One-loop integrals with XLOOPS-GiNaC

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Abstract
We present a new algorithm for the reduction of one-loop tensor Feynman integrals within the framework of the XLOOPS project, covering both mathematical and programming aspects. The new algorithm supplies a clean way to reduce the one-loop one-, two- and three-point Feynman integrals with arbitrary tensor rank and powers of the propagators to a basis of simple integrals. We also present a new method of coding XLOOPS in C++ using the GiNaC library.

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1 Introduction

The purpose of this paper is twofold. First, we introduce a new method to code XLOOPS in C++ with the help of the GiNaC library [1]. This also serves as an introduction to GiNaC for symbolic computations.

Second, we introduce a new procedure for tensor reduction which allows us to manipulate one-loop one-, two- and three-point Feynman integrals with arbitrary tensor rank and arbitrary powers of propagators. Working in orthogonal and parallel space of momentum configuration [2], this procedure allows one to reduce the one-loop tensor integrals to a basis of simpler integrals without using the Passarino-Veltman procedure. This procedure is implemented in the new version of the XLOOPS program, called XLOOPS-GiNaC, that on the one hand supplies a powerful tool for one-loop analytical Feynman integrations and on the other hand serves as a prerequisite for two-loop problems.

In sections 2 and 3 we introduce briefly the motivation to rewrite XLOOPS in C++, and the GiNaC library. In section 4 the procedure used to reduce one-loop two- and three-point integrals is presented. In two appendices simple examples of C++ programs using the GiNaC library and the XLOOPS-GiNaC interface for one-loop integrals are presented.

2 XLOOPS-GiNaC: The Motivation

In the past, the XLOOPS package [3] has been developed in a heterogenous environment: The core routines for transforming Feynman graphs into the basic integrals and for analytic integration are implemented in the language of the Maple [4] computer algebra system, the graphical user interface is written in Tcl/Tk [5], and the numerical integration is done by C programs that are generated and compiled at run-time by XLOOPS.

This way of implementation has a couple of drawbacks:

- While Maple and other computer algebra systems provide sophisticated mathematical capabilities, it is not suited as an environment for developing large applications such as XLOOPS, as it was not primarily intended as a programming language and only offers limited support for modern software engineering. For example, the only structured data type in Maple is the list, the distinction between local and global variables is not consistently enforced by all language constructs, and tools such as debuggers are very rudimentary.

- Different versions of Maple are in places incompatible with respect to the language. This leads to XLOOPS having to provide two versions of all program parts written in Maple one for Maple V Release 1 and one for Release 3. With Release 4 and later releases XLOOPS cannot currently be run. Not only does the parallel development of multiple program versions require higher maintenance effort, but it is also impractical for the user to require the installation of a specific version of Maple just to run XLOOPS.

- Maple is a commercial system, which prevents XLOOPS from getting a wide reach of distribution, especially in academic institutions.
The communication between different program parts is difficult because Maple has insufficient support for embedding programs written in other languages, it requires higher efforts for the conversion of data and, in some cases, for the multiple storing of redundant information (for example, the information about graph topologies has to be duplicated for the calculation and for the graphical interface).

In general, it is more difficult to maintain and develop a heterogenous program package. For the programmer it is necessary to become acquainted with three different environments (Maple, Tcl/Tk, and C).

The listed drawbacks and the observation of bugs related to the internal structure of XLOOPS led to the conclusion that the program had reached a state in which further development was almost impossible (approx. 15000 lines of badly documented source code, in some places very obscure programming techniques like identifiers whose meaning depends on capitalization). Thus, the decision was made to rewrite XLOOPS from scratch, putting it on more solid grounds.

In particular, it was decided that the new version of XLOOPS should be written in one uniform programming language. As the essential part of XLOOPS are the analytical calculations, traditional computer algebra systems (Maple, Mathematica [6], Reduce [7] and MuPAD [8]) were envisaged at first. But all these share more or less the same deficiencies as Maple, especially the low suitability of the built-in language for the development of large systems. It was therefore decided to use an established programming language as the foundation and extend it by the required algebraic capabilities. During the development of XLOOPS it was noted that only a small subset of the mathematical functions provided by Maple are actually needed. These are:

- complex arithmetic with arbitrary precision;
- simple manipulation of symbolic expressions, like expansion, collection, substitution of variables;
- simplification of rational functions;
- symbolic differentiation;
- special functions like polylogarithms and the Gamma function;
- Laurent series expansion of expressions containing these functions;
- solving systems of linear equations for the analysis of the tensor structure of the integrals;
- handling expressions containing elements of some particular non-commutative algebras such as Clifford and SU(3) Lie algebras;
- numeric integration.

Features that are particularly not required are:

- symbolic integration since only few master integrals need to be really integrated and they will be done by hand;
calculation of limits;

- treatment of domains and assumptions about the range of values of variables.

Due to the following reasons, C++ has been chosen as the programming language:

- C++ is officially standardized [9], so fewer complications are expected from the future development of the language.

- C++ allows to write down symbolic expressions in their natural mathematical notation by means of operator overloading (e.g. \(4*a+b\) instead of something like `add(mul(4,a),b)`).

- C++ is available for virtually all computer platforms. In particular, there are free compilers for the Unix systems predominant in the academic area.

