Explicit Derivation of the Fluctuation-Dissipation Relation of the Vacuum Noise in the $N$-dimensional de Sitter Spacetime

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(Jan. 2001)

Abstract

Motivated by a recent work by Terashima (Phys. Rev. D60 084001), we revisit the fluctuation-dissipation (FD) relation between the dissipative coefficient of a detector and the vacuum noise of fields in curved spacetime. In an explicit manner we show that the dissipative coefficient obtained from classical equations of motion of the detector and the scalar (or Dirac) field satisfies the FD relation associated with the vacuum noise of the field, which demonstrates that the Terashima’s prescription works properly in the $N$-dimensional de Sitter spacetime. This practice is useful not only to reconfirm the validity of the use of the retarded Green function to evaluate the dissipative coefficient from the classical equations of motion but also to understand why the derivation works properly, which is discussed in connection with previous investigations on the basis on the Kubo-Martin-Schwinger (KMS) condition. Possible application to black hole spacetime is also briefly discussed.

04.62+v, 04.70.Dy, 98.80.Hw
I. INTRODUCTION

It is well-known that the vacuum fluctuations of the Minkowski spacetime can be a thermal ambience when viewed by uniformly accelerated observers. This effect, which is called the Unruh effect, has been investigated by many authors. Recently Terashima [1] has proposed a new method to evaluate the Rindler noise indirectly by using the fluctuation-dissipation (FD) relation (or theorem) formulated by Callen and Welton [2]. He has explicitly shown that the FD relation holds between the Rindler noise and the dissipative coefficient of the De Witt detector which is obtained from classical equations of motion of the detector and the (scalar and Dirac) fields in $N$-dimensional Rindler spacetime.

The FD relation is a general relation between the spontaneous fluctuations of generalized forces in thermal equilibrium and dissipation in linear dissipative system. Studies on the FD relation has a long history, and provide a basis of statistical physics for dissipative systems (e.g., [3,4] and references therein). The Callen and Welton’s formula [2] is one of the general formulations of the FD relation developed in a system of many microscopic degrees of freedom in the thermal equilibrium state and a macroscopic degree of freedom, which are linearly coupled with each other.

Though several authors have investigated the vacuum noise of quantum fields in curved spacetime from a view point of the FD relation [5–8], however, Terashima’s approach is slightly different from those previous approaches. Following the Terashima’s prescription, the dissipative coefficient of the detector’s inner motion is explicitly calculated from the classical equations of motion of the detector and the fields. Then the Rindler noise is simply obtained by inserting the dissipative coefficient into the FD relation, which relates the dissipative coefficient to the noise of vacuum fluctuations. He has shown that this method is successful in the system of the De Witt detector coupled to the scalar field in $N$-dimensional Rindler spacetime, and he has generalized the method to the Dirac field by using the fermionic FD relation.

In two and four dimensional spacetime, it is well-known that the spectrum of the Rindler
noise is completely equivalent to the thermal noise spectrum with the temperature $T = \alpha/(2\pi)$, where $\alpha$ is the acceleration of the detector. Therefore one might think that the existence of the FD-relation is trivial. In general spacetime dimensions, however, there are differences between the thermal noise and the Rindler noise, which is well recognized by previous investigations. For example, Takagi found the 'statistical inversion' phenomenon [5], that is, the Rindler noise spectrum of massless particle in odd dimension becomes the Fermi-Dirac distribution instead of the Plank one, on the contrary to naive expectation.

