LINEARIZED GRAVITY IN ISOTROPIC COORDINATES IN THE BRANE WORLD

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Abstract

We solve the Einstein equations in the Randall-Sundrum framework using an isotropic ansatz for the metric and obtain an exact expression to first order in the gravitational coupling. The solution is free from metric singularities away from the source and it satisfies the Israel matching condition on a straight brane. At distances far away from the source and on the physical brane this solution coincides with the 4-D Schwarzschild metric in isotropic coordinates. Furthermore we show that the extension of the standard Schwarzschild horizon in the bulk is tubular for any diagonal form of the metric while there is no restriction for the extension of the Schwarzschild horizon in isotropic coordinates.

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1. Introduction.

Recently there have been several attempts to achieve localization of gravity. In the brane world ordinary matter and its gauge interactions are confined within a 4-D hypersurface, referred to subsequently as the physical brane. The graviton, however, is allowed to propagate in extra space dimensions. One implementation of such a brane world scenario was proposed by Randall and Sundrum [1] within the framework of General Relativity. The physical brane in their model is the junction of two pieces of 5-D spacetime manifolds that are asymptotically anti-de Sitter. The Gauss-normal form of the metric in this space is

\[ ds^2 = e^{-2\kappa|y|} \bar{g}_{\mu\nu} dx^\mu dx^\nu + dy^2, \]

with the brane located at \( y = 0 \) and \( \kappa > 0 \) sets the energy scale of the extra space dimension. The metric \( \bar{g}_{\mu\nu} \) is determined by the 5-D Einstein equations

\[ R_{mn} - \frac{1}{2} R \bar{g}_{mn} - \Lambda \bar{g}_{mn} = -4\pi G_5 T_{\mu\nu} \delta^\mu_m \delta^\nu_n \delta(y) + 6\kappa \bar{g}_{\mu\nu} \delta^\mu_m \delta^\nu_n \delta(y) \]

where the cosmological constant \( \Lambda = -6\kappa^2 \), \( G_5 \) is the 5-D gravitational constant and \( T_{\mu\nu} \) the energy-momentum tensor on the brane. Here and throughout the paper, we adopt the convention that the Greek indices take values 0-3 and the Latin indices 0-4. The 4-D gravitational constant is given by \( G \sim \frac{G_5}{\kappa} \). In the absence of matter, \( T_{\mu\nu} = 0 \), \( \bar{g}_{\mu\nu} = \eta_{\mu\nu} \) is a solution of equation (2). Subsequently, the metric in equation (1) becomes that of \( AdS_5 \).

The solution to equation (2) can also be obtained from the solution to the sourceless equation, \( T_{\mu\nu} = 0 \), subject to the appropriate Israel matching condition [2] determined by \( T_{\mu\nu} \). The perturbative solution of (2) to the linear order in \( G \) for an arbitrary \( T_{\mu\nu} \) [3] and to second order \( G^2 \) for a static spherical mass distribution on the brane [4] reveals no tangible disagreement with the classical tests of 4-D General Relativity at large distances, i.e., \( \kappa r >> 1 \). Discussions on different aspects of linearized gravity in the Randall-Sundrum framework appeared in [5].

A drawback of the Gauss-normal coordinates is the failure to compromise between a straight brane (i.e. Israel matching condition at \( y = 0 \)) and a non-singular boundary condition for the metric \( \bar{g}_{\mu\nu} \) as \( y \to \infty \). This was shown explicitly in the weak field approximation ( to first order [1] or to quadratic order [4] in the gravitational coupling ) for which the deviation of \( \bar{g}_{\mu\nu} \) from its Minkowski value \( \eta_{\mu\nu} \), i.e. \( h_{\mu\nu} = \bar{g}_{\mu\nu} - \eta_{\mu\nu} \), grows exponentially with increasing \( y \) if the Israel matching condition is imposed at \( y = 0 \). On the other hand, if the condition \( \lim_{y\to\infty} h_{\mu\nu} = 0 \) is imposed, the Israel matching condition fails to hold at \( y = 0 \) and the brane is bent. Beyond the weak field approximation, the lack of compromise is reflected in the rigorous statement that the \( 5 - D \) extension of the Schwarzschild horizon of a physical black hole is always tubular, parallel to the \( y \)-axis.
It was suggested in [6] that the Israel matching condition at \( y = 0 \) and a non-singular boundary condition for the metric as \( y \to \infty \) can be achieved by introducing the graviscalar component of the metric (the coefficient in front of \( dy^2 \) in the metric \( \text{ansätze} \)). In the present paper we explore this possibility for a physical black hole. By using a particular diagonal metric \( \text{ansätze} \) we find an explicit solution to the linearized Einstein equations which satisfies the Israel matching condition (straight brane) while behaves well at the \( AdS_5 \) horizon \((y \to \infty)\). Since the metric on the brane approximates the \( 4-D \) Schwarzschild metric in isotropic coordinates for \( \kappa \rho >> 1 \) and \( \kappa GM >> 1 \)

\[
\begin{align*}
 ds^2 &\approx -(1 - \frac{GM}{2\rho})^2 dt^2 + \left(1 + \frac{GM}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right), \\
\end{align*}
\]