- There is a large assortment of development tools available like powerful source-level debuggers and systems for version control and documentation.

- There is also a large number of existing libraries, especially for arbitrary precision arithmetics.

- As a compiled language, C++ is also suitable for numeric integration.

Based on C++ we developed a system called "GiNaC"\(^1\) that is primarily aimed at the re-implementation of XLOOPS, but is also suited for the development of other systems that integrate algebraic and numeric calculations with user interfaces for methods of data acquisition.

3 GiNaC: A New Programming Environment For XLOOPS

GiNaC [1] is a C++ library for handling symbolic mathematical expressions \(^2\). Some of the capabilities of the library are

- complex arithmetic with arbitrary precision, based on the CLN library by Bruno Haible [10];

- manipulation of symbolic expressions;

- normalization of rational functions;

- matrices and systems of linear equations;

- numerous special functions (trigonometric and hyperbolic functions, exponential functions, logarithms, Gamma und polygamma functions);

- symbolic differentiation;

- series expansion of functions (Taylor and Laurent series);

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\(^1\)an acronym for "GiNaC is not a Computer Algebra System"

\(^2\)http://www.ginac.de
numeric  real and complex numbers (integers like 42, exact fractions like $\frac{2}{3}$ and floating point numbers like 7.319), based on CLN
symbol algebraic symbols
constant constants like $\pi$ which are treated similarly to symbols but also have a predefined numerical value
add sums of the form $\sum_{i=1}^{n} c_i x_i$, for $c_i \in \mathbb{C}$ and arbitrary non-numeric expressions $x_i$
mul product of the form $\prod_{i=1}^{n} x_i^{c_i}$, for $c_i \in \mathbb{C}$ and arbitrary non-numeric expressions $x_i$
power arbitrary expressions of the form $x^y$
pseries compact representation of Laurent and Taylor series (only contains the series coefficients, the expansion variable, and the expansion point)
function mathematical functions like $\sin()$ und $\cos()$, where the individual functions are not implemented as subclasses of function but are distinguished by a function index
lst lists of expressions
matrix matrices
relational equations and unequations

<table>
<thead>
<tr>
<th>Table 1: The most important GiNaC classes</th>
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<td>- non-commutative algebras.</td>
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GiNaC is designed in an object-oriented fashion. The central class of GiNaC is the class ex that stores a symbolic expression. Strictly speaking, ex only represents a "smart" pointer to the real expression which is stored as a tree whose nodes are objects subclassed from the abstract base class basic. The operators $+$, $-$, $*$ und $/$ are overloaded to simplify the creation of expressions in the program code.

Table 1 gives an overview of the most important subclasses of basic. The classes can be categorized into the "atomic" classes numeric, symbol and constant which are at the leaves of an expression tree, and the container classes (all others) which themselves contain expressions. The representation of sums and products with numeric coefficients and powers was chosen for reasons of efficiency (see [11] and [12]).
4 One-Loop One-, Two-, And Three-Point Integrals

In this section we present our algorithm for tensor reduction. To regularize UV-divergences, we are working in $D = 4 - 2\epsilon$ dimensional space-time. The main idea is to split the space of integration in a parallel and an orthogonal space. We define the parallel space to be the linear span of the $n$-external momenta $q_{i\mu}$ ($i = 1, \ldots, n$) involved in the integrand. This parallel space has a finite dimension $J \leq D$. The remaining $D - J$ dimensions span an orthogonal complement of the parallel space called the orthogonal space [2].

Once an explicit configuration of external momenta is chosen, the dimension of the parallel space $J$ is known, and the scalar products are written explicitly in terms of the components of the external momenta

$$l^2 = l_0^2 - l_1^2 - \cdots - l_{J-1}^2 - l_1\cdot l_{J-1},$$

$$l\cdot q_i = l_0 q_{i0} - l_1 q_{i1} - \cdots - l_{J-1} q_{iJ-1}. \tag{1}$$

A general one-loop integral can be written as

$$\int d^D l F(l^2, \{l\cdot q_i\}) = \frac{2\pi^{D-J}}{\Gamma((D - J)/2)} \int_{-\infty}^{\infty} dl_0 \cdots \int_{-\infty}^{\infty} dl_{J-1} \int_0^{l_0^2 - l_1^2 - \cdots - l_{J-1}^2} F(l_\perp, l_0, \ldots, l_{J-1}). \tag{2}$$

In the usual Passarino-Veltman approach, to keep an explicitly covariant form, one expands tensor integrals of the type

$$T^{(n)}_{\mu_1\ldots\mu_N} = \int d^D l \frac{l_{\mu_1} \cdots l_{\mu_N}}{\prod_{i=1}^n (l + q_i)^2 - m_i^2 + i\rho} \tag{3}$$

in a basis of Lorentz tensors constructed from the metric tensor $g_{\mu\nu}$ and the external momenta $q_{i\mu}$. Since $T^{(n)}_{\mu_1\ldots\mu_N}$ is symmetric, one can re-group indices belonging to the parallel space and to the orthogonal space together

$$T^{(n)}_{\mu_1\ldots\mu_N} = T^{(n)}_{\mu_J\ldots\mu_D 0\ldots 0 1\ldots 1 J-1\ldots J-1 \begin{array}{c} p_0 \quad \vdots \quad p_1 \quad \vdots \quad p_{J-1} \\ p_{||} \end{array}} \tag{4}$$