In the meanwhile, de Sitter spacetime is one of the most famous curved spacetime which has been studied by many theorists, e.g., [9,10]. The reason of this comes from the fact that de Sitter spacetime is the unique maximally symmetric curved spacetime and also that quantum fluctuations in de Sitter spacetime is considered to be the origin of the cosmic structure in the inflationary universe scenario. In the quantum field theory in the de Sitter spacetime, it is well-known that the vacuum noise exhibits similar character of the Rindler noise including the phenomenon of the statistical inversion. Motivated by the investigation by Terashima, we apply his method to the vacuum noise in the de Sitter spacetime. We will show that the FD relation of Callen and Welton holds for the de Sitter noise of the scalar and the Dirac fields in $N$-dimensional de Sitter spacetime following the prescription done in the Rindler spacetime by Terashima. This practice is useful not only to understand the reason why Terashima’s method works properly irrespective of the existence of the statistical inversion but also to clarify the relation between the Terashima’s investigation and previous investigations of the FD relation of the vacuum noise in curved spacetime [5–8], in which the periodicity of Green functions in imaginary time is attributed to the thermal nature. Our investigation is also instructive to demonstrate the validity of the use of the retarded Green function to evaluate the dissipative coefficient from the classical equations of motion, which gives a principle to avoid singular poles in the Terashima’s prescription.

This paper is organized as follows: In §2 we calculate the dissipative coefficient of the detector from the classical equations of motion of the detector and the fields in $N$-dimensional de Sitter spacetime. Then we explicitly show that the bosonic and fermionic FD relations
exist between the vacuum noise and the dissipative coefficient. In §3 we rephrase the calculations in §2 from a general point of view by summarizing relations that hold between the Green functions, where key conditions so that the prescription of Terashima works properly are made clear. §4 is devoted to summary and discussions. Throughout this paper we use the natural unit $c = \hbar = k_B = 1$, and adopt the convention $(1, -1, -1, -1)$ for metric signature and curvature tensor and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ for gamma matrices [10].

II. EXPLICIT CALCULATION OF THE FD-RELATION IN DE SITTER SPACETIME

A. real scalar field

In this section we apply the Terashima’s prescription to the $N$-dimensional de Sitter spacetime. We first consider a real scalar field $\phi$ with mass $m$ and a detector which couples linearly with the field with the action:

$$S = S_0(q) + S_{int}(q, \phi) + S_0(\phi),$$

where

$$S_0(q) = \int d\tau L(q, \dot{q}),$$

$$S_0(\phi) = \int d^n x \sqrt{-g} \frac{1}{2} \left\{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (m^2 + \xi R)\phi^2 \right\},$$

$$S_{int}(q, \phi) = \int d\tau d^n x q(\tau) \phi(x) \delta^n(x - x_0(\tau)) = \int d\tau q(\tau) \phi(x_0(\tau)),$$

where $L$ denotes the Lagrangian of the degree of freedom of the detector $q(\tau)$, $\tau$ is the proper time, $x_0(\tau)$ denotes the world line of the detector, $R$ is the Ricci scalar, $\xi$ is the non-minimal coupling constant, and $\delta^n(x - y)$ is the $n$-dimensional Dirac delta function. From this action we have equations of motion,

$$\left( \frac{\delta S_0}{\delta \dot{q}} \right)(\tau) + \int d^n x \delta^n(x - x_0(\tau)) \phi(x) = 0,$$

$$\left( \nabla^\mu \nabla_\mu + m^2 + \xi R \right) \phi(x) = \frac{1}{\sqrt{-g}} \int d\tau q(\tau) \delta^n(x - x_0(\tau)).$$
where we defined

$$\left( \frac{\delta S_0}{\delta \phi} \right) (\tau) \equiv - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\phi}} \right) + \frac{\partial L}{\partial \phi}. \quad (2.7)$$

The line element of the $N$-dimensional de Sitter spacetime can be written as

$$ds^2 = dt^2 - e^{2Ht} \sum_{m=1}^{n-1} (dx^m)^2 = (-H\eta)^{-2} \eta_{\mu \nu} dx^\mu dx^\nu, \quad (2.8)$$

where $H$ is the Hubble parameter, $\eta_{\mu \nu}$ is the metric of the flat spacetime, and we introduced the conformal time $\eta$ in the last equality, which is related to the cosmic time $t$ as