(3)

we shall refer to these \( 5-D \) coordinates as the isotropic coordinates (this is not to be confused with any Gauss-normal coordinates which approximate the \( 4-D \) isotropic Schwarzschild metric). Beyond the linear approximation we demonstrate that the \( 5-D \) extension of the horizon in the isotropic coordinates is closed and might be of a pancake shape. On the other hand if one requires the standard Schwarzschild metric to be implemented on the physical brane

\[
\begin{align*}
 ds^2 &\approx -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \\
\end{align*}
\]

(4)

a straight brane and a non-singular boundary condition at the \( AdS_5 \) horizon can never be compromised with any diagonal metric \( \text{ansätze} \), since the off brane extension of the horizon in this case is always tubular.

In the next section, we shall present the explicit solution to the linearized Einstein equations in the isotropic coordinates. The rigorous statements on the horizon with an arbitrary diagonal metric \( \text{ansätze} \) will be discussed in section 3. In the final section, we shall summarize our results and speculate on the form of the solution in isotropic coordinates beyond the linear approximation. For the benefit of the readers, we present in the appendix the explicit form of all non-zero components of the Ricci tensor with a diagonal metric \( \text{ansätze} \).

2. Linearized Solution in Isotropic Coordinates.

The 5-D Einstein sourceless equations can be rewritten as

\[
\begin{align*}
 R_{\mu\nu} - 4\kappa^2 g_{\mu\nu} &= 0, \\
 R_{y\mu} &= 0, \\
 R_{yy} - 4\kappa^2 &= 0. \\
\end{align*}
\]

(5)
The most general metric in $D = 4 + 1$ dimensions produced by a static, spherically symmetric matter distribution on the physical brane or equivalently, axially symmetric in the bulk can always be brought to Gauss normal form:

$$ds^2 = e^{-2\kappa|y|}(-e^a dt^2 + e^b dr^2 + e^c r^2 d\Omega^2) + dy^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the solid angle on $S^2$ and $a$, $b$ and $c$ are functions of $r$ and $y$.

In this paper we will consider an alternative form of the metric in $D = 4 + 1$, written in isotropic coordinates

$$ds^2 = e^{-2\kappa\eta}(-e^\alpha dt^2 + e^\beta (d\rho^2 + \rho^2 d\Omega^2)) + e^f d\eta^2,$$

where $\alpha$, $\beta$ and $f$ are functions of $\rho$ and $\eta$, and we will obtain an exact solution of the Einstein equations to first order in the gravitational coupling $G$. Since the linearized solution is known in the Gauss normal coordinates [4] our objective initially would be to determine the form of the coordinate transformations that transform the Gauss normal form of the metric, equation (6), to the isotropic one, equation (7).

Let’s perform a coordinate transformation (gauge transformation) generated by $u, v$, functions of $\rho$ and $\eta$ such that $r = \rho + u(\rho, \eta)$ and $y = \eta + v(\rho, \eta)$. We find that the Gauss-normal form of the metric transforms to linear order as follows

$$ds^2 = e^{-2\kappa\eta}[-(1 + a - 2\kappa v)dt^2 + (1 + b + 2u' - 2\kappa v)d\rho^2 + (1 + c - 2\kappa v + \frac{2u}{\rho})\rho^2 d\Omega^2]
+ (1 + 2\dot{v})d\eta^2 + 2(e^{-2\kappa\eta}\dot{u} + v')d\rho d\eta,$$

where $u'$ and $\dot{u}$ indicate differentiation with respect to $\rho$ and $\eta$ respectively. Since we would like to transform the metric from the Gauss-normal form to the isotropic form we seek appropriate $u$ and $v$ such that they satisfy

$$e^{-2\kappa\eta}\dot{u} + v' = 0, \quad b + 2u' = c + 2u. \quad (9)$$

Furthermore with the identification

$$\alpha = a - 2\kappa v, \quad \beta = b - 2\kappa v + 2u', \quad f = 2\dot{v} \quad (10)$$