where $p_0, p_1, \ldots$ denote the numbers of indices for the 0-, 1-, \ldots components. The indices $\mu_J, \ldots, \mu_D$ in the orthogonal space have to result in a symmetric combination of metric tensors in the orthogonal space $g^\perp_{\mu\nu}$

$$T^{(n)}_{\mu_1\ldots\mu_N} = (-1)^{\frac{D-J}{2}} K \left( g_{\mu_J\mu_{J+1}} \cdots g_{\mu_{D-1}\mu_D} \right)_{\text{symm.}} \cdot T^{(n)}_{(p_0 \ldots p_{J-1} p_{\perp})} \tag{5}$$

with $N = p_0 + \ldots + p_{J-1} + p_{\perp}$. The normalization can be derived by looking at contractions with $g^\perp_{\mu\nu}$ and observing that

$$g^\perp_{\mu_J\mu_{J+1}} g_{\mu_{J+1}\mu_{J+2}} = D - J. \tag{6}$$
One obtains

\[ K = \prod_{i=J}^{D-J-2} (D - J + 2i). \] (7)

The coefficients needed for the calculation of a specific component of a general one-loop tensor integral therefore have the form

\[ T^{(n)}_{(p_0 \ldots p_{J-1} p_{\perp})} = \int d^D l \prod_{i=1}^n \frac{p_0^{p_0} \ldots p_{J-1}^{p_{J-1}} \perp^{p_{\perp}}}{(l + q_i)^2 - m_i^2 + i\rho}. \] (8)

In the next sections, we present an algorithm to calculate \( T^{(n)}_{(p_0 \ldots p_{J-1} p_{\perp})} \) for the one-loop two- and three-point tensor integrals. In the rest of this paper, we call \( T^{(n)}_{(p_0 \ldots p_{J-1} p_{\perp})} \) tensor integrals.

### 4.1 One-Loop Two-Point Tensor Integrals

In this section, the algorithm for an automatic calculation of one-loop two-point tensor integrals is presented. We first consider the case where \( q^2 \) is timelike, \( q^2 > 0 \). Then one can choose a reference frame where \( q_\mu = (q_0, 0, 0, 0) \). The general integral for a tensor Feynman diagram shown in Fig.1 has the form

\[ I_{t_1, t_2}^{(2)ij} (q^2) = \int d^D l \frac{l_0^t l_{\perp}^{t}}{P_1^t P_2^{t}}, \] (9)

with

\[ P_1 = (l_0 + q_0)^2 - l_\perp^2 - m_1^2 + i\rho, \]
\[ P_2 = l_0^2 - l_\perp^2 - m_2^2 + i\rho, \] (10)

where \( l_0 \) and \( l_\perp \) span the parallel and orthogonal subspaces respectively and the integral vanishes unless \( j \) is even. In the spacelike and lightlike cases where \( q^2 < 0 \) or \( q^2 = 0 \) one can choose a reference frame where \( q_\mu = (0, q_01, 0, 0) \) or \( q_\mu = (q_01, q_01, 0, 0) \) respectively and the integral space can be split into a two-dimensional parallel and a \( D - 2 \) dimensional orthogonal subspace. We will consider these cases later in section 4.2.3. A genuine one-loop integral has \( t_1 = t_2 = 1 \), but the more general case is needed in the case of the reduction of integrals with more than one loop.
The strategy now is to express $I^{(2)ij}_{t_1t_2}(q^2)$ as a polynomial of simpler integrals. It turns out that the usual scalar one- and two-point integrals $A_0$ and $B_0$ (in Passarino-Veltman notation \([14, 15]\)) are sufficient. This expansion is always possible except for some special cases that we will consider later.

Firstly consider the general case where $q_0 \neq 0$. We express the numerator of the integral in Eq.(9) as a function of $P_1$ and $P_2$. From Eq.(10) we get:

$$l_0 \rightarrow l_0(P_1, P_2, q_0, m_1, m_2) = \frac{1}{2q_0}(P_1 - P_2 - C_1),$$
$$l^2_\perp \rightarrow l^2_\perp(P_1, P_2, q_0, m_1, m_2) = l^2_0 - P_2 + C_2,$$
$$C_1 = q^2_0 - m^2_1 + m^2_2,$$
$$C_2 = -m^2_2 + i\rho.$$  

(11)

Inserting Eq.(11) into Eq.(9) and expanding the numerator of the integrand, one obtains

$$I^{(2)ij}_{t_1t_2}(q^2) = \sum_{n,m} C_{nm} \int d^Dl P^{n-t_1}_1 P^{m-t_2}_2$$

(12)

with $C_{nm}$ being simple functions of $q_0, m_1, m_2,$ and

$$\int d^Dl P^{n-t_1}_1 P^{m-t_2}_2 = \begin{cases} 
0 & \text{if } n - t_1 \geq 0 \text{ and } m - t_2 \geq 0, \\
\int d^Dl \frac{1}{P^{1-n}_1 P^{2-m}_2} & \text{if } n - t_1 < 0 \text{ and } m - t_2 < 0, \\
\int d^Dl \frac{P^{n-t_1}_1}{P^{2-m}_2} & \text{if } n - t_1 \geq 0 \text{ and } m - t_2 < 0, \\
\int d^Dl \frac{P^{m-t_2}_2}{P^{1-n}_1} & \text{if } n - t_1 < 0 \text{ and } m - t_2 \geq 0. 
\end{cases}$$