$$\eta \equiv - \frac{1}{H} e^{-Ht}, \quad (-\infty < \eta < 0). \quad (2.9)$$

For simplicity let us consider the massless field with the conformal coupling and the detector at rest of the coordinate $(t = \tau, \mathbf{x} = \mathbf{x}_0)$, where $\mathbf{x} = (x^1, x^2, \cdots, x^{n-1})$. In this case equations of motion (2.5) and (2.6) can be rewritten as

$$\left( \frac{\delta S_0}{\delta \phi} \right) (\tau) + (-H\eta(\tau))^{n/2-1} \tilde{\phi}(\mathbf{x}_0) = 0, \quad (2.10)$$

$$\eta^{\mu \nu} \partial_\mu \partial_\nu \phi = (-H\eta)^{n/2-2} \delta^{n-1}(\mathbf{x} - \mathbf{x}_0) q(t(\eta)), \quad (2.11)$$

by introducing the conformal field $\tilde{\phi}$, which is defined with a conformal weight as

$$\phi = (-H\eta)^{n/2-1} \tilde{\phi}. \quad (2.12)$$

Next we consider equation (2.11). According to the Terashima’s method, equations for Fourier coefficients are first derived, then the equations are solved. To obtain the dissipative coefficient, he used an $i\epsilon$-prescription. Namely, imaginary part arising from this $i\epsilon$ prescription leads to the dissipative coefficient. To solve equation (2.11), we here adopt the Green function method. It is natural to adopt the retarded Green function when treating classical equations of motion. Terashima’s method is equivalent to adopting this Green function. The retarded Green function satisfies the equation

$$\left( \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \mathbf{x}^2} \right) G_R(x, x') = \delta^n(x - x'), \quad (2.13)$$
and the boundary condition \( \check{G}_R(x, x') = 0 \) for \( \eta - \eta' < 0 \). Then equation (2.11) can be solved, and \( \check{\phi}(x) \) can be written,

\[
\check{\phi}(x) = \int_{-\infty}^{0} d\eta' \int_{-\infty}^{\infty} d^{n-1}x' \check{G}_R(x, x') \left(-H\eta'\right)^{n/2-2} \delta^{n-1}(x' - x_0)q(t'(\eta')) \\
= - \int_{-\infty}^{0} d\eta' \int_{-\infty}^{\infty} \frac{d^n p}{(2\pi)^n} \frac{e^{ip(\eta-\eta')}}{p^2} (-H\eta')^{n/2-2} q(t'(\eta')), 
\]

(2.14)

where \( p^2 = (p^0)^2 - |\mathbf{p}|^2 \), and the contour of the integral of \( p^0 \) is specified in Figure 1 (see e.g., [10]). We substitute (2.14) into (2.10), then we obtain

\[
\left( \frac{\delta S_0}{\delta q} \right) (\tau) - (-H\eta(\tau))^{n/2-1} \int_{-\infty}^{0} d\eta' \int_{-\infty}^{\infty} \frac{d^n p}{(2\pi)^n} \frac{e^{ip(\eta-\eta')}}{p^2} (-H\eta')^{n/2-2} q(t'(\eta')) = 0. 
\]

(2.15)

After inserting the relation

\[
q(\tau') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \check{q}(\omega') e^{i\omega'\tau'},
\]

(2.16)

into (2.15), the Fourier transformation of equation (2.15) yields

\[
\left( \frac{\delta \tilde{S}_0}{\delta q} \right) (\omega) - \int_{-\infty}^{0} d\eta \int_{-\infty}^{0} d\eta' \int_{-\infty}^{\infty} \frac{d^n p}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \\
\times (-H\eta)^{n/2-2} (-H\eta')^{n/2-2} \check{q}(\omega') \frac{e^{i(\omega'\tau' - \omega\tau)}}{p^2} e^{ip(\eta-\eta')} = 0,
\]

(2.17)

where we defined

\[
\left( \frac{\delta \tilde{S}_0}{\delta q} \right) (\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \left( \frac{\delta S_0}{\delta q} \right) (\tau).
\]

(2.18)