we transform the metric into the isotropic form

$$ds^2 = e^{-2\kappa\eta}[-(1 + \alpha)dt^2 + (1 + \beta)(d\rho^2 + \rho^2 d\Omega^2)] + (1 + f)d\eta^2.$$
Substituting the metric (11) into equations (5), we obtain the following components of the linearized Einstein equation outside the source:

\[ R_{tt} + 4\kappa^2 e^{-2\kappa \eta + \alpha} = -\frac{1}{2} \alpha'' - \frac{1}{\rho} \alpha' - \frac{1}{2} e^{-2\kappa \eta} [\ddot{\alpha} - 5 \kappa \dot{\alpha} - 3 \kappa \dot{\beta} + \kappa \ddot{f} - 8 \kappa^2 f] = 0 \]  
(12)

\[ R_{\rho \rho} - 4\kappa^2 e^{-2\kappa \eta + \beta} = \frac{1}{2} \alpha'' + \frac{1}{\rho} f'' + \beta'' + \frac{1}{\rho} \beta' + \frac{1}{2} e^{-2\kappa \eta} [\ddot{\beta} - 7 \kappa \dot{\beta} - \kappa \dot{\alpha} + \kappa \ddot{f} - 8 \kappa^2 f] = 0 \]  
(13)

\[ R_{\theta \theta} - 4\kappa^2 \rho^2 e^{-2\kappa \eta + \beta} = \frac{1}{2} \rho^2 [\beta'' + \frac{3}{\rho} \beta' + \frac{1}{\rho} \alpha' + \frac{1}{2} f' + e^{-2\kappa \eta} [\ddot{\beta} - \kappa (\dot{\beta} - \dot{f}) - 7 \kappa \dot{\beta} - 8 \kappa^2 f] = 0 \]  
(14)

\[ R_{\eta \eta} - 4\kappa^2 e^{\beta} = \frac{1}{2} e^{2\kappa \eta} (f'' + \frac{2}{\rho} f') + \frac{1}{2} (\ddot{\alpha} + 3 \ddot{\beta}) - \kappa \dot{\alpha} - 3 \kappa \dot{\beta} + 2 \kappa \ddot{f} + 8 \kappa^2 f = 0 \]  
(15)

These equations apply to the positive side of the brane, \( \eta > 0 \), the corresponding equations to the negative side of the brane, \( \eta < 0 \), are obtained by switching the sign of \( \kappa \).

In reference [4] solutions to the linearized Einstein equations using the Gauss-normal form of the metric, equation (1), were obtained in two different coordinate systems, or equivalently, in two different gauges. The solution in the coordinate system based on the \( AdS \) horizon is free from metric singularities far away from the source but fails to satisfy the Israel matching condition at \( y = 0 \), while when it is transformed to a coordinate system based on the brane it satisfies the Israel matching condition at \( y = 0 \) but is not free from metric singularities anymore. In this coordinate system the physical brane appears bent [7]. In the latter case the solution is given by the expression

\[ a^P (r, \zeta) = -\frac{8GM\kappa}{3\pi} \zeta^2 \int_0^\infty dp j_0 (pr) \frac{K_2 (p\zeta)}{K_1 (p)} + \frac{2GM}{3r} \]  
(16)

\[ b^P (r, \zeta) = \frac{8GM\kappa}{3\pi} \zeta^2 \int_0^\infty dp \frac{j_1 (pr)}{pr} \frac{K_2 (p\zeta)}{K_1 (p)} + \frac{2GM}{3r} + \frac{2GM}{3r^3} \zeta^2 \]

\[ c^P (r, \zeta) = \frac{8GM\kappa}{3\pi} \zeta^2 \int_0^\infty dp \frac{1}{2} [j_0 (pr) - \frac{j_1 (pr)}{pr}] \frac{K_2 (p\zeta)}{K_1 (p)} - \frac{2GM}{3r} - \frac{GM}{3r^3} \zeta^2 \]

where we have introduced \( \hat{p} = \frac{p}{\kappa}, \zeta = \frac{1}{\kappa} e^{\kappa y}, K_\nu (\zeta) \) is the modified Bessel function of the second kind and \( j_0 (x) \) is the spherical Bessel function. The superscript \( P \) indicates that \( a^P, b^P \) and \( c^P \) satisfy the Neumann boundary condition (Israel matching condition away from the source) on the brane located at \( \zeta = \frac{1}{\kappa} \)

\[ \frac{\partial}{\partial \zeta} a^P |_{\zeta = \frac{1}{\kappa}} = \frac{\partial}{\partial \zeta} b^P |_{\zeta = \frac{1}{\kappa}} = \frac{\partial}{\partial \zeta} c^P |_{\zeta = \frac{1}{\kappa}} = 0. \]  
(17)