(13)

We see that the second case actually corresponds to a scalar two-point function. For the last two cases, from Eq.(10) one can insert

$$P_1 = P_1(P_2, l_0, q_0) = 2l_0q_0 + P_2 + C_1,$$

(14)

or

$$P_2 = P_2(P_1, l_0, q_0) = P_1 - 2l_0q_0 - C_1$$

(15)

and expand the numerator of the integrands in Eq.(13). Then Eq.(10) can be completely reduced to the one-point functions

$$I^{(1)i}_l = \int d^Dl \frac{(l^2)^{\frac{4}{i}}}{(l^2 - m^2 + i\rho)^l}. $$

(16)
4.1.1 The case $q_0 = 0$

If $q_0 = 0$, Eq.(13) must be rewritten using

\[ P_1 = l_0^2 - l_\perp^2 - m_1^2 + i\rho, \]
\[ P_2 = l_0^2 - l_\perp^2 - m_2^2 + i\rho. \]  \hspace{1cm} (17)

If $m_1 = m_2$ then $P_1 = P_2$ and $I_{t_1 t_2}^{(2)ij}(q^2)$ has the simple form

\[ I_{t_1 t_2}^{(2)ij}(q^2) = \int d^Dl \frac{l_0^i l_0^j}{P_1^{t_1 + t_2}} \]  \hspace{1cm} (18)

that is actually the one-loop one-point function.

If $m_1 \neq m_2$, one performs partial fraction decomposition and finds

\[ I_{t_1 t_2}^{(2)ij}(q^2) = \int d^Dl \frac{l_0^i l_0^j}{P_1(m_1^2)^{t_1} P_2(m_2^2)^{t_2}} \]
\[ = \frac{1}{(t_1 - 1)!} d(m_1^2)^{t_1 - 1} \left( \frac{1}{m_1^2 - m_2^2} \int d^Dl \frac{l_0^i l_0^j}{P_1(m_1^2)} \right) + \]
\[ \frac{1}{(t_2 - 1)!} d(m_2^2)^{t_2 - 1} \left( \frac{1}{m_2^2 - m_1^2} \int d^Dl \frac{l_0^i l_0^j}{P_2(m_2^2)} \right), \]  \hspace{1cm} (19)

which is a combination of one-loop one-point functions.

4.2 One-Loop Three-Point Tensor Functions

The notation we use for one-loop three-point functions is shown in Fig.2. We are working in the frame of reference where the external momentum configuration is $q_{1\mu} = (q_{10}, 0, 0, 0)$, $q_{2\nu} = (q_{20}, q_{21}, 0, 0)$. The parallel space is now two-dimensional and the general form of the one-loop three-point tensor function is

\[ I_{t_1 t_2 t_3}^{(3)ijk} = \int d^Dl \frac{l_0^i l_0^j l_0^k}{P_1^{t_1} P_2^{t_2} P_3^{t_3}} \]  \hspace{1cm} (20)
with
\[ P_1 = l_0^2 - l_1^2 - l_2^2 + 2l_0 q_{10} + q_{10}^2 - m_1^2 + i\rho, \]
\[ P_2 = l_0^2 - l_1^2 - l_2^2 + 2l_0 q_{20} - 2l_1 q_{21} + q_{20}^2 - q_{21}^2 - m_2^2 + i\rho, \]
\[ P_3 = l_0^2 - l_1^2 - l_2^2 - m_3^2 + i\rho \]

where \( \{l_0, l_1\} \) and \( l_\perp \) span the parallel and orthogonal subspaces, respectively.

### 4.2.1 The general case

Firstly we consider the case where \( q_{10} \neq 0 \) and \( q_{21} \neq 0 \). Then we are always able to express \( l_0, l_1, l_\perp \) in terms of \( P_1, P_2, P_3 \):

\[ l_0 = \frac{1}{2q_{10}}(P_1 - P_3 - c_{00}), \]
\[ l_1 = \frac{1}{2q_{10}q_{21}} \left[ q_{20} \left(P_1 - P_3 - (q_{10}^2 - m_1^2 + m_3^2)\right) + q_{10} \left(P_3 - P_2\right) + q_{10} \left(m_3^2 - m_2^2 + q_{20}^2 - q_{21}^2\right) \right] \]
\[ = \frac{1}{2q_{21}} \left[c_{10}l_0 + (P_3 - P_2) + c_{11}\right], \]
\[ l_\perp^2 = l_0^2 - l_1^2 - P_3 - c_{20} \]

with
\[ c_{00} = q_{10}^2 - m_1^2 + m_3^2, \]
\[ c_{10} = 2q_{20}, \]
\[ c_{11} = m_3^2 - m_2^2 + q_{20}^2 - q_{21}^2, \]
\[ c_{20} = m_3^2 - i\rho. \]

Again, as in the case of two-points functions, one substitutes Eq.(22) into Eq.(20) and obtains a combination of scalar three-point functions

\[ I_{ijk}^{(3)}(t_1, t_2, t_3) = \sum_{m,n,r} \int d^D l \frac{1}{P_1^{m-t_1} P_2^{n-t_2} P_3^{r-t_3}} \]

with \( C_{mnr} \) being simple functions of masses and components of external momenta. More explicitly, the integrand on the right hand side of Eq.(24) contains terms like

\[ \frac{1}{P_1^{n_1} P_2^{n_2} P_3^{n_3}}, \frac{P_i^{n_i} P_j^{n_j}}{P_k^{n_k}}, \text{ and } \frac{P_i^{n_i}}{P_j^{n_j} P_k^{n_k}} \]

with \( i \neq j \neq k \) and positive \( n_i \). Other possible combinations lead to a vanishing integral.

