendix.

$Tr$ over matrices can be mapped to the integration over functions as

$$\frac{1}{N} Tr(\hat{a}) \rightarrow \int \frac{d\Omega}{4\pi} a(\Omega). \quad (27)$$

Let us expand the action (1) around the classical solution (4) and apply these mapping rules. The action becomes

$$S = -\frac{1}{g^2} Tr\left( \frac{\alpha^4}{4} \rho^4 \hat{F}_{ij} \hat{F}^{ij} + \frac{i \alpha^3 \rho^2}{2} \epsilon_{ijk} \hat{F}_{ij} A_k - \frac{\alpha^2}{2} A_i A_i + \frac{\alpha^2}{2} \hat{L}_i \rho + \hat{a}_i \hat{\phi} \right) \left[ \frac{\hat{L}_i}{\rho} + \hat{a}_i \phi \right] - \frac{\alpha^2}{g^2} Tr A_\mu A_\mu$$

$$\rightarrow -\frac{\rho^2}{4g^2}_Y \int d\Omega (F_{ij} F^{ij} + 2 \left[ \frac{L_i}{\rho} + a_i, \phi \right] \left[ \frac{L_i}{\rho} + a_i, \phi \right] + 4 \frac{\phi^2}{\rho^2}).$$
\[ -\frac{3i}{2g_{YM}^2} \epsilon_{ijk} \int d\Omega((L_ia_j) a_k + \frac{\rho}{3}[a_i, a_j] a_k - \frac{i}{2} \epsilon_{ijk} a_i a_k), \]

\[ -\frac{\pi}{g_{YM}^2} \frac{N^2}{2\rho^2} \]

(28)

where \( \hat{F}_{ij} \) is the field strength on the sphere and given by

\[ \hat{F}_{ij} = \frac{1}{\alpha^2 \rho^2} ([A_i, A_j] - i\alpha \epsilon_{ijk} A_k) \]

\[ = \left[ \frac{\hat{L}_i}{\rho}, \hat{a}_j \right] - \left[ \frac{\hat{L}_j}{\rho}, \hat{a}_i \right] + \left[ \hat{a}_i, \hat{a}_j \right] - \frac{1}{\rho} i \epsilon_{ijk} \hat{a}_k \]

(29)

and this is mapped to the following function

\[ F_{ij}(\Omega) = \frac{1}{\rho} L_i a_j(\Omega) - \frac{1}{\rho} L_j a_i(\Omega) + [a_i(\Omega), a_j(\Omega)] - \frac{1}{\rho} i \epsilon_{ijk} a_k(\Omega). \]

(30)

Gauge covariance of \( F_{ij} \) is manifest from the viewpoint of the matrix model and \( F_{ij} \) becomes zero when the fluctuations are set to zero. \( (\quad) \) means that the products should be taken as the star product. The Yang-Mills coupling \( g_{YM}^2 \) is defined by

\[ g_{YM}^2 = \frac{4\pi g^2}{N \alpha^4 \rho^2}. \]

(31)

Thus we have obtained U(1) noncommutative gauge theory on the fuzzy sphere by expanding the matrix model around the fuzzy sphere solution and mapping the matrix model to the field theory.

We have so far discussed the U(1) noncommutative gauge theory on the fuzzy sphere. A generalization to U(m) gauge group is realized by the following replacement:

\[ \hat{x}_i \rightarrow \hat{x}_i \otimes 1_m. \]

(32)

\( \hat{a} \) is also replaced as follows:

\[ \hat{a} \rightarrow \sum_{a=1}^{m^2} \hat{a}^a \otimes T^a, \]

(33)

where \( T^a(a=1, \cdots, m^2) \) denote the generators of U(m). Then we obtain a U(m) noncommutative gauge theory by the same procedure as the U(1) case:

\[ S = -\frac{\rho^2}{4g_{YM}^2} tr \int d\Omega (F_{ij} F_{ij} + 2[\frac{L_i}{\rho} + a_i, \phi] [\frac{L_i}{\rho} + a_i, \phi] + \frac{4}{\rho^2} \phi^2)_* \]

\[ -\frac{3i}{2g_{YM}^2} \epsilon_{ijk} tr \int d\Omega ((L_ia_j) a_k + \frac{\rho}{3}[a_i, a_j] a_k - \frac{i}{2} \epsilon_{ijk} a_i a_k)_* \]

\[ -\frac{\pi}{g_{YM}^2} \frac{mN^2}{2\rho^2} \]

(34)
where $tr$ is taken over $m \times m$ matrices.

We next focus on the gauge symmetry of this action. The action (1) is invariant under the unitary transformation (2). Gauge symmetry of noncommutative gauge theories is included in the unitary transformation (2) of the matrix model. For an infinitesimal transformation $U = \exp(i\hat{\lambda}) \sim 1 + i\hat{\lambda}$ in (2) where $\hat{\lambda} = \sum_{lm} \lambda_{lm} \hat{Y}_{lm}$, the fluctuation around the fixed background transforms as

$$\hat{a}_i \rightarrow \hat{a}_i - \frac{i}{\rho} [\hat{L}_i, \hat{\lambda}] + i[\hat{\lambda}, \hat{a}_i].$$  \hspace{1cm} (35)

After mapping to functions, we have local gauge symmetry

$$a_i(\Omega) \rightarrow a_i(\Omega) - \frac{i}{\rho} [\lambda(\Omega), a_i(\Omega)]_\star.$$ \hspace{1cm} (36)

Let us discuss a scalar field which is a normal component of $a_i$. Three fields $a_i$ are defined in three dimensional space $\mathbb{R}^3$ and contain a gauge field on $S^2$ and a scalar field as well. We define it by

$$\hat{\phi} \equiv \frac{1}{2\alpha \rho} (\hat{A}_i A_i - \hat{x}_i \hat{x}_i)$$
$$= \frac{1}{2} (\hat{x}_i \hat{a}_i + \hat{a}_i \hat{x}_i + \alpha \rho \hat{a}_i \hat{a}_i).$$ \hspace{1cm} (37)

It transforms covariantly as an adjoint representation under the gauge transformation

$$\hat{\phi} \rightarrow \hat{\phi} + i[\hat{\lambda}, \hat{\phi}].$$ \hspace{1cm} (38)

Since the scalar field should become the radial component of $\hat{a}_i$ in the commutative limit, a naive choice is $\hat{\phi}_0 = (\hat{x}_i \hat{a}_i + \hat{a}_i \hat{x}_i)/2$. For small fluctuations this field is the correct component of the field $\hat{a}_i$ but large fluctuations of $\hat{a}_i$ deform the shape of the sphere and $\hat{\phi}_0$ can be no longer interpreted as the radial component of $\hat{a}_i$. This is a manifestation of the fact that matrix models or noncommutative gauge theories naturally unify spacetime and matter on the same footing. An addition of the non-linear term $\hat{a}_i \hat{a}_i$ makes $\hat{\phi}$ transform correctly as the scalar field in the adjoint representation.