We would like to find an exact solution to the linearized Einstein equations, expressed in the isotropic coordinates, equations (12)-(15). We shall proceed by determining the form of
the parameters of coordinate transformations \( u \) and \( v \) that transform the metric from the Gauss-normal form to the isotropic one. The second of equations (9) can be solved and provides us with the following expression for \( u \)

\[
u(\rho, \zeta) = -\rho \int_{\rho}^{\infty} \frac{b^P - c^P}{2s} ds + \rho \chi(\zeta)
\]

where \( \chi \) is an arbitrary function of \( \zeta \). Using the expressions for \( b^P \) and \( c^P \) from equations (16) we find the following expression for \( u \) and consequently for \( v \) by substituting into the first of equations (9)

\[
u(\rho, \zeta) = \rho \chi(\zeta) + \frac{2GM}{3} + \frac{GM\zeta^2}{6\rho^2} + \frac{2GM\kappa}{3\pi} \zeta^2 \int_{0}^{\infty} dpj_1(\rho p) \frac{K_2(p\zeta)}{K_1(\hat{p})} \]

where \( \phi \) is another arbitrary function of \( \zeta \).

Consequently we substitute equation (19) into equation (10) and demand that the expressions for \( \alpha, \beta \) and \( f \) are free from metric singularities far away from the source

\[
\lim_{\zeta, \rho \to \infty} \alpha = \lim_{\zeta, \rho \to \infty} \beta = \lim_{\zeta, \rho \to \infty} f = 0.
\]

We find that \( \chi(\zeta) = \phi(\zeta) = 0 \) and consequently the solution takes the form

\[
\alpha(\rho, \zeta) = -\frac{8GM\kappa}{3\pi} \zeta^2 \int_{0}^{\infty} dpj_0(\rho p) \frac{K_2(p\zeta)}{K_1(\hat{p})} + \frac{4GM\kappa}{3\pi} \zeta \int_{0}^{\infty} dpj_0(\rho p) \frac{K_1(p\zeta)}{K_1(\hat{p})} \]

\[
\beta(\rho, \zeta) = \frac{4GM\kappa}{3\pi} \zeta^2 \int_{0}^{\infty} dpj_0(\rho p) \frac{K_2(p\zeta)}{K_1(\hat{p})} + \frac{4GM\kappa}{3\pi} \zeta \int_{0}^{\infty} dpj_0(\rho p) \frac{K_1(p\zeta)}{K_1(\hat{p})}. \]

\[
f(\rho, \zeta) = \frac{4GM\kappa}{3\pi} \zeta^2 \int_{0}^{\infty} dpj_0(\rho p) \frac{K_0(p\zeta)}{K_1(\hat{p})}
\]

In this coordinate system the brane remains straight since \( v(\rho, \frac{1}{\kappa}) = 0 \), in contrast with the Gauss-normal coordinates in which the brane was bent in the system in which the solution was free of singularities off brane.

We still need to check whether the solution \( \alpha(\rho, \zeta), \beta(\rho, \zeta) \) and \( f(\rho, \zeta) \) satisfies the equations of motion (12)-(15). Lets define

\[
\Phi(\rho, \zeta) = \frac{4GM\kappa}{3\pi} \zeta^2 \int_{0}^{\infty} dpj_0(\rho p) \frac{K_2(p\zeta)}{K_1(\hat{p})}
\]

\[
\Psi(\rho, \zeta) = \frac{4GM\kappa}{3\pi} \zeta \int_{0}^{\infty} dpj_0(\rho p) \frac{K_1(p\zeta)}{K_1(\hat{p})}
\]
such that
\[
\alpha(\rho, \zeta) = -2\Phi(\rho, \zeta) + \Psi(\rho, \zeta), \quad \beta(\rho, \zeta) = \Phi(\rho, \zeta) + \Psi(\rho, \zeta), \quad f(\rho, \zeta) = -\zeta \frac{\partial \Psi}{\partial \zeta}(\rho, \zeta). \tag{23}
\]

Equation (12) then becomes
\[
-2\Phi'' + \Psi'' + \frac{2}{\rho}(-2\Phi' + \Psi') - 2\frac{\partial^2 \Phi}{\partial \zeta^2} + \frac{5}{\zeta} \frac{\partial \Phi}{\partial \zeta} = 0 \tag{24}
\]
Subsequently we substitute the expressions for the solution into equation (24) and taking into account that \(x^2K_2''(x) + xK_2'(x) = (4 + x^2)K_2(x)\) together with \([x^2K_2(x)]' = -2xK_1(x)\), we verify that they satisfy equation (12). Similarly we demonstrate that the expressions for \(\alpha, \beta\) and \(f\) satisfy the remaining Einstein equations.

