Matching the observed cosmological constant with vacuum energy density in AdS

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Abstract

The question of why the observational cosmological constant is so small in comparison with the vacuum energy density in particle physics has become a celebrated puzzle for a long time. We calculate the vacuum energy density by taking into account of different massive scalar fields in AdS spacetime. It is found that the mass spectrum of a scalar field in AdS spacetime is discrete because of a natural boundary condition. After supposed a reasonable magnitude ($6.93 \times 10^{26}$ m) of the radius of AdS, we match well the observed cosmological constant with the energy density of vacuum.

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1 Introduction

The cosmological constant has evoked much controversy both in astronomy and particle physics communities[1, 2]. Recent observations of high-redshift supernovae seem to suggest that the global geometry of the universe may be affected by a positive cosmological constant[3]. And all kinds of cosmological observations, such as background radiation, redshifts of the supernovae and quasars[4, 5, 6], give a very tiny vacuum energy density as $10^{-47}\text{GeV}^4$[7].

In particle physics, the vacuum is used to mean the ground state of quantum fields. A relativistic field may be thought of as a collection of harmonic oscillators of all possible frequencies, and each possible mode devotes $\frac{1}{2}\hbar\omega$ energy to the vacuum. In this way, particle physicists[1] get a huge vacuum energy density as $2 \times 10^{71}\text{GeV}^4$, which is over 120 orders of magnitude in excess of the value allowed by cosmological observations. It is a more challenging problem to explain why the cosmological constant is so small but non-zero, than to build theoretical models where it exactly vanishes[8].

About twenty years ago, a number of authors discovered that Anti-de Sitter (AdS) spacetime generically arose as ground states in supergravity theory, which at the time was considered to be among the most promising candidates for quantum gravity[9, 10, 11]. The interest on AdS spacetime was revived by a conjectured duality between string theory in the bulk of AdS and conformally invariant field theory (CFT) living on the boundary of AdS[12]. The AdS/CFT correspondence gives an explicit relation between Yang-Mill theory and string theory[13, 14]. More recently, there has been a renewed interest in AdS spacetime since progresses in theories of extra dimensions present us with the enticing possibility to explain some long-standing particle physics problem by geometrical means[15, 16]. Several groups even have begun to work on possible experimental signatures of the extra dimensions. The cosmology of the Randall-Sundrum model can be very different from ordinary inflationary cosmology[17].

In this paper, we calculate the energy density of vacuum by taking into account of
different massive scalar fields in AdS spacetime. It is found that the mass spectrum of a scalar field in AdS spacetime is discrete because of a natural boundary condition\cite{18}. After supposed a reasonable magnitude \((6.93 \times 10^{26} \text{ m})\) of the radius of AdS spacetime, we can match well the observed cosmological constant with the energy density of vacuum.

2 Equations of motion

AdS spacetime can be described as a submanifold of a pseudo-Euclidean five-dimensional embedding space with Cartesian coordinates \(\xi^a\) and metric \(\eta_{ab} = \text{diag}(1, -1, -1, -1, 1)\),

\[
(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 + (\xi^5)^2 = -\frac{1}{\lambda},
\]

\[
ds^2 = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 + (d\xi^5)^2, \tag{1}
\]

where \(\lambda\) is a constant \((\lambda < 0)\). It is obvious that the symmetry group of AdS is the conformal group \(SO(3, 2)\). It is convenient to introduce the Beltrami coordinates \(\{x^i\} \ (i = 0, 1, 2, 3)\) as

\[
\sqrt{-\lambda}x^i = (\xi^5)^{-1}\xi^i. \tag{2}
\]

In terms of the Beltrami coordinates, AdS is of the form

\[
\sigma(x) = 1 - \lambda \eta_{ij}x^ix^j > 0, \quad \eta_{ij} = \text{diag}(1, -1, -1, -1). \tag{3}
\]

The Beltrami metric can be deduced directly

\[
ds^2 = (\eta_{ij}\sigma^{-1} + \lambda \eta_{ik}\eta_{js}x^r x^s \sigma^{-2}) dx^i dx^j. \tag{4}
\]

The Beltrami metric is invariant under the coordinate transformations
\[ x^i \rightarrow \bar{x}^i = \sigma^{1/2}(a)(1 - \lambda \eta_{rs}a^r x^s)^{-1}(x^j - a^j)D^i_j, \]
\[ D^i_j = L^i_j + \lambda \eta_{kl}a^k a^l \left[ \sigma(a) + \sigma^{1/2}(a) \right]^{-1} L^k_j, \quad (5) \]
\[ (L^i_j)_{i,j=0,1,2,3} \in SO(3, 1), \quad a^i \text{ are constants.} \]