We now consider a commutative limit. From (11), the commutative limit is realized by

$$\rho = \text{fixed}, \quad g_{YM} = \text{fixed}, \quad N \rightarrow \infty.$$ \hspace{1cm} (39)

In the commutative limit, the star product becomes the commutative product. The action becomes

$$S = -\frac{\rho^2}{4g_{YM}} tr \int d\Omega (F_{ij} F_{ij} + 2 \lambda_i \rho + a_i, \phi)[\frac{\lambda_i}{\rho} + a_i, \phi] + \frac{4}{\rho^2} \phi^2)$$

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Since the noncommutativity deforms the shape of the sphere as explained in the previous paragraph, the scalar field cannot be separated from the gauge field. In the commutative case, however, the scalar field $\varphi$ and the gauge field are separable from each other as in

$$\rho a_i(\Omega) = = K_i^a b_a(\Omega) + \frac{x_i}{\rho} \varphi(\Omega)$$

(41)

where $i = 1, 2, 3$ and $a = \theta, \phi$. $b_a$ is a gauge field on the sphere. The field strength is expressed in terms of the gauge field $b_a$ and the scalar field $\varphi$ as

$$F_{ij}(\Omega) = \frac{1}{\rho^2} K_i^a K_j^b F_{ab} + \frac{i}{\rho^2} \varepsilon_{ijk} x_k \varphi + \frac{1}{\rho^2} x_j K_i^b D_a \varphi - \frac{1}{\rho^2} x_i K_j^a D_a \varphi$$

(42)

where $D_a = \frac{1}{\iota} \partial_a [b_a, \cdot]$ and $F_{ab} = \frac{1}{\iota} \partial_a b_b - \frac{1}{\iota} \partial_b b_a + [b_a, b_b]$. The action in (40) is also rewritten as

$$S = -\frac{3i}{2g_{YM}^2} \frac{\varepsilon_{ijk} \rho}{\pi} \frac{1}{N^2} \int d\Omega ((L_i a_j) a_k + \frac{\rho}{3} [a_i, a_j] a_k - \frac{i}{2} \varepsilon_{ijk} a_i a_k)$$

(40)

$$- \frac{\pi}{g_{YM}^2 2\rho^2}$$

$$- \frac{3i}{2g_{YM}^2} \frac{\varepsilon_{ijk} \rho}{\pi} \frac{1}{N^2} \int d\Omega ((L_i a_j) a_k + \frac{\rho}{3} [a_i, a_j] a_k - \frac{i}{2} \varepsilon_{ijk} a_i a_k)$$

where $\varepsilon^{\theta \phi} = -\varepsilon^{\phi \theta} = -1$. The gauge transformation (36) becomes

$$b_a \rightarrow b_a - \partial_a \lambda$$

$$\varphi \rightarrow \varphi + \iota[\lambda, \varphi]$$

$$\phi \rightarrow \phi + \iota[\lambda, \phi]$$

(44)

for $U(1)$ gauge group and

$$b_a \rightarrow b_a - \partial_a \lambda + i[\lambda, b_a]$$

$$\varphi \rightarrow \varphi + i[\lambda, \varphi]$$

$$\phi \rightarrow \phi + i[\lambda, \phi]$$

(45)

for $U(m)$ gauge group. For $U(1)$ gauge group case, (43) is simplified:

$$S = -\frac{1}{4g_{YM}^2 \rho^2} \int d\Omega (F_{ab} F^{ab} + 8i \varepsilon^{ab} F_{ab} \varphi$$

(43)
\[-2(\partial_a \varphi)(\partial^a \varphi) - 8\varphi^2 - 2\rho^2(\partial_a \phi)(\partial^a \phi) + 4\rho^2\phi^2) = -\frac{1}{4g_Y^2 M^2 \rho^2} \int d^2\sigma \sqrt{g}(F_{ab}F^{ab} + 8i\epsilon^{ab}_{\phi}F_{ab}\varphi \\
+ 2\varphi \Delta \varphi - 8\varphi^2 + 2\rho^2\phi \Delta \phi + 4\rho^2\phi^2) \] (46)

where
\[\Delta = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \] (47)
is the Laplacian on a unit sphere. The scalar field \(\phi\) has a negative mass term. This negative mass term is originated from the mass term in the action (1). This action is invariant under the diffeomorphism. Since \(\Delta \sim l(l + 1)\) respects \(SO(3)\) symmetry, this action is \(SO(3)\) invariant.

Let us investigate another large \(N\) limit with \(\alpha\) fixed. In this limit, the radius of the fuzzy sphere becomes large and the noncommutative sphere is expected to become a noncommutative plane. By virtue of the \(SO(3)\) symmetry of the fuzzy sphere, we may consider the theory around the north pole without loss of generality. Around the north pole, \(\hat{L}_3\) can be approximated as \(\hat{L}_3 \sim (N - 1)/2\). By defining \(\hat{L}'_i = \sqrt{\frac{2}{N-1}} \hat{L}_i\), the commutation relation (9) becomes
\[[\hat{L}'_1, \hat{L}'_2] \sim i. \] (48)
By further defining \(\hat{x}'_i = \alpha \hat{L}'_i\) and \(\hat{p}'_i = \alpha^{-1} \varepsilon_{ij} \hat{L}'_j (i, j = 1, 2)\), we have
\[[\hat{x}'_1, \hat{x}'_2] = i\alpha^2, \quad [\hat{p}'_1, \hat{p}'_2] = i\alpha^{-2}, \quad [\hat{x}'_i, \hat{p}'_j] = i\delta_{ij} \] (49)
and
\[\rho'^2 = \hat{x}'_i \hat{x}'_i = \frac{2}{N-1} \rho^2 = \frac{N + 1}{2} \alpha^2 \sim \frac{N}{2} \alpha^2. \] (50)
In the coordinates of \(\hat{x}'_i\), the Plank constant is given by
\[\frac{4\pi \rho'^2}{N} \sim 2\pi \alpha^2. \] (51)
We take the following limit to decompactify the sphere around the north pole,
\[\alpha = \text{fixed}, \quad \rho' \to \infty \quad (N \to \infty). \] (52)
\(a \ast b\) which is defined in (21) becomes the Moyal product \(a \ast_M b\) because of (48) and the Weyl type ordering property in (17). The following replacement holds in this limit,
\[\frac{1}{N} Tr \to \int \frac{d\Omega}{4\pi} = \int \frac{d^2x}{4\pi \rho^2} = \int \frac{d^2x'}{4\pi \rho'^2} \] (53)
and the rotation around the 1(2) axis corresponds to the translation in the 2(1) direction around the north pole,

$$Ad(\hat{p}_i') = \frac{1}{\rho} \varepsilon_{ij} Ad(\hat{L}_j) \rightarrow \frac{1}{\iota} \partial'_l \ (i = 1, 2). \quad (54)$$

We can regard $\hat{a}_3$ as the scalar field $\hat{\varphi}$ around the north pole. Since the mass term in the action (1) drops in this limit, the action becomes

$$S_B = -\frac{\alpha^4}{2g^2} Tr([\hat{L}_1 + \rho \hat{a}_1, \hat{L}_2 + \rho \hat{a}_2]^2) + \frac{\alpha^4}{2g^2} Tr(\rho \hat{\varphi}[\hat{L}_i + \rho \hat{a}_i, [\hat{L}_i + \rho \hat{a}_i, \rho \hat{\varphi}]])$$

$$+ \frac{\alpha^4}{2g^2} Tr(\rho \hat{\varphi}[\hat{L}_i, \rho \hat{a}_i, [\hat{L}_i + \rho \hat{a}_i, \rho \hat{\varphi}]] - \frac{\alpha^4}{2g^2} Tr([\rho \hat{\varphi}, [\rho \hat{\varphi}, [\rho \hat{\varphi}, \rho \hat{\varphi}]]))$$

$$= \frac{\alpha^4}{2g^2} (\rho')^4 \left\{ -Tr([-\hat{p}_2 - \hat{a}_2, \hat{p}_1' + \hat{a}_1'][^2] + Tr(\hat{\varphi}'[\hat{p}_i' + \hat{a}_i', [\hat{p}_i' + \hat{a}_i', \hat{\varphi}']]\right.$$

$$+ Tr(\hat{\varphi}'[, \hat{\varphi}', \hat{\varphi}')] - Tr([\hat{\varphi}', \hat{\varphi}', \hat{\varphi}']))) \quad (55)$$