If either \(\rho\) or \(\zeta \equiv \frac{1}{\kappa}e^{\kappa y}\) becomes large, i.e, \(\kappa \rho >> 1\) or \(\kappa \zeta >> 1\), the integrals (22), are dominated by the region where \(\hat{p} \ll 1\). The modified Bessel function in the denominator, \(K_1(\hat{p}) \approx \frac{1}{\hat{p}}\), and the integrals can be carried out explicitly. We find that
\[
\Phi(\rho, \zeta) = \frac{2GM}{3} \frac{2\rho^2 + 3\zeta^2}{(\rho^2 + \zeta^2)^{\frac{3}{2}}}, \quad \Psi(\rho, \zeta) = \frac{2GM}{3} \frac{1}{\sqrt{\rho^2 + \zeta^2}} \tag{25}
\]
which leads to the following approximate expressions for \(\alpha, \beta\) and \(f\)
\[
\alpha(\rho, \zeta) = -\frac{2GM}{3} \frac{3\rho^2 + 5\zeta^2}{(\rho^2 + \zeta^2)^{\frac{3}{2}}}, \quad \beta(\rho, \zeta) = \frac{2GM}{3} \frac{3\rho^2 + 4\zeta^2}{(\rho^2 + \zeta^2)^{\frac{3}{2}}}, \quad f(\rho, \zeta) = \frac{4GM}{3} \frac{\zeta^2}{(\rho^2 + \zeta^2)^{\frac{3}{2}}}. \tag{26}
\]

It is straightforward to verify that the metric on the physical plane \((\zeta = \frac{1}{\kappa})\) becomes for \(\rho >> 1/\kappa\)
\[
ds^2 = -(1 - \frac{2GM}{\rho} + \cdots)dt^2 + (1 + \frac{2GM}{\rho} + \cdots)(d\rho^2 + \rho^2 d\Omega^2), \tag{27}
\]
thus reproducing the standard form of the Schwarzschild metric in isotropic coordinates.
The dots in equation (27) represent terms of order \(O(\frac{G^2M^2}{\rho^2})\) and higher. The weak field expansion of a general static spherical metric in its isotropic form which is not necessarily determined by the Einstein equations is [8]
\[
ds^2 = -(1 - 2\alpha_1 \frac{GM}{\rho} + 2\alpha_2 \frac{G^2M^2}{\rho^2} + \cdots)dt^2 + (1 + 2\alpha_3 \frac{GM}{\rho} + \cdots)(d\rho^2 + \rho^2 d\Omega^2). \tag{28}
\]

General Relativity predicts that \(\alpha_1 = \alpha_2 = \alpha_3 = 1\). Comparing equations (27) and (28) we note that the Randall-Sundrum scenario is consistent with all the experimental tests of linearized General Relativity.
Let’s now check whether our solution $\alpha, \beta$ and $f$ satisfies the Israel matching condition on the brane. The Israel matching condition in the case of the metric in Gauss normal form and far away from the source was simply the Neumann boundary condition for the components of the metric. We can derive the Israel matching condition for the components of the metric in isotropic coordinates by performing a coordinate transformation to the Neumann boundary condition, generated by $u$ and $v$, subjected to constraints (9). Furthermore with the identifications (10) we derive the Israel matching condition in the isotropic coordinates

$$\frac{\partial \alpha}{\partial \eta}|_{\eta \to 0} = \frac{\partial \beta}{\partial \eta}|_{\eta \to 0} = -\kappa f|_{\eta \to 0}$$

(29)

It is straightforward then to verify that our linearized solution in isotropic coordinates satisfies the Israel matching condition. For example

$$\frac{\partial \alpha}{\partial \eta}|_{\eta \to 0} = k\zeta \frac{\partial \alpha}{\partial \zeta}|_{\zeta \to \frac{1}{\kappa}} = -2k\zeta \frac{\partial \Phi}{\partial \zeta}|_{\zeta \to \frac{1}{\kappa}} + k\zeta \frac{\partial \Psi}{\partial \zeta}|_{\zeta \to \frac{1}{\kappa}} =$$

$$= -\frac{4GM\kappa}{3\pi} \int_{0}^{\infty} dppj_0(pp) \frac{K_0(\hat{p})}{K_1(\hat{p})} = -\kappa f(\rho, \frac{1}{\kappa}).$$

(30)

We have thus derived an exact solution to linear order in the gravitational coupling which is both free from singularities and satisfies the Israel matching condition.