In the coordinate \((\xi^0, x^a)\), the \(SO(3, 2)\) invariant metric can be written as
\[
ds^2 = \frac{1}{1 + \lambda \xi^0 \xi^0} d\xi^0 d\xi^0 - \frac{(1 + \lambda \xi^0 \xi^0) d\mathbf{x}(I + \lambda \mathbf{x}' \mathbf{x})^{-1} d\mathbf{x}'}{1 + \lambda \mathbf{x} \mathbf{x}'}, \quad (6)\]
where the vector \(\mathbf{x}\) denotes \((x^1, x^2, x^3)\) and \(\mathbf{x}'\) the transpose of the vector \(\mathbf{x}\).

In the spherical coordinate \((x^1, x^2, x^3) \rightarrow (\rho, \theta, \phi)\), the \(SO(3, 2)\) invariant metric (6) is of the form
\[
ds^2 = \frac{1}{1 + \lambda \xi^0 \xi^0} d\xi^0 d\xi^0 - (1 + \lambda \xi^0 \xi^0) \left[ (1 + \lambda \rho^2)^{-2} d\rho^2 + (1 + \lambda \rho^2)^{-1} \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (7)\]

There are singularities in the transformation from \((x^0, \mathbf{x})\) to \((\xi^0, \mathbf{x})\). So that, structures of AdS in the coordinate \((\xi^0, \mathbf{x})\) should be analyzed carefully. The plot of the Penrose diagram of AdS shows that there is a horizon\[19\] in the coordinate \((\xi^0, \mathbf{x})\). We can limit us in the region of \(|\xi^0| < \frac{1}{\sqrt{-\lambda}}\). In this region of AdS, we can introduce a time like variable \(\tau\) as
\[ \sqrt{-\lambda} \xi^0 \equiv \sin(\sqrt{-\lambda} \tau). \quad (8) \]

Then we have a Robertson-Walker-like metric
\[
ds^2 = d\tau^2 - R^2(\tau)[(1 + \lambda \rho^2)^{-2} d\rho^2 + (1 + \lambda \rho^2)^{-1} \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (9)\]
where we have used the notation \(R(\tau) = \cos(\sqrt{-\lambda} \tau)\).

The equation of motion for a massive scalar field in AdS spacetime (with \(\hbar = c = 1\)) is of the form

4
\[
\left[ \frac{1}{R^3} \frac{\partial}{\partial \tau} \left( R^3 \frac{\partial}{\partial \tau} \right) - \frac{(1 + \lambda \rho^2)^2}{R^2 \rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) \right.
\]
\[- \frac{1 + \lambda \rho^2}{R^2 \rho^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) + m^2 \right] \Phi(\tau, \rho, \theta, \phi) = 0 .
\]

(10)

3 Discrete mass spectrum

We can solve the equation of motion, which was obtained in the last section, for a massive scalar field[18] by writing

\[
\Phi(\tau, \rho, \theta, \phi) = T(\tau)U(\rho)Y_{lm}(\theta, \phi) .
\]

(11)

The reduced equations of motion in terms of \( T(\tau) \), \( U(\rho) \) and \( Y_{lm}(\theta, \phi) \) are of the form

\[
\begin{align*}
\frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial U}{\partial \rho} + \frac{k^2}{(1 + \lambda \rho^2)^2} U - \frac{l(l + 1)}{\rho^2 (1 + \lambda \rho^2)} U &= 0 , \\
R^2 \frac{d^2 T}{d \tau^2} + 3R \frac{d R}{d \tau} \frac{dT}{d \tau} + (m^2 R^2 + k^2) T &= 0 , \\
\frac{\partial^2 Y_{lm}}{\partial \theta^2} + \cot \theta \frac{\partial Y_{lm}}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_{lm}}{\partial \phi^2} + l(l + 1) Y_{lm} &= 0 .
\end{align*}
\]

(12)

It is obvious that, at \( \theta = 0 \); \( \phi = \pi \), the radial function is singular. One can set \( U(\varphi) \) as following

\[
\frac{d^2 U(\varphi)}{d \varphi^2} + \frac{2}{\varphi} \frac{d U(\varphi)}{d \varphi} + \left[ \frac{k^2 a^2}{(1 - \varphi^2)^2} - \frac{l(l + 1)}{\varphi^2 (1 - \varphi^2)} \right] U(\varphi) = 0 .
\]

(13)