where we have defined $\hat{a}_i = \sqrt{\frac{\alpha}{N}} \hat{a}_j \varepsilon_{ji} \ (i, j = 1, 2)$, $\hat{\varphi} = \sqrt{\frac{\alpha}{N}} \hat{\varphi}'$ and $\hat{\phi} = \sqrt{\frac{\alpha}{N}} \hat{\varphi}'$. This action can be mapped to the following field theory action,

$$S_B = -\frac{\alpha^6 N^2}{16 \pi g^2} \left\{ \int d^2 x' F_{12}(x)^2 + \int d^2 x'(D_i \varphi'(x))(D_i \varphi'(x))\right.$$

$$+ \int d^2 x' (D_i \varphi'(x))(D_i \varphi'(x)) + \int d^2 x'[\varphi', \varphi'][\varphi', \varphi']. \quad (56)$$

where $D_i = \frac{1}{\iota} \partial'_l + [a'_i, \cdot]$ and $F_{12} = \frac{1}{\iota} \partial'_l a'_2 - \frac{1}{\iota} \partial'_2 a'_1 + [a'_1, a'_2]$. It is found that the Yang-Mills coupling is $g_{YM}^2 = 4\pi g^2 / N^2 \alpha^6$. We thus obtained an action of a gauge theory on a noncommutative plane by taking a large $N$ limit with noncommutativity fixed from a gauge theory on a noncommutative sphere.

The gauge transformation (35) becomes

$$a'(x) \rightarrow a'(x) - \partial'_l \lambda(x) + i[\lambda(x), a'_l(x)],$$

$$\varphi'(x) \rightarrow \varphi'(x) + i[\lambda(x), \varphi'(x)],$$

$$\phi'(x) \rightarrow \phi'(x) + i[\lambda(x), \phi'(x)]. \quad (57)$$

In our previous paper[15], we consider a matrix model with Chern-Simons term. Before finishing this section, we remark on a difference between CS term and mass term. Action which is used in the previous paper is

$$S = \frac{1}{g^2} Tr(-\frac{1}{4}[A_i, A_j][A^i, A^j] + \frac{2}{3} i \alpha \varepsilon_{ijk} A^i A^j A^k). \quad (58)$$
The difference is given by the following action,
\[ S' = \frac{1}{g^2} Tr(\frac{2}{3} i \alpha \epsilon_{ijk} A^i A^j A^k + \alpha^2 A_i A_i) \]  
and it is mapped to the following field theory action as
\[ S' = \frac{i}{g_{YM}^2} \epsilon_{ijk} \int d\Omega ((L_i a_j) a_k + \frac{\rho}{3} [a_i, a_j] a_k - \frac{i}{2} \epsilon_{ijl} a_l a_k) + \frac{\pi}{3 g_{YM}^2} N^2 \rho^2. \]  
This represents a gauge invariant coupling between the gauge field and the scalar field. (Gauge invariance is manifest in (59).) In the commutative limit this becomes as follows
\[ S' = \frac{1}{g_{YM}^2} \int d\Omega (i \epsilon_{ijk} K_i^a K_j^b F_{ab} x_k \phi - \phi^2) \]
\[ = \frac{1}{g_{YM}^2} \int d\Omega \left( \frac{\epsilon_{ab}}{\sqrt{g}} F_{ab} \phi - \phi^2 \right). \]  

In considering a field theory on a sphere, we may add a term of this type because gauge invariance is not lost. While a matrix model with Chern-Simons term leads to a supersymmetric gauge theory on a fuzzy sphere as discussed in [15], a matrix model with mass term does not.

### 3 Noncommutative gauge theory on noncommutative torus

In this section we consider the second classical solution of our model, a noncommutative torus. It is described naturally in terms of unitary matrices (by imposing a condition (62)). Since eigenvalues of unitary matrices are distributed over \( S^1 \), a compact spacetime appears naturally from the unitary matrices. In this section, in order to treat a compact spacetime in terms of hermitian matrices, we decompose a unitary matrix into two hermitian matrices. This corresponds to a treatment of a two-dimensional torus embedded into \( \mathbb{R}^4 \).

The algebra on the fuzzy torus is generated by two unitary matrices \( U \) and \( V \) which satisfy the following relation,
\[ UV = e^{i\theta} VU \]  
where \( \theta = 2\pi/N \). It is well known that these two traceless unitary matrices are given by the following \( N \times N \) clock and shift matrices,
\[ U = \begin{pmatrix} 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ \omega & \cdots & \omega^2 & \cdots & \omega^{N-1} \\ \omega^2 & \cdots & \omega^2 & \cdots & \omega^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega^{N-1} & \cdots & \omega^{N-1} & \cdots & 1 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \]  
(63)
where \( \omega = e^{i\theta}(\omega^N = 1) \). \( U \) and \( V \) satisfy

\[
U^N = V^N = 1
\]  

(64)

and

\[
UU^\dagger = U^\dagger U = 1, \quad VV^\dagger = V^\dagger V = 1.
\]  

(65)

To embed two-dimensional torus into \( \mathbb{R}^4 \), we decompose these unitary matrices into hermitian matrices:

\[
U \equiv \hat{y}_1 + i\hat{y}_2 \equiv e^{i\hat{u}} \\
V \equiv \hat{y}_3 + i\hat{y}_4 \equiv e^{i\hat{v}}
\]  

(66)

or

\[
\hat{y}_1 = \frac{1}{2}(U + U^\dagger) \quad \hat{y}_2 = \frac{1}{2i}(U - U^\dagger) \\
\hat{y}_3 = \frac{1}{2}(V + V^\dagger) \quad \hat{y}_4 = \frac{1}{2i}(V - V^\dagger).
\]  

(67)

\( \hat{w} \equiv R_1\hat{u} \) and \( \hat{z} \equiv R_2\hat{v} \) are introduced as noncommutative coordinates on the torus. From (62) and (66), \( \hat{u} \) and \( \hat{v} \) satisfy the following commutation relation:

\[
[\hat{u}, \hat{v}] = \frac{2\pi i}{N} = i\theta.
\]  

(68)

This is the same relation as the usual flat case. Strictly speaking, such \( \hat{u} \) and \( \hat{v} \) does not exist for finite \( N \) while \( U \) and \( V \) exist for finite \( N \). (65) can be rewritten in terms of \( \hat{y}_i \) as

\[
\hat{y}_1^2 + \hat{y}_2^2 = 1, \quad [\hat{y}_1, \hat{y}_2] = 0, \\
\hat{y}_3^2 + \hat{y}_4^2 = 1, \quad [\hat{y}_3, \hat{y}_4] = 0.
\]  

(69)

Commutation relations of \( \hat{y}_i \)'s are represented by

\[
[\hat{y}_1, \hat{y}_2] = 0 \\
[\hat{y}_3, \hat{y}_4] = 0 \\
[\hat{y}_1, \hat{y}_3] = (\cos \theta - 1)\hat{y}_4\hat{y}_1 + i \sin \theta \hat{y}_3\hat{y}_2 \\
[\hat{y}_2, \hat{y}_3] = (\cos \theta - 1)\hat{y}_3\hat{y}_2 + i \sin \theta \hat{y}_4\hat{y}_1 \\
[\hat{y}_1, \hat{y}_4] = (\cos \theta - 1)\hat{y}_3\hat{y}_1 - i \sin \theta \hat{y}_4\hat{y}_2 \\
[\hat{y}_2, \hat{y}_4] = (\cos \theta - 1)\hat{y}_4\hat{y}_2 - i \sin \theta \hat{y}_3\hat{y}_1.
\]  

(70)

It can be shown that \( A_\mu = \hat{x}_\mu \equiv R_a\hat{y}_\mu \) \((a = 1 \text{ for } \mu = 1, 2 \text{ and } 2 \text{ for } \mu = 3, 4)\) satisfy the equation of motion (3) if we set
\[ \lambda^2 = \frac{\beta^2}{2} \equiv R_a^2 (1 - \cos \theta) \]  

(71)

where \( a = 2 \) for \( \mu = 1, 2 \) and \( a = 1 \) for \( \mu = 3, 4 \). \( \beta \) is a quantity which should be compared with \( \alpha \) in the previous section. \( R_a(a = 1, 2) \) are radii of two cycles of the torus. From (69), we obtain

\[ \hat{x}_1^2 + \hat{x}_2^2 = R_1^2, \quad \hat{x}_3^2 + \hat{x}_4^2 = R_2^2. \]  

(72)

This classical solution preserves \( SO(2) \times SO(2) \) symmetry. Hereafter we set \( R_1 = R_2 \) for simplicity.