3. Extensions of the Horizon in the Bulk.

Beyond the linear approximation, the Einstein equations are difficult to solve. The presence of Bessel functions in the linear approximation makes it implausible that a closed form exact solution exists. Nevertheless, some rigorous statements can still be made. An interesting issue is the 5-D extension of the 4-D Schwarzschild horizon for $\kappa r >> 1$. A discussion on black holes in the brane world can be found in [9] In a previous paper [10] we showed that within the Gauss-normal form of the metric, such an extension can only take a tubular shape. We shall examine this problem for a general diagonal form of the metric

$$ds^2 = -e^a dt^2 + e^b dr^2 + e^c r^2 d\Omega^2 + e^f dy^2.$$  

(31)

where we have absorbed the conformal factor $e^{-2\kappa y}$ into the definition of $a, b$ and $c$. The non-zero components of the Ricci tensor are presented in the Appendix. What we need here is

$$R_{yy} = \frac{1}{2}(\ddot{a} + \ddot{b} + 2\ddot{c}) + \frac{1}{4}(\dot{a}^2 + \dot{b}^2 + 2\dot{c}^2)$$

$$+ \frac{1}{2}e^{-b}[f'' + \frac{1}{2}(\frac{4}{r} + a' - b' + 2c' + f')f'] - \frac{1}{4}(\dot{a} + \dot{b} + 2\dot{c}) \dot{f}$$

(32)
Let $H(r, y) = 0$ describe the trajectory of the horizon on the $r - y$ parametric plane. Consider a point $P(r_0, y_0)$ on the trajectory at which the unit normal to the horizon $\vec{n} = (\cos \alpha, \sin \alpha)$ with

$$\cos \alpha = \frac{1}{\Delta} \left( \frac{\partial H}{\partial r} \right)_P, \quad \sin \alpha = \frac{1}{\Delta} \left( \frac{\partial H}{\partial y} \right)_P$$

and

$$\Delta = \sqrt{\left( \frac{\partial H}{\partial r} \right)_P^2 + \left( \frac{\partial H}{\partial y} \right)_P^2}.$$  \hspace{1cm} (34)

In the neighbourhood of $P$ we may introduce the normal and tangent coordinates $\xi$ and $\eta$

$$\xi = (r - r_0) \cos \alpha + (y - y_0) \sin \alpha$$
$$\eta = -(r - r_0) \sin \alpha + (y - y_0) \cos \alpha.$$  \hspace{1cm} (35)

For the standard form of the Schwarzschild horizon, we expect that

$$a \simeq \ln \xi, \quad b \simeq \ln \xi$$  \hspace{1cm} (36)

and $c$ nonsingular as $\xi, \eta \to 0$. Assuming that $f \simeq n \ln \xi$ with $n$ an even integer we have

$$\dot{a} \simeq \frac{1}{\xi} \sin \alpha, \quad \dot{b} \simeq -\frac{1}{\xi} \sin \alpha, \quad \dot{f} \simeq \frac{n}{\xi} \sin \alpha$$
$$a' \simeq \frac{1}{\xi} \cos \alpha, \quad b' \simeq -\frac{1}{\xi} \cos \alpha, \quad f' \simeq \frac{n}{\xi} \cos \alpha$$  \hspace{1cm} (37)

and

$$\ddot{a} \simeq -\frac{1}{\xi^2} \sin^2 \alpha, \quad \ddot{b} \simeq \frac{1}{\xi^2} \sin^2 \alpha, \quad \ddot{f} \simeq -\frac{n}{\xi^2} \sin^2 \alpha$$
$$a'' \simeq -\frac{1}{\xi^2} \cos^2 \alpha, \quad b'' \simeq \frac{1}{\xi^2} \cos^2 \alpha, \quad f'' \simeq -\frac{n}{\xi^2} \cos^2 \alpha$$  \hspace{1cm} (38)