With the use of the relation

\[
\int_{-\infty}^{\infty} dp^0 \frac{e^{ip(\eta-\eta')}}{p^2} = \theta(\eta - \eta') \frac{i\pi}{|\mathbf{p}|} \left\{ e^{i|\mathbf{p}|(\eta-\eta')} - e^{-i|\mathbf{p}|(\eta-\eta')} \right\},
\]

(2.19)

we have

\[
\left( \frac{\delta S_0}{\delta q} \right) (\omega) - i \frac{\pi^{(n-1)/2}}{(2\pi)^n \Gamma((n-1)/2)} \int_{-\infty}^{0} d\eta \int_{-\infty}^{0} d\eta' (-H\eta)^{n/2-2} (-H\eta')^{n/2-2} \\
\times \int_{-\infty}^{\infty} d\omega' \check{q}(\omega') e^{i(\omega'\tau' - \omega\tau)} \theta(\eta - \eta') \int_{0}^{\infty} dp \ p^{n-3} \left\{ e^{i|\mathbf{p}|(\eta-\eta' + i\epsilon)} - e^{-i|\mathbf{p}|(\eta-\eta' - i\epsilon)} \right\} = 0.
\]

(2.20)

where we inserted \( \pm i\epsilon \) for convergence of \( p \)-integration. Integration with respect to \( p \) yields
\[
\left( \frac{\delta S_0}{\delta q} \right)(\omega) - \frac{i\pi^{-(n+1)/2}}{2^n} \frac{\Gamma(n-2)}{\Gamma((n-1)/2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau d\tau' (-H\eta)^{(n-2)/2} (-H\eta')^{(n-2)/2} \\
\times \int_{-\infty}^{\infty} d\omega' \hat{q}(\omega') e^{i(\omega'\tau' - \omega\tau)} \left\{ i^{2-n} \frac{\eta - \eta'}{(\eta + \eta' - i\epsilon)^{n-2}} - i^{2-n} \frac{\eta - \eta' + i\epsilon}{(\eta + \eta' + i\epsilon)^{n-2}} \right\} = 0, \quad (2.21)
\]

where we used the next relations,
\[
\int_0^\infty x^n e^{-\alpha x} \, dx = \frac{\Gamma(n-1)}{\alpha^{n+1}}, \quad (2.22)
\]
\[
\Gamma(2z) = \frac{2^z}{2\pi^{1/2}} \Gamma(z) \Gamma(z + 1/2), \quad (2.23)
\]
and \(d\eta' = (-H\eta')d\tau'\). Then, by using the relation
\[
\frac{(-H\eta)^{(n-2)/2} (-H\eta')^{(n-2)/2}}{(\eta - \eta' - i\epsilon)^{n-2}} = \frac{H^{n-2}}{\{2 \sinh (H(\tau - \tau')/2) - i\epsilon\}^{n-2}}, \quad (2.24)
\]
and by introducing the variables
\[
\Delta\tau \equiv \tau - \tau', \quad T \equiv \frac{\tau + \tau'}{2}, \quad (2.25)
\]
we obtain the following equation after integrations
\[
\left( \frac{\delta S_0}{\delta q} \right)(\omega) + K(\omega) \hat{q}(\omega) = 0, \quad (2.26)
\]
where
\[
K(\omega) = \frac{i\Gamma(n/2 - 1)}{4\pi^{n/2}} \left( \frac{H}{i} \right)^{n-2} \int_0^\infty d(\Delta\tau) \ e^{-i\omega \Delta\tau} \\
\times \left[ \left\{ 2 \sinh \frac{H}{2} \Delta\tau - i\epsilon \right\}^{2-n} - (-1)^n \left\{ 2 \sinh \frac{H}{2} \Delta\tau + i\epsilon \right\}^{2-n} \right]. \quad (2.27)
\]
According to [1,3], the dissipative coefficient \(R(\omega)\) is obtained by
\[
R(\omega) = \frac{-1}{\omega} \text{Im}[K(\omega)]. \quad (2.28)
\]
After integration the dissipative coefficient is explicitly obtained from equation (2.27) as
\[
R(\omega) = \frac{1}{2^{n-1}\pi^{(n-3)/2}} \frac{\omega^{n-4}}{\Gamma((n-1)/2)} \ \exp \left( \frac{2\pi\omega}{H} \right) - \frac{1}{\exp \left( \frac{2\pi\omega}{H} \right) - (-1)^n}. \quad (2.29)
\]
On the other hand, following the formula by Callen and Welton [2], the FD relation for the thermal noise is written as
\begin{equation}
\langle \phi(x)\phi(x) \rangle \equiv \int_0^\infty dE \ \rho(E)f(E) \ \langle E|\phi(x)\phi(x)|E \rangle \\
= \frac{2}{\pi} \int_0^\infty E(\omega,T)R(\omega) \ d\omega, \tag{2.30}
\end{equation}