In the commutative limit, \( \hat{x}_\mu \) becomes the commutative coordinates on the torus:

\[ x_1 = R \cos u, \quad x_2 = R \sin u, \]
\[ x_3 = R \cos v, \quad x_4 = R \sin v. \]  

(73)

The metric tensor of the torus is

\[ ds^2 = R^2 du^2 + R^2 dv^2 \equiv R^2 g_{ab} d\sigma^a d\sigma^b. \]  

(74)

The Plank constant, which represents the area occupied by the unit quantum on the torus, is given by

\[ \frac{(2\pi R)^2}{N} = 2\pi R^2 \theta = N \beta^2 \]  

(75)

In the second equality, we have assumed large \( N \) such that \( \beta = R \theta \).

We derive a \( U(1) \) noncommutative gauge theory on a noncommutative torus by expanding the matrices around the classical solution:

\[ A_\mu = \hat{x}_\mu + R^2 \theta \hat{a}_\mu \]  

(76)

where \( \hat{a}_\mu \) are propagating fields on the torus. Matrices on the fuzzy torus can expanded as

\[ \hat{a} = \sum_{n=0}^N \sum_{m=0}^N a_{nm} e^{-\frac{\pi i}{N} nm U^n V^m} \]
\[ = \sum_{n=0}^N \sum_{m=0}^N a_{nm} e^{in\hat{u} + im\hat{v}} \]
\[ \equiv \sum_{n=0}^N \sum_{m=0}^N a_{nm} \hat{Z}_{nm}. \]  

(77)

\( \hat{Z}_{nm} \) forms a complete basis of \( N \times N \) hermitian matrices. \( a_{nm}^* = a_{-n-m} \) comes from a hermitian condition on \( \hat{a} \). The ordering of \( \hat{u} \) and \( \hat{v} \) corresponds to Weyl type ordering. Functions on the fuzzy torus is expanded in terms of the plane waves as

\[ a(u, v) = \sum_{n=0}^N \sum_{m=0}^N a_{nm} e^{i nu} e^{imv} \]
\[
\sum_{n=0}^{N} \sum_{m=0}^{N} a_{nm} Z_{nm}(u, v). \tag{78}
\]

where same momentum modes are used as (77). From the condition (64) which expresses the periodicity of the torus, the momentum \(n\) and \(m\) is bounded at \(N\).

Now we consider a map from matrices to functions,
\[
\hat{a} \rightarrow a(u, v) \tag{79}
\]

From (77) and (78), we obtain an explicit map from matrices to functions as
\[
a(u, v) = \sum_{nm} \frac{1}{N} Tr(\hat{Z}_{nm}\hat{a}) Z_{nm}(u, v). \tag{80}
\]

The product of matrices is also mapped to the so called star product,
\[
a \star b(u, v) = \sum_{nm} \frac{1}{N} Tr(\hat{Z}_{nm}\hat{a}\hat{b}) Z_{nm}(u, v). \tag{81}
\]

To guarantee that this map is one to one, we take large \(N\) limit such that typical \(n\) is much smaller than \(N\). \(Tr\) over matrices can be mapped to the integration over functions as
\[
\frac{1}{N} Tr(\hat{a}) \rightarrow \int \frac{dw dz}{(2\pi R)^2} a(w, z). \tag{82}
\]

We next consider differential operators corresponding to the adjoint action of \(\hat{x}_i\). We first investigate the adjoint action of \(U\) and \(V\). It is calculated as
\[
[U, \hat{a}] = \sum_{nm} a_{nm} e^{-\frac{i}{N} nm} (1 - e^{-im\theta}) U^{n+1} V^m
\]
\[
\rightarrow e^{iu} (1 - e^{-i\theta(\frac{1}{N} \partial_v)}) a \sim e^{iu} \theta \partial_v a,
\]

\[
[U^\dagger, \hat{a}] = \sum_{nm} a_{nm} e^{-\frac{i}{N} nm} (1 - e^{im\theta}) U^{n-1} V^m
\]
\[
\rightarrow e^{-iu} (1 - e^{i\theta(\frac{1}{N} \partial_v)}) a \sim -e^{iu} \theta \partial_v a,
\]

\[
[V, \hat{a}] = -\sum_{nm} a_{nm} e^{-\frac{i}{N} nm} (1 - e^{-i\theta}) U^n V^{m+1}
\]
\[
\rightarrow -e^{iv} (1 - e^{-i\theta(\frac{1}{N} \partial_u)}) a \sim -e^{iv} \theta \partial_u a,
\]

\[
[V^\dagger, \hat{a}] = -\sum_{nm} a_{nm} e^{-\frac{i}{N} nm} (1 - e^{i\theta}) U^n V^{m-1}
\]
\[
\rightarrow -e^{-iv} (1 - e^{i\theta(\frac{1}{N} \partial_u)}) a \sim e^{-iv} \theta \partial_u a. \tag{83}
\]

We have taken a large \(N\) limit such that \(n\theta \sim n/N \ll 1\). After taking this large \(N\) limit, compactness of the torus can not be observed. From these relations, we can obtain the adjoint action of \(\hat{x}_i\):
\[
[\hat{x}_1, \hat{a}] \rightarrow \frac{R}{2} (e^{iu} (1 - e^{-i\theta(\frac{1}{N} \partial_v)}) + e^{-iu} (1 - e^{i\theta(\frac{1}{N} \partial_v)})) a(u, v) \tag{84}
\]

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\[ \sim - R \theta \sin u \left( \frac{1}{i} \partial_u \right) a(u, v), \]

\[ [\hat{x}_2, \hat{a}] \rightarrow \frac{R}{2i} (e^{iu} (1 - e^{-i \theta (\frac{4}{3} \partial v)}) - e^{-iu} (1 - e^{i \theta (\frac{4}{3} \partial v)})) a(u, v) \]

\[ \sim R \theta \cos u \left( \frac{1}{i} \partial_v \right) a(u, v), \]

\[ [\hat{x}_3, \hat{a}] \rightarrow - \frac{R}{2} (e^{iv} (1 - e^{-i \theta (\frac{4}{3} \partial u)}) + e^{-iv} (1 - e^{i \theta (\frac{4}{3} \partial u)})) a(u, v) \]

\[ \sim R \theta \sin v \left( \frac{1}{i} \partial_u \right) a(u, v), \]

\[ [\hat{x}_4, \hat{a}] \rightarrow - \frac{R}{2i} (e^{iv} (1 - e^{-i \theta (\frac{4}{3} \partial u)}) - e^{-iv} (1 - e^{i \theta (\frac{4}{3} \partial u)})) a(u, v) \]

\[ \sim - R \theta \cos v \left( \frac{1}{i} \partial_u \right) a(u, v). \] (84)