The Einstein equation $R_{yy} - 4\kappa^2 e^f = 0$ demands the absence of the singularity on the left hand side. For $n < -1$, the leading singularity stems from the term in the bracket of equation (32), i.e.,

$$R_{yy} \sim n^2 \xi^{-1+n} \cos^2 \alpha$$  \hspace{1cm} (39)

which implies $\cos \alpha = 0$. This horizon is parallel to the brane and will not join the 4D Schwarzschild horizon on the brane. For $n > -1$, the leading singularity comes from the second term of equation (32), i.e.,

$$R_{yy} \sim \frac{1}{2\xi^2} \sin^2 \alpha$$  \hspace{1cm} (40)

and we have $\sin \alpha = 0$ which implies a tubular horizon. Therefore we conclude that the 5-D extension of the standard Schwarzschild horizon with any diagonal form of the metric is tubular.
The situation changes, however, for the 5-D extension of the isotropic form of the 4-D Schwarzschild horizon, for which \( a \simeq 2 \ln \xi \) and \( b = c \) nonsingular. With the assumption that \( f = n \ln \xi \), we find

\[
R_{yy} \sim C n^2 \xi^{n-2} \cos^2 \alpha + D n \xi^{-2} \sin^2 \alpha \tag{41}
\]

as \( \xi, \eta \to 0 \) and \( C, D \) being constants. It follows that \( \cos \alpha = 0 \) for \( n = 0 \) and \( \sin \alpha = 0 \) for \( n > 0 \). There is no restriction on the horizon shape for \( n = 0 \) and in this case \( f \) is nonsingular as well. The coexistence of a straight brane and the \( AdS_5 \) horizon within the coordinate patch specified by \( b = c \), as it is revealed by the linear gravity analysis, implies a non-trivial shape- including a pancake) for the 5-D extension of the 4-D horizon.

### 4. Concluding Remarks.

In this section we will recapitulate what we have done in this paper. We considered a static, axially symmetric metric in \( D = 4 + 1 \) dimensions in the Randall-Sundrum framework in isotropic coordinates and derived the Einstein equations. We subsequently found an exact solution of the equations to first order in the gravitational coupling. This solution is free from singularities as \( y \to \infty \) and satisfies the Israel matching condition at \( y = 0 \). It describes the gravitational field of a spherically symmetric mass distribution confined on the physical brane which is straight in the isotropic coordinates. At distances far away from the material source and on the physical brane this solution coincides with the four dimensional Schwarzschild metric in isotropic coordinates. Thus we confirmed that all tests of Linearized General Relativity are satisfied. We have also found that the vacuum Einstein equations together with the form of the metric place fairly stringent restrictions on the shape of the event horizon. More specifically we showed that the \( 5 - D \) extension of the standard Schwarzschild horizon with any diagonal form of the metric is tubular while there is no restriction for the \( 5 - D \) extension of the Schwarzschild horizon in isotropic coordinates.

Finally we would like to speculate on the structure of the solution in isotropic coordinates beyond the linearized approximation. In accordance with the discussion in the previous section, the metric near the horizon can be approximated by

\[
ds^2 = -\xi^2 dt^2 + d\xi^2 + \zeta^2 \tag{42}\]

where \( d\zeta^2 \) is a \( 3 - D \) Euclidean metric. We notice that the determinant of the metric vanishes on the horizon. The transformation to a local inertial frame, \( T = \xi \cosh t \) and \( X = \xi \sinh t \) does not cover the region with \( T^2 > X^2 \) and therefore the isotropic coordinate patch does not cover the entire spacetime.
To be more specific we consider the unrealistic case \( \kappa GM \ll 1 \). The \( 4-D \) brane gravity joins the \( 5-D \) gravity in the region where the linearized approximation is still valid. For \( \rho, \eta \) satisfying the inequality

\[
\left( \frac{GM}{\kappa} \right)^{\frac{1}{2}} \ll l \ll \left( \frac{GM}{\kappa^2} \right)^{\frac{1}{2}}
\]

(43)

with \( l = (\rho^2 + \eta^2)^{\frac{1}{2}} \), the metric (11), with \( \alpha, \beta \) and \( f \) given by (21), becomes approximately

\[
ds^2 = -(1 - \frac{8GM}{3\pi \kappa l^2}) dt^2 + (1 + \frac{4GM}{3\pi \kappa l^2})(d\rho^2 + \rho^2 d\Omega^2 + d\eta^2).
\]

(44)

The lower bound of (43) results from the linear approximation while the upper one from ignoring the conformal factor \( e^{-2\kappa \eta} \). As we go beyond the lower bound of (43), \( \alpha, \beta \) and \( f \) are expected to coincide with the corresponding expressions of the isotropic form of the \( 5-D \) Schwarzschild metric

\[
ds^2 = -\left( \frac{l^2 - l_0^2}{l^2 + l_0^2} \right)^2 dt^2 + (1 + \frac{l_0^2}{l^2})^2 (d\rho^2 + \rho^2 d\Omega^2 + d\eta^2).
\]