The first group of terms can be obtained from derivatives of scalar three-point functions. For the last two cases, using Eq.(21) one can expand the numerator in terms of propagators \( P_{n_j} \) in the denominator. This expansion is always possible and reduces the second group of terms to one-point
functions. Similarly the third group of terms can be reduced to one-loop two-point functions. Note that in this step we meet two kinds of one-loop two-point functions. The first kind is the one-loop two-point function with one parallel dimension that was already found in the previous section. The second kind is the one-loop two-point function with two parallel dimensions of the internal momentum $l$ that is not trivial and will be given in the section 4.2.3 as a separate case.

4.2.2 The case $q_{10} q_{21} = 0$

In this case the expansions in Eq.(22) cannot be used. However, Eq.(20) can be reduced by partial fraction decomposition. First consider the case $q_{10} = 0$ with arbitrary values of $q_{21}$.

4.2.2.1 The case $q_{10} = 0$

Eq.(21) simplifies to

$$P_1 = l_0^2 - l_2^2 - l_1^2 - m_1^2 + i\rho,$$
$$P_2 = l_0^2 - l_2^2 - 2l_0 q_{20} - 2l_1 q_{21} + q_{20}^2 - q_{21}^2 - m_2^2 + i\rho,$$
$$P_3 = l_0^2 - l_2^2 - m_3^2 + i\rho.$$  \hspace{1cm} (26)

If $m_1 \neq m_3$, partial fraction decomposition leads to a separation into terms which contain only two propagators

$$I^{(3)}_{t_1 t_2 t_3} = \prod_{f=1}^{3} \left( \frac{1}{(t_f - 1)!} \frac{d^{t_f-1}}{d(m_f^2)^{t_f-1}} \right) \left\{ \frac{1}{m_1^2 - m_3^2} \left( \int dP_l \frac{l_{01} l_{11} l_{k1}}{P_1(m_1^2) P_2(m_2^2)} - \int dP_l \frac{l_{01} l_{11} l_{k1}}{P_2(m_2^2) P_3(m_3^2)} \right) \right\}.$$  \hspace{1cm} (27)

If $m_1 = m_3$, two propagators are equal and it is sufficient to calculate

$$I^{(3)}_{t_1 t_2 t_3} = \int dP_l \frac{l_{01} l_{11} l_{k1}}{P_1^{t_1+t_3}(m_1^2) P_2^{t_2}(m_2^2)}.$$  \hspace{1cm} (28)

The three-point integrals are then reduced to combinations of two-point integrals with two parallel dimensions. We will treat this class of two-point functions in section 4.2.3.

4.2.2.2 The case $q_{10} \neq 0$ and $q_{21} = 0$

In this special case, the integrals collapse from two parallel dimensions to one and Eq.(21) reads

$$P_1 = (l_0 + q_{10})^2 - l_2^2 - m_1^2 + i\rho,$$
$$P_2 = (l_0 + q_{20})^2 - l_2^2 - m_2^2 + i\rho,$$
$$P_3 = l_0^2 - l_2^2 - m_3^2 + i\rho.$$  \hspace{1cm} (29)

where the components $l_1$ and $l_\perp$ can be combined into $l'_\perp$ to form a new $D - 1$ dimensional orthogonal subspace with

$$l'_\perp = l_1^2 + l_\perp^2.$$  \hspace{1cm} (30)
Then the integral can be reduced to a simpler integral with only one parallel dimension

\[ I_{t_1 t_2 t_3}^{(3)ij+k} = \int d^Dl \frac{l_0^i l_1^j l_2^k}{P_1^i P_2^j P_3^k} \]  \hspace{1cm} (31)

Again, one can use the same procedure as in the general case

\[ l_0 \rightarrow l_0(P_1, P_3, m_i, q) = \frac{1}{2q_{10}}(P_1 - P_3 + m_1^2 - m_3^2 - q_{10}^2), \]

\[ l_\perp \rightarrow l_\perp(P_1, P_3, m_i, q) = l_0^2 - P_3 - m_3^2 + ip \]  \hspace{1cm} (32)

that reduces Eq.(31) to a form similar to Eq.(24).