These are expressed in terms of Killing vectors on the torus as

\[ [\hat{x}_\mu, \hat{a}] \equiv \beta [\hat{T}_\mu, \hat{a}] \]

\[ \rightarrow - i \beta K^a_\mu \partial_a a(u, v) \]

\[ \equiv \beta T_\mu a(u, v) \] (85)

where \( \mu = 1, 2, 3, 4 \) and \( a = u, v \). The metric tensor on the torus is also expressed in terms of the Killing vectors as

\[ g^{ab} = K^a_i K^b_i. \] (86)

The explicit forms of \( K^a_\mu \) are summarized in appendix. The Laplacian on the torus is given by

\[ \frac{1}{\theta^2 R^3} (Ad(\hat{x}))^2 \hat{a} \rightarrow \frac{1}{\theta^2 R^4} \{ R^2 (1 - \cos (\theta \partial_v)) + R^2 (1 - \cos (\theta \partial_u)) \} a(u, v) \]

\[ \sim \frac{1}{2 R^2} (\partial_u^2 + \partial_v^2) a(u, v). \] (87)

Then we expand the action (1) around the classical solution (67) as in (76) and apply this mapping rule. The action becomes

\[ S = - \frac{1}{4 g_{YM}^2} \int dwdz F_{\mu \nu} \ast F_{\mu \nu} \]

\[ + \frac{i}{2 g_{YM}^2 R^4} \int dwdz \{ R^2 F_{13}(x_4 a_2 + a_4 x_2) - R^2 F_{14}(x_3 a_2 + a_3 x_2) \]

\[ - R^2 F_{23}(x_4 a_1 + a_4 x_1) + R^2 F_{24}(x_3 a_1 + a_3 x_1) \]

\[ + (T_2 a_1 - T_1 a_2)(a_3 x_4 - a_4 x_3) \]

\[ + (T_4 a_1) x_2 a_3 - (T_3 a_1) x_2 a_4 - (T_2 a_4) x_1 a_3 + (T_3 a_4) x_1 a_4 \]

\[ + i a_4 a_2 x_4 x_2 + i a_3 a_2 x_3 x_2 + i a_4 a_1 x_4 x_1 + i a_3 a_1 x_3 x_1 \} \ast \]

\[ + \frac{1}{2 g_{YM}^2 R^4} \int dwdz \{ 2(x_1 x_3 a_1 a_3 + x_3 x_1 a_3 a_1 + x_1 x_4 a_1 a_4 + x_4 x_1 a_4 a_1) \].

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\[
+ x_2 x_3 a_2 a_3 + x_3 x_2 a_3 a_2 + x_2 x_4 a_2 a_4 + x_4 x_2 a_4 a_2 \\
+ x_3 a_1 x_3 a_1 + x_1 a_3 x_1 a_3 + x_4 a_1 x_4 a_1 + x_1 a_4 x_1 a_4 \\
+ x_3 a_2 x_3 a_2 + x_2 a_3 x_2 a_3 + x_4 a_2 x_4 a_2 + x_2 a_4 x_2 a_4 \\
+ O(\theta) \}
\]
\[
- \frac{1}{2 g_{YM}^2 R^2} \int d\omega d\alpha \mu \ast \alpha - \frac{1}{g_{YM}^2} \frac{N^2}{2 R^2} \tag{88}
\]

where \( O(\theta) \) contains terms which are proportional to \( \theta \). \( F_{\mu\nu} \) is the field strength on the torus:

\[
\hat{F}_{12} = \frac{1}{\theta^2 R^4} ([A_1, A_2]), \\
\hat{F}_{13} = \frac{1}{\theta^2 R^4} ([A_1, A_3] - (\cos \theta - 1) A_3 A_1 + i \sin \theta A_4 A_2), \\
\hat{F}_{14} = \frac{1}{\theta^2 R^4} ([A_1, A_4] - (\cos \theta - 1) A_4 A_1 - i \sin \theta A_3 A_2), \\
\hat{F}_{23} = \frac{1}{\theta^2 R^4} ([A_2, A_3] - (\cos \theta - 1) A_3 A_2 - i \sin \theta A_4 A_1), \\
\hat{F}_{24} = \frac{1}{\theta^2 R^4} ([A_2, A_4] - (\cos \theta - 1) A_4 A_2 + i \sin \theta A_3 A_1), \\
\hat{F}_{34} = \frac{1}{\theta^2 R^4} ([A_3, A_4]). \tag{89}
\]

These are gauge covariant and equal to zero when the fluctuating fields vanish. The Yang-Mills coupling is defined by

\[
g_{YM}^2 = (2\pi R)^2 g^2 \frac{g_{YM}^2 N^2}{\theta^4 R^8 N} = \frac{g^2 N^3}{(2\pi)^2 R^6}. \tag{90}
\]

We have so far discussed the \( U(1) \) noncommutative gauge theory on the fuzzy torus. A generalization to \( U(m) \) gauge group is realized by the same way as the sphere case.

Here we note the gauge transformation of this noncommutative gauge theory. The gauge symmetry of the noncommutative gauge theories is embedded in the unitary transformation of the matrix model. The gauge transformation in noncommutative gauge theories is obtained from the transformation around the fixed background. For \( U = \exp(i\hat{\lambda}) \sim 1 + i\hat{\lambda} \) in (2), we obtain

\[
\hat{a}_\mu \rightarrow \hat{a}_\mu - i \frac{1}{R} [\hat{T}_\mu, \hat{\lambda}] + i [\hat{\lambda}, \hat{a}_\mu]. \tag{91}
\]

After mapping to functions, we have local gauge symmetry

\[
a_\mu(u, v) \rightarrow a_\mu(u, v) - i \frac{1}{R} T_\mu \lambda(u, v) + i [\lambda(u, v), a_\mu(u, v)], \tag{92}
\]

where \( T_\mu \) is a derivative operator which is defined in (85).
In the same way as the fuzzy sphere case, two scalar fields which are transverse components of two cycles of the fuzzy torus is given as follows:

\[ \hat{\phi}_1 = \frac{1}{2\theta R^2}(A_1A_1 + A_2A_2 - \hat{x}_1\hat{x}_1 - \hat{x}_2\hat{x}_2) \]
\[ \hat{\phi}_2 = \frac{1}{2\theta R^2}(A_3A_3 + A_4A_4 - \hat{x}_3\hat{x}_3 - \hat{x}_4\hat{x}_4). \] (93)

These scalar fields transform covariantly as adjoint scalars under the gauge transformation:

\[ \hat{\phi}_1 \rightarrow \hat{\phi}_1 + i[\lambda, \hat{\phi}_1] \]
\[ \hat{\phi}_2 \rightarrow \hat{\phi}_2 + i[\lambda, \hat{\phi}_2]. \] (94)

We then look at a gauge theory on a commutative torus. From (75), a commutative limit is taken by

\[ R = \text{fixed}, \quad g_{YM} = \text{fixed}, \quad N \rightarrow \infty. \] (95)