(45)

with \( l_0^2 = \frac{2GM}{3\pi \kappa} \). In this case we have a trivial case of a closed \( 5-D \) horizon. It is well known for \( \kappa = 0 \) that the isotropic coordinates for \( l < l_0 \) represent a copy of the spacetime with \( l > l_0 \) as it is obvious from the mapping \( l \rightarrow l' = \frac{l^2}{l_0^2} \) which amounts to

\[
\rho \rightarrow \rho' = \frac{l_0^2 \rho}{\rho^2 + \eta^2}, \quad \eta \rightarrow \eta' = \frac{l_0^2 \eta}{\rho^2 + \eta^2}.
\]

(46)

The approximation of ignoring \( \kappa \) continues to hold with decreasing \( l \) for \( l < l_0 \) until \( \alpha \sim 2\kappa \eta \) beyond which we shall find ourselves in the region where \( 4-D \) gravity dominates again. Therefore the physical black hole solution specified by the isotropic coordinates for \( \kappa GM \ll 1 \) describes only the spacetime outside the Schwarzschild horizon, as for the case \( \kappa = 0 \). This property might pertain as the parameter \( \kappa GM \) is continued to more realistic regions, i. e. \( \kappa GM \gg 1 \). In that case, the solution to the brane world Einstein equations expressed in isotropic coordinates never probes the spacetime interior to the Schwarzschild horizon of a physical black hole, in particular the coordinate patch never extends to the curvature singularity.

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5. Appendix.

In this appendix, we tabulate all non-zero components of the Ricci tensor for the general diagonal metric ansatz (31). Our sign conventions follows that of reference [8]. We find

\[
R_{tt} = \frac{1}{2} e^{a-b} \left[ -a'' - \frac{2}{r} a' + \frac{1}{2} a'(-a' + b' - 2c') - \frac{1}{2} a' f' \right] + \frac{1}{2} e^{a-f} \left[ -\ddot{a} - \frac{1}{2} \dot{a}(\dot{a} + \dot{b} + 2\dot{c} - \dot{f}) \right]
\]

\[
R_{rr} = \frac{1}{2} a'' + c'' + \frac{1}{2} f'' - \frac{1}{2} b' + \frac{2}{r} c' + \frac{1}{4} a'(a' - b') - \frac{1}{2} c'(b' - c') + \frac{1}{4} f'^2 - \frac{1}{4} b' f' \\
+ \frac{1}{2} e^{b-f} \left[ \dot{b} + \frac{1}{2} b(\dot{a} + \dot{b} + 2\dot{c} - \dot{f}) \right]
\]

\[
R_{\theta\theta} = -1 + e^{c-b} + r^2 e^{c-b} \left[ \frac{1}{2} c'' + \frac{2}{r} c' + \frac{a' - b' + f'}{2r} + \frac{1}{2} c'^2 + \frac{1}{4} (a' - b') c' + \frac{1}{4} c f' \right] \\
+ e^{c-f} r^2 \left[ \frac{1}{2} \ddot{c} + \frac{1}{4} \dot{c}(\dot{a} + \dot{b} + 2\dot{c} - \dot{f}) \right]
\]

\[
R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta
\]

\[
R_{yy} = \frac{1}{2}(\ddot{a} + \ddot{b} + 2\ddot{c}) + \frac{1}{4}(\dot{a}^2 + \dot{b}^2 + 2\dot{c}^2) - \frac{1}{4} \dot{f}(\dot{a} + \dot{b} + 2\dot{c}) \\
+ \frac{1}{2} e^{f-b} \left[ f'' + \frac{1}{2} (\dot{a}^2 + \dot{b}^2 + 2\dot{c}^2 + f') f' \right]
\]

\[
R_{ry} = R_{yr} = \frac{1}{2} \left[ \dot{a}' + 2c' - \frac{2}{r} (\dot{b} - \dot{c}) + \frac{1}{2} a'(\dot{a} - \dot{b}) - c'(\dot{b} - \dot{c}) - \frac{1}{2}(\dot{a} + 2\dot{c}) f' \right]
\]

The Gauss normal ansatz corresponds to the identification

\[
a \to a - 2\kappa y, \quad b \to b - 2\kappa y, \quad c \to c - 2\kappa y, \quad f = 0,
\]

while the isotropic form of the metric, employed in this work corresponds to \( r \to \rho, y \to \eta \) and

\[
a = \alpha - 2\kappa \eta, \quad b = \beta - 2\kappa \eta, \quad c = -2\kappa \eta, \quad f \neq 0.
\]

References.