\section{4.2.3 The two-point integral with two parallel dimensions \( J_2^{ijk} \)}

In the preceding section we have shown that general one-loop three-point tensor functions can be reduced to the usual scalar integrals and one new two-point function corresponding to a tensor component in a two-dimensional parallel space. Explicitly, this integral reads

\[ J_2^{ijk} = \int d^Dl \frac{l_0^i l_1^j l_2^k}{[(l_0 + q_{10})^2 - l_1^2 - l_2^2 - m_1^2 + ip]^i \ (l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_1^2 - m_2^2 + ip]^j \ (l_0 + q_{30})^2 - (l_1 + q_{31})^2 - l_1^2 - m_3^2 + ip]^k} \]  \hspace{1cm} (33)

with \( Q_0 = q_{20} - q_{10}, Q_1 = q_{21}; Q_\mu = q_{2\mu} - q_\mu \). If \( Q_1 = 0 \), this integral reduces to the one-loop two-point function in one-dimensional parallel space as found in the previous section. If, on the other hand, \( Q_1 \neq 0 \), one can always find a Lorentz boost which transforms into a reference frame where the transformed 4-momentum \( Q' \) has either only one non-zero component \( (Q' \) or \( Q' \) if \( Q \) is timelike or spacelike) or where \( Q'_0 = Q'_1 \) if \( Q \) is lightlike. The loop momentum has to be boosted accordingly which, however, modifies only the numerator of the integrand in Eq.(33). Explicitly, consider the boost

\[ \begin{pmatrix} l_0 \\ l_1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma \beta \gamma \end{pmatrix} \begin{pmatrix} l_0' \\ l_1' \end{pmatrix}. \]  \hspace{1cm} (34)

Then the three sub-cases are treated as follows.

\subsection{4.2.3.1 The timelike case \( Q_0^2 - Q_1^2 > 0 \)}

In this case, under the transformation in Eq.(34) the integral will be reduced to one-loop two-point functions with a one-dimensional parallel space as found in the previous section:

\[ J_2^{ijk} = \int d^Dl' \frac{[\gamma (l_0' + \beta l_1') - q_1]^i \ [\gamma (\beta l_0' + l_1')]^j l_2^k}{[l_0'^2 - l_1'^2 - l_2^2 - m_1^2 + ip]^i \ [(l_0' + P)^2 - l_1'^2 - l_2^2 - m_2^2 + ip]^j \ [(l_0' + P)^2 - l_1'^2 - l_2^2 - m_3^2 + ip]^k} \]  \hspace{1cm} (35)

with \( P = \sqrt{Q_0^2 - Q_1^2}, \gamma = Q_0/P \) and \( \beta = Q_1/Q_0 \).
4.2.3.2 The spacelike case, $Q_0^2 - Q_1^2 < 0$

In this case the boost with $P = \sqrt{Q_1^2 - Q_0^2}$, $\gamma = Q_1/P$ and $\beta = Q_0/Q_1$ transforms to a reference frame in which the integral reads

$$J^{ijk}_2 = \int d^{D-1}l' \frac{[\gamma (l'_0 + \beta l'_1) - q_1]^i [\gamma (\beta l'_0 + l'_1)]^j l'_k}{[l'_0^2 - l'_1^2 - l'_1^\perp - m_1^2 + i\rho]^l} \frac{[l'_0^2 - (l'_1 + P)^2 - l'_1^\perp - m_2^2 + i\rho]}{[l'_1^2 - m_2^2 + i\rho]^2}. \quad (36)$$

The components $l'_0$ and $l'_1$ can be combined to form a new $D-1$ dimensional orthogonal subspace while $l_1$ spans the parallel subspace. Using the same procedure as in the previous sections one can reduce the integral completely to the scalar one- and two-point functions.

4.2.3.3 The lightlike case, $Q_0^2 - Q_1^2 = 0$

In this case the transformation in Eq.(34) is singular and the integral $J^{ijk}_2$ becomes

$$J^{ijk}_2 = \int d^{D-1}l \frac{(l_0 - q_1)^i l'_1 l'_k}{[l_0^2 - l_1^2 - l_1^\perp - m_1^2 + i\rho]^l} \frac{[(l_0 + Q_0)^2 - (l_1 + Q_0)^2 - l_1^\perp - m_2^2 + i\rho]}{[l_1^2 - m_2^2 + i\rho]^2}. \quad (37)$$

By solving the system of equations

$$P_1 = l_0^2 - l_1^2 - l_1^\perp - m_1^2 + i\rho,$$
$$P_2 = (l_0 + Q_0)^2 - (l_1 + Q_0)^2 - l_1^\perp - m_2^2 + i\rho, \quad (38)$$

one obtains

$$l_1^2 = l_0^2 - l_1^2 - P_1 - m_1^2 + i\rho,$$
$$l_1 = \frac{P_1 - P_2 + m_2^2 - m_1^2}{2Q_0} + l_0. \quad (39)$$

Inserting Eq.(39) into Eq.(37) and expanding the numerator of the integrand, the integral will be reduced to scalar one-point functions and a simpler tensor integral

$$J^{i}_2 = \int d^{D-1}l \frac{(l_0)^i}{[l_0^2 - l_1^2 - l_1^\perp - m_1^2 + i\rho]^l} \frac{[(l_0 + Q_0)^2 - (l_1 + Q_0)^2 - l_1^\perp - m_2^2 + i\rho]}{[l_1^2 - m_2^2 + i\rho]^2}. \quad (40)$$

The explicit calculation of this integral is given in [13].