In this limit, \( O(\theta) \rightarrow 0 \) and the star product becomes the ordinary product. Four fields \( a_\mu \) contain a gauge field on \( T^2 \) and two scalar fields \( \phi_1 \) and \( \phi_2 \). In the commutative theory, we can separate \( a_\mu \) in terms of Killing vectors and \( x_\mu \) as

\[ a_\mu(u, v) = \frac{1}{R}K_\mu^a b_a(u, v) + \frac{x_\mu}{R^2}\phi_1(u, v). \] (96)

where \( i = 1 \) for \( \mu = 1, 2 \) and \( i = 2 \) for \( \mu = 3, 4 \). Field strength in (89) can be rewritten in terms of the gauge field \( b_a \) and the scalar fields \( \phi_1 \) and \( \phi_2 \) as

\[ F_{12} = \frac{1}{R^2}D_v\phi_1, \]
\[ F_{13} = \frac{1}{R^2}(x_2x_4F_{uv} - x_2x_3(D_v\phi_2) - x_1x_4(D_u\phi_1) + x_1x_3[\phi_1, \phi_2]), \]
\[ F_{14} = \frac{1}{R^2}(-x_2x_3F_{uv} - x_2x_4(D_v\phi_2) + x_1x_3(D_u\phi_1) + x_1x_4[\phi_1, \phi_2]), \]
\[ F_{23} = \frac{1}{R^2}(-x_1x_4F_{uv} + x_1x_3(D_v\phi_2) - x_2x_4(D_u\phi_1) + x_2x_3[\phi_1, \phi_2]), \]
\[ F_{24} = \frac{1}{R^2}(x_1x_3F_{uv} + x_1x_4(D_v\phi_2) + x_2x_3(D_u\phi_1) + x_2x_4[\phi_1, \phi_2]), \]
\[ F_{34} = \frac{1}{R^2}D_u\phi_2, \] (97)

where \( F_{uv} = \frac{1}{i}\partial_u b_v - \frac{1}{i}\partial_v b_u + [b_u, b_v] \) and \( D_a = \frac{1}{i}\partial_a + [b_a, \cdot] \). Then the action becomes

\[ S = -\frac{1}{4g_{YM}^2R^4}tr \int dwdz(2(D_a\phi_1)(D^a\phi_1) + 2(D_a\phi_2)(D^a\phi_2) + 2[\phi_1, \phi_2]^2 + F_{ab}F^{ab}) \]
\[ + \frac{i}{g_{YM}} \int dwdz\epsilon^{ab}F_{ab}(\phi_1 + \phi_2) + \frac{1}{g_{YM}^2} \frac{1}{R^4}tr \int dwdz(\phi_1\phi_2) \]
\[
S = -\frac{1}{4g_{YM}^2 R^4} tr \int dwdz(2(D_a \chi)(D^a \chi) + 2(D_a \psi)(D^a \psi) + 2[\chi, \psi]^2 + F_{ab}F^{ab})
+ \frac{\sqrt{2}i}{g_{YM}} tr \int dwdz \epsilon^{ab} F_{ab} \chi + \frac{1}{2g_{YM}^2 R^4} tr \int dwdz(\chi^2 - \psi^2),
\]

(98)

where \(\epsilon^{uv} = -\epsilon^{vu} = 1\). In the second equality, we changed the variable as \(\chi = (\phi_1 + \phi_2)/\sqrt{2}\) and \(\psi = (\phi_1 - \phi_2)/\sqrt{2}\). In the commutative case, the gauge transformation becomes

\[
\begin{align*}
    b_a(u, v) &\to b_a(u, v) - \partial_a \lambda(u, v) + i[\lambda(u, v), b_a(u, v)] \\
    \phi_i(u, v) &\to \phi_i(u, v) + i[\lambda(u, v), \phi_i(u, v)] \quad (i = 1, 2)
\end{align*}
\]

(99)

for \(U(m)\) case and

\[
\begin{align*}
    b_a(u, v) &\to b_a(u, v) - \partial_a \lambda(u, v) \\
    \phi_i(u, v) &\to \phi_i(u, v)
\end{align*}
\]

(100)

for \(U(1)\) case. (98) is further simplified for \(U(1)\) gauge group,

\[
S = -\frac{1}{4g_{YM}^2 R^4} \int d^2 \sigma \sqrt{g}(F_{ab}F^{ab} - 4\sqrt{2}i \epsilon^{ab} \sqrt{g} F_{ab} \chi + 2\chi \Delta \chi - 2\chi^2 + 2\psi \Delta \psi + 2\psi^2)
\]

(101)

where

\[
\Delta = \partial_u^2 + \partial_v^2
\]

(102)

is the Laplacian on a unit torus. A tachyonic scalar field \(\psi\) appeared. This tachyonic mode may be related to the instability of the matrix model. This action is invariant under the general coordinate transformation. Since \(\Delta \sim -(n^2 + m^2)\) respects \(SO(2) \times SO(2)\) symmetry, this action is also invariant under \(SO(2) \times SO(2)\) symmetry.

We next consider another large \(N\) limit with \(\beta\) fixed. In this limit, two radii of a torus become large and the noncommutative torus is expected to become a noncommutative plane. By virtue of the \(SO(2) \times SO(2)\) symmetry we may consider the theory around \(\hat{x}_1 = R\) and \(\hat{x}_3 = -R\). We rescale as \(\hat{x}_i = \sqrt{\frac{N}{2\pi}} \hat{x}'_i = \frac{1}{\sqrt{\theta}} \hat{x}'_i\). The commutation relations (70) become

\[
[\hat{x}'_2, \hat{x}'_4] = i\beta^2
\]

(103)

and other commutation relations vanish. Defining \(\hat{x}'_i = \beta \hat{T}'_i\), \(\hat{p}'_2 = -\beta^{-1} \hat{T}'_4\) and \(\hat{p}'_4 = \beta^{-1} \hat{T}'_2\), we get

\[
[\hat{x}'_2, \hat{x}'_4] = i\beta^2, \quad [\hat{p}'_2, \hat{p}'_4] = i\beta^{-2}, \quad [\hat{x}'_2, \hat{p}'_2] = -i, \quad [\hat{x}'_4, \hat{p}'_4] = -i.
\]

(104)

In the coordinates of \(\hat{x}'_i\), the Plank constant is

\[
\frac{(2\pi R')^2}{N} = 2\pi \beta^2,
\]

(105)
where
\[ R^2 = \theta R^2 = \frac{1}{\theta} \beta^2. \] (106)

We take the following limit which leads to a two-dimensional noncommutative plane,
\[ \beta = \text{fixed,} \quad R' \to \infty \quad (N \to \infty). \] (107)

The following replacements hold in this case,
\[ \frac{1}{N} Tr \to \int \frac{dw dz}{(2\pi R)^2} = \int \frac{dw' dz'}{(2\pi R')^2} = \int \frac{dx' dx}{(2\pi R')^2} \] (108)
and
\[ \text{Ad}(\hat{\rho}_i') \to \frac{1}{i} \frac{\partial}{\partial x_i'} \] (109)
where we have used the fact that \( x_2 \sim Ru \) and \( x_4 \sim -R(v - \pi), (u, v - \pi \ll 1) \).