where \( \rho(E) \) is the density of states at the energy \( E \) and \( f(E) \) is the normalized Boltzmann factor, which satisfies

\begin{equation}
\frac{f(E + \omega)}{f(E)} = \exp \left( -\frac{\omega}{T} \right) \tag{2.31}
\end{equation}

with the temperature \( T \), and \( E(\omega, T) \) is defined by

\begin{equation}
E(\omega, T) = \frac{1}{2} \omega + \frac{\omega}{\epsilon^{\omega/T} - 1}. \tag{2.32}
\end{equation}

Here \( E(\omega, T) \) can be regarded as the energy of a Bosonic harmonic oscillator with zero-point oscillation at the temperature \( T \).

We can easily show that the FD relation (2.30) is satisfied between the vacuum fluctuations and the dissipative coefficient. The quantum field theory of a real scalar field in de Sitter spacetime has been well investigated [11]. In the case of the massless and conformally coupling, it is well-known that the two-point function is in proportion to that in the Minkowski spacetime. With the use of equation (2.24), the Wightman function leads to

\begin{equation}
\langle 0_c|\phi(x(\tau))\phi(x'(\tau'))|0_c \rangle = \frac{\Gamma(n/2 - 1)}{4\pi^{n/2}} \left( \frac{H}{i} \right)^{n-2} \left\{ 2 \sinh \frac{H}{2} \Delta \tau - i\epsilon \right\}^{2-n}, \tag{2.33}
\end{equation}

where \( |0_c \rangle \) denotes the conformal vacuum. With the relation

\( \delta(\tau - \tau') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(\tau - \tau')} \),

the fluctuation of the real scalar field is expressed as follows

\begin{equation}
\langle 0_c|\phi(x)^2|0_c \rangle = \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(\tau - \tau')} \langle 0_c|\phi(x(\tau))\phi(x'(\tau'))|0_c \rangle \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_n(\omega) \ d\omega, \tag{2.34}
\end{equation}

where the power spectrum, \( F_n(\omega) \), is defined by

\begin{equation}
F_n(\omega) = \int_{-\infty}^{\infty} d\tau' e^{-i\omega(\tau - \tau')} \langle 0_c|\phi(x(\tau))\phi(x'(\tau'))|0_c \rangle. \tag{2.35}
\end{equation}
Inserting equation (2.33) into (2.35) we obtain
\[
F_n(\omega) = \frac{1}{2^{n-2}\pi^{(n-3)/2}} \frac{\omega^{n-3}}{\Gamma((n-1)/2)} \exp\left(\frac{2\pi\omega}{H}\right) - (-1)^n.
\] (2.36)

We finally conclude the existence of the FD relation in \(N\)-dimensional de Sitter spacetime
\[
\langle 0_c|\phi(x(\tau))|0_c \rangle = \frac{2}{\pi} \int_0^\infty E(\omega, H/2\pi)R(\omega)d\omega.
\] (2.37)

From equation (2.36) we can see that the statistical inversion occurs as in the case of the \(N\)-dimensional Rindler spacetime [1]. The FD relation (2.37) holds irrespective of the statistical inversion.