This completes the description of our algorithm for tensor reduction. We did not reproduce in this paper explicit expression for the basic scalar integrals, since these can be found in the literature [2, 15].

5 Conclusion

Due to the limitations of Maple and the internal structure of XLOOPS, it was decided to rewrite XLOOPS from scratch, based on GiNaC, a C++ library to replace Maple as an algebraic programming environment [1]. A new algorithm for one-loop one-, two- and three-point tensor reduction was also
successfully implemented to optimize the performance of the package. At this stage of the project, a
package for calculating one-loop one-, two- and three-point tensor integrals is completely coded\(^3\).

In the next step, we plan to rewrite the module for two-loop one-, two- and three-point integrals
and to completely code XLOOPS in GiNaC.

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**Appendix A**

In this section we introduce the XLOOPS-GiNaC definitions for the one-loop integral functions. Com-
plete explicit expressions for the scalar (one-, two-, and three-point) integrals can be found in [2, 15].

- The one-point function

\[
\text{OneLoop1Pt}(i, m, t, \rho) = I^{(1)}_t = \int d^D l \frac{(l^2)^{\frac{D}{2}}}{[l^2 - m^2 + i\rho]^t}.
\]

- The two-point function

\[
\text{OneLoop2Pt}(i, j, q, m_1, m_2, t_1, t_2, \rho) = I^{(2)}_{ij} = \int d^D l \frac{l_i^0 l_j^i}{[(l_0 + q)^2 - l_1^2 - m_1^2 + i\rho]^{t_1} [l_0^2 - l_1^2 - m_2^2 + i\rho]^{t_2}}.
\]

- The three-point function

\[
\text{OneLoop3Pt}(i, j, k, q_10, q_{20}, q_{21}, m_1, m_2, m_3, t_1, t_2, t_3, \rho) = I^{(3)}_{ijk} = \int d^D l \frac{l_i^0 l_j^1 l_k^3}{P_1^{t_1} P_2^{t_2} P_3^{t_3}}
\]

with

\[
P_1 = l_0^2 - l_1^2 - l_2^2 + 2 l_0 q_{10} + q_{10}^2 - m_1^2 + i\rho,
\]

\[
P_2 = l_0^2 - l_1^2 - l_2^2 + 2 l_0 q_{20} - 2 l_1 q_{21} + q_{20}^2 - q_{21}^2 - m_2^2 + i\rho,
\]

\[
P_3 = l_0^2 - l_1^2 - l_2^2 - m_3^2 + i\rho.
\]

\(^3\)The package can be downloaded at [http://wwwthep.physik.uni-mainz.de/~xloops](http://wwwthep.physik.uni-mainz.de/~xloops)
The arguments and return values of the above functions are objects of type \texttt{ex} so the return values as well as input parameters can be a symbol, expression or a real number. A detailed description of the internal functionality of the basic classes and methods of 	extsc{GiNaC} is given in [12].

In order to illustrate the output of 	extsc{XLOOPS-GiNaC}, we give one example program that calculates and prints out both analytical and numerical results of the UV-divergent and the finite terms of the integral \texttt{OneLoop2Pt(1, 0, q, m1, m2, 1, 1, \rho)}

```cpp
#include <iostream>
#include <ex>
#include <xloops.h>
using namespace GiNaC;
using namespace xloops;

int main()
{
  symbol q("q"), m1("m1"), m2("m2") eps("eps"), rho("rho");
  ex a = OneLoop2Pt(1, 0, q, m1, m2, 1, 1, rho);
  a = a.series(eps == 0, 4);
  ex a1 = a.coeff(eps, -1).subs(rho == 0);
  ex a2 = a.coeff(eps, 0).subs(rho == 0);
  cout << "Order eps^-1 is " << endl << a1.normal() << endl;
  cout << "Order eps^0 is " << endl << a2.normal() << endl;
  ex b1 = a.subs(rho==0).subs(m1==80).subs(m2==80).subs(q==100).evalf();
  cout << "Numerical value up to order eps^2 is " << endl << b1 << endl;
  return 0;
}
```

The output of this program reads

```
Order eps^-1 is
-1/2*I*Pi^2*q

Order eps^0 is
(-1/2*q^(-1)*m2^2-1/2*q+1/2*q^(-1)*m1^2)*(I*Pi^2*q^(-1)*((-1/2*q-1/2*q^(-1)*(m1^2-m2^2))*R2ex1(-m2^2,-(1/2*q-1/2*q^(-1)*(m1^2-m2^2))^2)+R2ex1(-m1^2,-(1/2*q+1/2*q^(-1)*(m1^2-m2^2))^2)*(1/2*q+1/2*q^(-1)*m1^2)-(I*Pi^2*log(Pi)*q^(-1)*(1/2*q+1/2*q^(-1)*m1^2)+Pi^2*(I-I*Euler)*m1^2)-I*Pi^2*log(m1^2)-1/2*q^(-1)*m1^2))*R2ex1(-m1^2,-(1/2*q+1/2*q^(-1)*(m1^2-m2^2))^2)+R2ex1(-m2^2,-(1/2*q-1/2*q^(-1)*(m1^2-m2^2))^2)*(1/2*q+1/2*q^(-1)*m1^2)-(I*Pi^2*log(Pi)*q^(-1)*(1/2*q+1/2*q^(-1)*m1^2)+Pi^2*(I-I*Euler)*m1^2)-I*Pi^2*log(m1^2)-1/2*q^(-1)*m1^2))

Numerical value up to order eps^2 is
(-493.48022005446793098*I)*eps^(-1)+(5019.9161138633880865*I)+(-4.6074255521949439496E-15-25944.085010687200793*I)*eps+Order(eps^2)
```

The output looks ugly but one can easily convert it to \textit{T\AE X} format to get a better look.