We can regard \( \hat{a}_1 \) and \( \hat{a}_3 \) as \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \) respectively. Since the mass term drops in this limit, the bosonic part of the action (1) becomes (only \( U(1) \) case is treated for simplicity in the present discussions.)

\[
S_B = -\frac{\beta^4}{2g^2} Tr([\hat{T}_2 + R\hat{a}_2, \hat{T}_4 + R\hat{a}_4]^2) + \frac{\beta^4}{2g^2} Tr(R\hat{\phi}_1[\hat{T}_i + R\hat{a}_i, [\hat{T}_i + R\hat{a}_i, \hat{R}\hat{\phi}_1]]) \\
+ \frac{\beta^4}{2g^2} Tr(R\hat{\phi}_1[\hat{T}_i + R\hat{a}_i, [\hat{T}_i + R\hat{a}_i, \hat{R}\hat{\phi}_2]]) - \frac{\beta^4}{2g^2} Tr([\hat{R}\hat{\phi}_1, \hat{R}\hat{\phi}_2][\hat{R}\hat{\phi}_1, \hat{R}\hat{\phi}_2]) \\
= \frac{\beta^6}{2g^2}(\frac{1}{\sqrt{\theta}})^4\{-Tr([\hat{p}_1' + \hat{a}_1', \hat{p}_2' + \hat{a}_2']^2) + Tr(\hat{\phi}_1[\hat{p}_1' + \hat{a}_1', [\hat{p}_1' + \hat{a}_1', \hat{\phi}_1']]) \\
+ Tr(\hat{\phi}_2[\hat{p}_1' + \hat{a}_1', [\hat{p}_1' + \hat{a}_1', \hat{\phi}_2']]) - Tr([\hat{\phi}_1, \hat{\phi}_2][\hat{\phi}_1, \hat{\phi}_2])\} \] (110)

where we have defined \( \hat{a}_2 = \sqrt{\theta}\hat{a}_4', \hat{a}_4 = \sqrt{\theta}\hat{a}_2' \), \( \hat{\phi}_1 = \sqrt{\theta}\hat{\phi}_1' \) and \( \hat{\phi}_2 = \sqrt{\theta}\hat{\phi}_2' \). Repeated index \( i \) takes 2 and 4. This action can be mapped to the following field theory action,
\[
S_B = -\frac{\beta^6 N^2}{16\pi^3 g^2} \left\{ \int d^2 x' F_{24}(x)^2 + \int d^2 x'(D_i \phi_1(x))(D_i \phi_1'(x)) \\
+ \int d^2 x'(D_i \phi_2(x))(D_i \phi_2'(x)) + \int d^2 x'[\phi_1(x, y),\phi_2(x, y)]_y \right\}, \] (111)
where \( D_i = \frac{1}{g'}[a_i', \cdot] \) and \( F_{24} = \frac{1}{g}[a_2', a_2] - \frac{1}{g}[a_4', a_4] + [a_2', a_2']. \) It is found that the Yang-Mills coupling is \( g_{YM}^2 = 4\pi^3g^2/N^2\beta^6. \) We thus obtained an action of a gauge theory on a noncommutative plane by taking a large \( N \) limit with fixed noncommutativity from a gauge theory on a noncommutative torus.

The gauge transformation (92) becomes
\[
a_i'(x) \to a_i'(x) - \partial_i' \lambda(x) + i[\lambda(x), a_i'(x)]_y, \\
\phi_1'(x) \to \phi_1'(x) + i[\lambda(x), \phi_1'(x)]_y, \\
\phi_2'(x) \to \phi_2'(x) + i[\lambda(x), \phi_2'(x)]_y. \] (112)
4 Gauge invariant operators on sphere and torus

It is shown in [12] that a gauge invariant operator in noncommutative gauge theories can have non-vanishing momentum which is proportional to the distance between the end-points of the path. (See also [14, 28, 29, 30].) This section is devoted to the analysis of such gauge invariant operators on a sphere and a torus. We are treating two manifolds which have different topology, genus zero and genus one. The difference of the topology is discussed.

We first consider a gauge invariant operator on a noncommutative sphere. From (8) and (9), translation on sphere, that is rotation, is realized by the following unitary transformation

\[ e^{iL \cdot \Delta \omega} \hat{a}_i(\hat{x}_i) e^{-iL \cdot \Delta \omega} \]

for infinitesimal small value \( \Delta \omega \). This rotation is obtained from the gauge transformation (36) if we set \( \lambda = L_i \Delta \omega_i \):

\[ a_i(x) \rightarrow a_i(x) + i \Delta \omega_j L_j a_i(x) + \frac{1}{\rho} \epsilon_{ijk} \Delta \omega_j x_k. \]  

(114)

Now let us study a gauge invariant operator on the sphere. For simplicity we consider a rotation in \( \hat{x}_3 = d = \text{constant} \) plane. The generator is \( \hat{L}_3 \). A gauge invariant operator is made by covariantizing \( \exp (i \hat{L}_3 \omega_3) \) as

\[ W = \frac{1}{N} Tr \exp (\frac{i}{\alpha} A_3 \omega_3). \]  

(115)

We now rewrite this as

\[ W = \frac{1}{N} Tr \prod_{m=1}^{n} e^{i \hat{m} \hat{a}_3 (\hat{x} + m \hat{\omega} \times \hat{x})} e^{i \hat{L}_3 \omega_3} \]

\[ \rightarrow \int \frac{d\Omega}{4\pi} \exp (i \int_{0}^{\omega} d\tilde{\omega} \rho a_3 (\tilde{x} + \tilde{\omega} \times \hat{x})) * e^{ik_3 x_3} \]  

(116)

where \( k_3 = \omega_3 / \alpha \). Taking account of the fact that the value of \( x_3 \) takes the integer multiple of \( \alpha \) (or the half integer multiple of \( \alpha \)), total length of the Wilson line is expressed in terms of \( k'_3 = \omega'_3 / \alpha = (\omega_3 - 2\pi) / \alpha \):

\[ l = 2\pi \sqrt{\rho^2 - d^2 n_{\text{win}}^2} + \sqrt{\rho^2 - d^2} \alpha k'_3. \]  

(117)

\( n_{\text{win}} \) is an integer and represent the winding number. This shows that this operator has the momentum which is proportional to the distance between two end-points up to winding...
modes and the direction of the momentum is orthogonal to the direction of the path. These are characteristic features of noncommutative gauge theories. If we take the commutative limit, the contour becomes closed and \( l \) vanishes when \( d \) approaches \( \rho \). This shows that the Wilson loops on a commutative sphere is contractable. On the other hand, Wilson loops on a torus show a different behavior as considered in the next part.

We next have a discussion of the noncommutative torus. Translation on the torus is generated by the unitary operators \( U \) and \( V \):

\[
U \hat{a}(\hat{x}) U^\dagger = \hat{a}(\hat{x}_1, \hat{x}_2, \cos \theta \hat{x}_3 - \sin \theta \hat{x}_4, \sin \theta \hat{x}_1 + \cos \theta \hat{x}_4),
\]

\[
V \hat{a}(\hat{x}) V^\dagger = \hat{a}(\cos \theta \hat{x}_1 + \sin \theta \hat{x}_2, - \sin \theta \hat{x}_1 + \cos \theta \hat{x}_2, \hat{x}_3, \hat{x}_4,).
\]

(118)

This is also expressed in terms of \( \hat{u} \) and \( \hat{v} \),

\[
U \hat{a}(\hat{u}, \hat{v}) U^\dagger = \hat{a}(\hat{u}, \hat{v} + \theta),
\]

\[
V \hat{a}(\hat{u}, \hat{v}) V^\dagger = \hat{a}(\hat{u} - \theta, \hat{v}),
\]

(119)

where \( \theta = 2\pi/N \). These show that \( U \) and \( V \) are translation operators in the \( v \) and \( u \) direction on the torus by angle \( \theta \) respectively. \( U^N = 1 \) and \( V^N = 1 \) are operators which perform full translation around two cycles of the torus.