It is obvious that, at \( \varphi = 0, \pm 1 \), the radial function is singular. One can set \( U(\varphi) \) as following
\[ U(\varrho) = \varrho^l (1 - \varrho^2)^{\mu/2} F(\varrho), \quad (14) \]

where \( \mu \) is a solution of the index equation \( \mu(\mu - 2) + k^2a^2 = 0 \). \( F(\varrho) \) is a solution of the hypergeometric equation

\[
(1 - \varrho^2) \frac{d^2 F}{d\varrho^2} + \left[ \frac{2(l + 1)}{\varrho} - 2(l + \mu + 1) \varrho \right] \frac{dF}{d\varrho} + \left[ \frac{1}{4} - \left( \mu + l + \frac{1}{2} \right)^2 \right] F = 0. \quad (15)
\]

Therefore, we get the radial function of the form

\[
U(\rho) = C \left( \frac{\rho}{a} \right)^l \left( 1 - \frac{\rho^2}{a^2} \right)^{\frac{1}{2} - \frac{1}{2} \sqrt{1 - k^2a^2}} \\
\times \ _2F_1 \left( \frac{1}{2} l + \sqrt{1 - k^2a^2} + 2, \frac{1}{2} l + \sqrt{1 - k^2a^2} + 1, l + \frac{3}{2}, \frac{\rho^2}{a^2} \right), \quad (16)
\]

where \( C \) is the normalization constant.

In terms of the variable \( \zeta (\equiv \sin \frac{\tau}{a}) \), the time-like evolution function \( T \) satisfies the equation

\[
(1 - \zeta^2) \frac{d^2T}{d\zeta^2} - 4\zeta \frac{dT}{d\zeta} + \left( a^2m^2 + \frac{a^2k^2}{1 - \zeta^2} \right) T = 0. \quad (17)
\]

By introducing \( T(\zeta) = (1 - \zeta^2)^{-\frac{1}{2}} P(\zeta) \), we transform the time-like evolution equation as the standard associated Legendre equation

\[
(1 - \zeta^2) \frac{d^2P}{d\zeta^2} - 2\zeta \frac{dP}{d\zeta} + \left( a^2m^2 + 2 - \frac{1 - a^2k^2}{1 - \zeta^2} \right) P = 0. \quad (18)
\]

Therefore, the solutions of the time-like evolution equation can be presented as

\[
T_1(\tau) \propto \frac{1}{\cos \frac{\tau}{a}} P^{\sqrt{1 - k^2a^2}} \left( \sin \frac{\tau}{a} \right),
\]

\[
T_2(\tau) \propto \frac{1}{\cos \frac{\tau}{a}} Q^{\sqrt{1 - k^2a^2}} \left( \sin \frac{\tau}{a} \right), \quad (19)
\]
where $P_I^N(\xi^0)$ and $Q_I^N(\xi^0)$ are associated Legendre functions. The natural boundary condition on $|\xi^0| = \frac{1}{\sqrt{\lambda}}$ requires $I, N$ to be integers. This gives the discrete mass spectrum of scalar fields in AdS spacetime

$$\begin{align*}
a^2m^2 + 2 &= I(I + 1), \\
-k^2a^2 + 1 &= N^2, \quad |N| \leq I.
\end{align*}$$

\section{Cosmological constant}

In quantum theory of fields, a relativistic field may be thought of as a collection of harmonic oscillators of all possible frequencies. A simple example is provided by a scalar field of mass $m$. For this system, the vacuum energy is simple a sum of contributions

$$E_0 = \sum_i \frac{1}{2} \hbar \omega_i,$$

where the sum is over all possible modes of the field. In Minkowskian space, summing the expression (21) up to a wave number cutoff $\Lambda \gg m$ yields a vacuum energy density,

$$\langle \rho \rangle = \int_0^\Lambda \frac{4\pi k^2 dk}{(2\pi)^3} \hbar \frac{k^2 + m^2}{2} \simeq \hbar \frac{\Lambda^4}{16\pi^2}.$$  \hspace{1cm} (22)

If we believe general relativity up to the Planck scale\cite{1}, then we might take $\Lambda \simeq (8\pi G)^{-\frac{1}{2}}$, which would give

$$\langle \rho \rangle \simeq 2^{-10} \pi^{-4} G^{-2} = 2 \times 10^{71}\text{GeV}^4.$$  \hspace{1cm} (23)

It shows that the energy density of vacuum got by the quantum field theory in Minkowskian spacetime is over 120 orders of magnitude in excess of the value of astronomical observations\cite{4, 5}.