Appendix B

In order to illustrate the use of \textsc{GiNaC} for calculating one-loop Feynman diagrams, we give one example program that is actually part of the automated regression tests of the \textsc{XLOOPS} package. It
checks that the longitudinal part of the vacuum polarization in QED vanishes on the one-loop level as required by gauge invariance, i.e.

\[ q_\mu \Pi^{\mu\nu}(q^2) = 0, \quad (41) \]

where

\[ \Pi^{\mu\nu}(q^2) = -e^2 \int d^D l \, \text{Tr} \frac{\gamma^\mu(l + q + m)\gamma^\nu(l + m)}{((l + q)^2 - m^2)(l^2 - m^2)}. \]

The general tensor structure of \( \Pi^{\mu\nu} \) is

\[ \Pi^{\mu\nu} = A g^{\mu\nu} + B \frac{q_\mu q_\nu}{q^2} \]

with functions \( A \) and \( B \) that, because of (41), satisfy

\[ A + B = 0. \]

This expression is obtained by contracting \( \Pi^{\mu\nu} \) with \( \frac{q_\mu q_\nu}{q^2} \):

\[ \Pi^{\mu\nu} q_\mu q_\nu \frac{1}{q^2} = A + B \]

\[ = -\frac{e^2}{q^2} \int d^D l \, \text{Tr} \frac{q(l + q + m)q(l + m)}{((l + q)^2 - m^2)(l^2 - m^2)} \]

\[ = -\frac{e^2}{q^2} \int d^D l \, \text{Tr} \frac{qlqI + q^2q + m^2q^2}{((l + q)^2 - m^2)(l^2 - m^2)} \]

\[ = \frac{4e^2}{q^2} \int d^D l \, 2(l \cdot q)^2 - q^2l^2 + q^2(l \cdot q) + m^2q^2 \]

\[ = \frac{4e^2}{q^2} \int d^D l \, q^2l_0^2 + q^2l_1^2 + q^2l_2^2 + m^2q^2 \]

\[ = -4e^2(I_{11}^{(2)20} + I_{11}^{(2)02} + qI_{11}^{(2)10} + m^2I_{11}^{(2)00}). \]

The following program verifies that this expression vanishes for the first three orders of the regularization parameter:

```cpp
#include <iostream>
#include <ginac/ginac.h>
#include "xloops.h"
using namespace GiNaC;

int main()
{
    symbol q("q"), m("m"), eps("eps"), rho("rho");

    ex a = OneLoop2Pt(2, 0, q, m, m, 1, 1, rho)
        + OneLoop2Pt(0, 2, q, m, m, 1, 1, rho)
        + q * OneLoop2Pt(1, 0, q, m, m, 1, 1, rho)
        + OneLoop2Pt(0, 0, q, m, m, 1, 1, rho);
```
+ pow(m, 2) * OneLoop2Pt(0, 0, q, m, m, 1, 1, rho);

a = a.series(eps == 0, 4);

ex a1 = a.coeff(eps, -1).subs(rho == 0);
ex a2 = a.coeff(eps, 0).subs(rho == 0);
ex a3 = a.coeff(eps, 1).subs(rho == 0);

cout << "Order eps^{-1} is " << a1.expand() << endl;
cout << "Order eps^0 is " << a2.expand() << endl;
cout << "Order eps^1 is " << a3.expand() << endl;

return 0;
}

### Line 2-4
Include the header files for the GiNaC and XLOOPS libraries.

### Line 8
The external parameters mass \( m \) and momentum \( q \), the dimensional regularization parameter \( \epsilon \) and the infinitesimal imaginary part of the propagator \( \rho \) are declared as symbols. It is necessary to specify the name used for printing expressions because C++ does not provide the names of variables at run-time.

### Line 10-13
The right hand side of Eq.(42) is declared as an object of type \( \text{ex} \). \text{pow()} \) is used for exponentiation because the C++ operator \(^\) has the wrong precedence in relation to the operator \(*\).

### Line 14
To get the UV divergence as well as the finite part, one needs to take the series expansion of the right hand side of Eq.(42) at the pole \( \epsilon = 0 \). Expressions are usually manipulated in the C++ oriented notation \( \text{expression.function(parameters)} \), but the functional notation \( \text{function(expression, parameters)} \) is also available.

### Line 16-18
The coefficients of the series for the orders \( \epsilon^{-1}, \epsilon^{0} \) and \( \epsilon^{1} \) are extracted and the limit of \( \rho \to 0 \) is taken by calling the function \( \text{subs()} \).

### Line 20-22
The three coefficients are simplified with \( \text{expand()} \) and printed to the standard output stream. When run, this program will output all three coefficients as 0.

A detailed description of the internal functionality of the basic classes and methods of GiNaC is given in [12].
BIBLIOGRAPHY