For simplicity, we treat a Wilson line operator whose path is extended only with \( v \) direction. The generalization to an arbitrary path is straightforward. Translation along \( v \) direction is generated by the unitary operator \( U \). A gauge invariant Wilson line operator is obtained by covariantizing the translation operator \( U \) as

\[
W = \frac{1}{N} Tr(U + R \theta \hat{a}_+)^M
\]

(120)

where \( MR\theta \) is length of the path (\( M \) is a integer). We rewrite it as follows and show that this operator can have momentum which is proportional to the distance between the two ends of the contour.

\[
W = \lim_{n \to \infty} \frac{1}{N} Tr(U + R \theta \hat{a}_+)^n
\]

\[
= \lim_{n \to \infty} \frac{1}{N} Tr(U + R \theta \hat{a}_+)^M(U + R \theta \hat{a}_+)^M \cdots (U + R \theta \hat{a}_+)^M
\]

\[
= \lim_{n \to \infty} \frac{1}{N} Tr \left( 1 + \frac{M}{n} U^\dagger R \theta \hat{a}_+ \left( \hat{v} + \frac{M}{n} \theta \right) \right) \left( 1 + \frac{M}{n} U^\dagger R \theta \hat{a}_+ \left( \hat{v} + \frac{M}{n} 2\theta \right) \right) \cdots \left( 1 + \frac{M}{n} U^\dagger R \theta \hat{a}_+ \left( \hat{v} + \frac{M}{n} n\theta \right) \right) U^M
\]

\[
= \lim_{n \to \infty} \frac{1}{N} Tr \prod_{m=1}^{n} e^{\frac{M}{n} U^\dagger R \theta \hat{a}_+ (\hat{v} + \frac{M}{n} m\theta)} U^M
\]

23
\[ \int \frac{dwdz}{(2\pi R)^2} \exp \left( \int_0^{M\theta} Rd\vec{v}e^{-iu}a_+(v + \tilde{v}) \right) \star e^{ik_ww} \] (121)

where \( k_w = M/R \) and the following manipulation are done in the above calculations,

\[
\begin{align*}
(U + R\theta\hat{a}_+)^{\frac{M}{n}} &= U^{\frac{M}{n}}(1 + U^\dagger R\theta\hat{a}_+)^{\frac{M}{n}} \\
&= U^{\frac{M}{n}}\left(1 + \frac{M}{n} U^\dagger R\theta\hat{a}_+\right) \\
&= \left(1 + \frac{M}{n} U^\dagger R\theta\hat{a}_+ \left(\hat{v} + \frac{M}{n}\theta\right)\right) U^{\frac{M}{n}}. \tag{122}
\end{align*}
\]

We have assumed that \( R\theta = \beta \) is small. The momentum which is carried by the Wilson line operator is not \( k_w \) but also \( k'_w \) which is defined by the following equation,

\[
l = MR\theta = k_w R\beta \\
= 2\pi Rn_{win} + k'_w R\beta, \tag{123}
\]

because the space of \( w \) looks like lattice with lattice spacing \( R\theta(= \beta) \). \( n_{win} \) represents the winding number. We find that the momentum is proportional to the distance between the two end-points of the contour up to winding modes and the direction of the path and the momentum is orthogonal. Although there seems to appear a extra factor \( e^{-iu} \) in front of \( a_+ \) in (121), it cancels with a factor which comes from \( a_+ \) in the commutative limit:

\[
\int \frac{dwdz}{(2\pi R)^2} \exp \left( \int_0^{l/R} d\vec{v} (ib_v(v + \tilde{v}) + \phi_1(v + \tilde{v})) \right) e^{ik_ww}. \tag{124}
\]

In the commutative limit the path becomes closed, that is \( l = 2\pi Rn_{win} \). This operator is the usual Polyakov loop operator. The path is not contractable and is classified by the winding number \( n_{win} \).

5 Summary and discussions

In this paper, we have investigated two noncommutative gauge theories on a fuzzy sphere and a fuzzy torus in terms of a four-dimensional bosonic matrix model. By adding a mass term to the original matrix model, this matrix model can describe curved spacetime (fuzzy sphere and fuzzy torus). By expanding matrices into backgrounds and fields propagating on them, we obtained noncommutative gauge theories on the backgrounds. A characteristic feature of noncommutative gauge theories or the matrix model is that spacetime and fields are treated on the same footing and due to this properties, classical backgrounds are deformed by fields on them.
In the analysis of these noncommutative gauge theories, we discussed two large $N$ limits. One corresponds to a commutative limits and another corresponds to a large radius limits. From the matrix model point of view, these two gauge theories are equivalent. However, some differences appeared by taking two large $N$ limits. We first studied the commutative limit. By taking this limit, we can obtain commutative gauge theories. Comparing these two gauge theories, the difference of the symmetry appeared.

The advantage of the compact manifolds is that one can construct the solutions in terms of finite size matrices while a solution which represents a noncommutative plane cannot be constructed by finite size matrices. (From the viewpoint of the field theories, $N$ plays the role of the cut off parameter.) We have shown that gauge theories on a noncommutative plane are reproduced from gauge theories on a noncommutative sphere and a noncommutative torus in a large $N$ limit.

We also discussed a gauge invariant operator on a sphere and a torus. It is well known that in noncommutative gauge theories, a Wilson line operator can be gauge invariant by carrying momentum which is proportional to the distance between the two ends of the contour. In this paper we checked whether this fact holds on sphere and torus cases. Because these manifolds are compact, the Wilson lines on them have winding modes, which are topologically different in these two cases. One is contractable and another is noncontractable. The discussion of the topological feature is meaningful only in the commutative limit. Since the propagating fields deforms the classical background in noncommutative field theories, the concept of the topology in noncommutative field theories is different from commutative field theories. From the viewpoint of the matrix model, noncommutative field theories on these manifolds are equivalent. By taking the commutative limit, the difference of the symmetry or the difference of the topology appeared.

One of the future problems is to consider noncommutative gauge theories on the other noncommutative curved manifolds. The extension to higher dimensional manifolds is a interesting problem. Especially four dimensional case is one of them. Since construction of four-dimensional sphere in matrix models is already known in [31], it may be possible to do the same discussions as this paper. Other way of the extension to a curved manifold is to map onto a complex plane. As analyzed in our previous paper [15], we have obtained the normal ordered type basis on the fuzzy sphere by mapping onto a complex plane. After mapping to field theory, a product of functions are written by the Berezin product[32]. The merits of using the Berezin product is that a star product of a noncommutative manifold is given if the Kähler potential of the manifold is given. Since the Kähler potential is known for more general manifold, it may be interesting problem to consider more general curved manifolds from the matrix model point of view.

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A Killing vectors

In this appendix, we summarize Killing vectors on $S^2$ and $T^2$.

A.1 $S^2$

Killing vector on $S^2$ is given by

$$L_i = -iK_i^a \partial_a$$  \hspace{1cm} (A.1)

where

$$K_1^\theta = - \sin \phi \quad K_1^\phi = - \cot \theta \cos \phi$$
$$K_2^\theta = \cos \phi \quad K_2^\phi = - \cot \theta \sin \phi$$
$$K_3^\phi = 0 \quad K_3^\theta = 1.$$  \hspace{1cm} (A.2)

The metric tensor is written in terms of these vectors as

$$g^{ab} = K_i^a K_i^b.$$  \hspace{1cm} (A.3)

$K_i^a$ further satisfy the following relation,

$$\epsilon_{ijk} K_i^\theta K_j^\phi x_k^\rho = \frac{1}{\sin \theta} = \frac{1}{\sqrt{g}}$$  \hspace{1cm} (A.4)

where $g = \det g_{ab}$.

A.2 $T^2$

Killing vectors on $T^2$ is given by

$$T_\mu = -iK_\mu^a \partial_a$$  \hspace{1cm} (A.5)

where

$$K_1^u = 0 \quad K_1^v = - \sin u$$
$$K_2^u = 0 \quad K_2^v = \cos u$$
$$K_3^u = \sin v \quad K_3^v = 0$$
$$K_4^u = - \cos v \quad K_4^v = 0.$$  \hspace{1cm} (A.6)

The metric tensor is written in terms of these vectors as

$$g^{ab} = K_i^a K_i^b.$$  \hspace{1cm} (A.7)
References