In AdS spacetime, as shown in the previous section, the natural boundary conditions at the points $\xi^0 = \pm \frac{1}{\sqrt{\lambda}}$ assure that the wavevector $k$ be discrete. We can compute
now the expression (21) exactly. In fact, any scalar field of arbitrary mass $m$ give contributions to the energy density of vacuum. Because the mass spectrum in the quantum field theory on Minkowskian spacetime is continuum, it is difficult to get a sum over different mass modes. Now a discrete mass spectrum has been obtained for scalar fields in AdS spacetime, thus we can sum the contributions of all scalar fields with different mass to the energy density of vacuum. We, therefore, get the energy density of AdS vacuum (here we resume the values of $\hbar$ and $c$)

$$
\langle \rho \rangle = 2\pi \sum_{m} \sum_{k} \frac{k^2}{(2\pi)^3} \delta k \sqrt{m^2 c^4 + \hbar^2 k^2 c^2},
$$

(24)

where $\delta k = |k(N = 1) - k(N = 0)| = \frac{1}{a}$ is the wavevector difference of two eigen states. Equation (20) is used to get the energy density of vacuum of the following form

$$
\langle \rho \rangle = \frac{\hbar c}{(2\pi)^3} I_{\text{max}} \sum_{I=1}^{I_{\text{max}}} \sum_{N=0}^{1} \frac{1 - N^2}{a^4} \sqrt{I(I+1) - N^2 - 1},
$$

(25)

where $I_{\text{max}}$ is the cutoff of mass spectrum. We would estimate $I_{\text{max}}$ as the Planck scale, $E_{\text{planck}} \approx 10^{19}\text{GeV}$, which is widely accepted as a point where conventional field theory breaks down due to quantum gravitational effects. The maximal energy $E_{\text{max}}$ will be the Planck energy corresponding to the cutoff $I_{\text{max}}$,

$$
E_{\text{max}} = E_{\text{planck}} = \frac{\hbar c}{a} \sqrt{I_{\text{max}}(I_{\text{max}} + 1) - N^2 - 1}, \quad N = 0.
$$

(26)

In terms of the Planck energy, we obtain a relation of the energy density of vacuum with the radius $a$ of AdS spacetime

$$
\langle \rho \rangle \approx \frac{1}{4\pi} \frac{(E_{\text{planck}})^2}{a^2 \hbar c}.
$$

(27)

A crude experimental upper bound on $\langle \rho \rangle$ is provided by measurements of cosmological redshifts as a function of distance, the program begun by Hubble in the 1920s. The present value is estimated by modern astronomical observations[3] as
\begin{equation}
\langle \rho \rangle \leq 10^{-47} \text{ GeV}^4 .
\end{equation}

If the radius of AdS spacetime has a reasonable value of $6.93 \times 10^{26}$ m, we can get by the Equation (27) almost the same cosmological constant with the observed one. What we obtain here is in good agreement with the hints of the redshift observations of supernovae and quasars. References[20, 21] gave a upper limitation of the curvature $\lambda$ from the estimation of the universe age and the Hubble constant: $|\lambda| \simeq 10^{-56}$ cm$^{-2}$. Our result is also in agreement with this upper limit for the radius of AdS spacetime got by different ways.

5 Concluding remarks

Since de Sitter found the de Sitter solution of Einstein’s equation in 1917, de Sitter and anti-de Sitter spacetime has been studied extensively by physicists and astronomers. Recent developments in AdS physics include the AdS/CFT correspondence and theory of extra dimensions. In this paper, we try to give a new understanding to the long standing cosmological constant problem. We assumed that the topological structure of the whole universe is AdS spacetime. This is consistent with the Randall-Sundrum model, where a slice of five-dimensional AdS was used[15]. We got a Robertson-Walker-like metric which keeps the space submanifold of AdS invariant under the transformation of $SO(3, 1)$. Equations of motion for massive scalar fields were solved exactly by variables separating method. Solutions indicate that the mass spectrum of scalar fields is discrete and possible normal modes of scalar fields are limited. These facts tell us clearly that we can sum the zero-point energies of all kinds of scalar fields with different mass. At last, a intrinsic relation between the energy density of vacuum and the curvature of AdS spacetime was obtained. By adopting a reasonable value $a = 6.327 \times 10^{30}$ km of the radius of AdS spacetime, we got the energy density of vacuum $\langle \rho \rangle \approx 10^{-47}$ GeV$^4$, which matches well with the observational cosmological constant. It should be pointed
out that our results is not dependent explicitly on dimensions of spacetime[22]. Only for definiteness, we presented the formalism for AdS$_4$.

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References