### B. Dirac field

Following the investigation by Terashima [1], we next consider the system of a Dirac field \(\psi\) and a detector of a spinor \(\Theta\) which is linearly coupled with the field. The action of the system is
\[
S = S_0(\Theta, \bar{\Theta}) + S_{\text{int}}(\Theta, \bar{\Theta}, \psi, \bar{\psi}) + S_0(\psi, \bar{\psi}),
\]
where we defined
\[
S_0(\Theta, \bar{\Theta}) = \int d\tau L(\Theta, \bar{\Theta}, \dot{\Theta}, \dot{\bar{\Theta}}),
\] (2.38)
\[
S_0(\psi, \bar{\psi}) = \int d^n x (\det V) \left[ \frac{i}{2} \left\{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) \gamma^\mu \psi \right\} - m\bar{\psi}\psi \right],
\] (2.39)
\[
S_{\text{int}}(\Theta, \bar{\Theta}, \psi, \bar{\psi}) = \int d\tau \left\{ \bar{\Theta}(\tau) \psi(x(\tau)) + \bar{\psi}(x(\tau)) \Theta(\tau) \right\}
= \int d\tau d^n x \left( \bar{\Theta}(\tau) \psi(x) + \bar{\psi}(x) \Theta(\tau) \right) \delta^n(x - x_0(\tau)),
\] (2.40)
where \(\Theta(\tau)\) and \(\bar{\Theta}(\tau) = \Theta(\tau)^\dagger \gamma^0\) are Grassmann numbers, and \(\nabla_\mu\) is defined by
\[
\nabla_\mu \equiv \partial_\mu + \Gamma_\mu, \quad \Gamma_\mu = \frac{1}{2} \Sigma^{\alpha\beta} V_\alpha^\nu(x)(\nabla_\mu V_{\beta\nu}),
\] (2.41)
with the vierbein $V_\alpha^\nu$ and the generator of Lorentz transformation $\Sigma^{\alpha\beta}$ [10]. Equations of the motion are written

$$\left(\frac{\delta S_0}{\delta \Theta}\right)(\tau) + \int d^n x \psi(x) \delta^n(x - x_0(\tau)) = 0, \quad (2.42)$$

$$(i\nabla - m)\psi = \frac{-1}{\det V} \int d\tau \Theta(\tau) \delta^n(x - x(\tau)), \quad (2.43)$$

where, as in the case of the scalar field, we defined

$$\left(\frac{\delta S_0}{\delta \Theta}\right)(\tau) \equiv -\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\Theta}}\right) + \frac{\partial L}{\partial \Theta}.$$

For simplicity we restrict $\psi$ is the massless field in this section. Similar to the case of the scalar field, we introduce the conformal field $\psi_c$ defined with a conformal weight as

$$\psi(x) = (-H\eta)^{(n-1)/2} \psi_c(x). \quad (2.44)$$

In terms of $\psi_c$, equations (2.42) and (2.43) reduce to

$$\left(\frac{\delta S_0}{\delta \Theta}\right)(\tau) + (-H\eta(\tau))^{(n-1)/2} \psi_c(x_0(\tau)) = 0, \quad (2.45)$$

and

$$i\tilde{\psi}_c(x) = -(-H\eta)^{(n-3)/2} \Theta(\tau(\eta)) \delta^{n-1}(x - x_0), \quad (2.46)$$

respectively, where we defined $\tilde{\psi} \equiv \gamma^0 \partial_0 + \gamma^i \partial_i$ with $(i = 1, 2, \ldots, n - 1)$, and assumed the detector at rest of the spatial coordinate $x = x_0$. As is done in the previous subsection, equation (2.46) can be solved by using the retarded Green function, which we denote by $S_R(x, x')$, as follows,

$$\psi_c(x) = \int_{-\infty}^{0} d\eta' \int_{-\infty}^{\infty} d^{n-1} x' S_R(x, x')(\eta'(\tau')) (\eta(\tau'))^{(n-3)/2} \Theta(\tau(\eta')) \delta^{n-1}(x' - x_0), \quad (2.47)$$

where we assume that the retarded Green function satisfies

$$i\tilde{\psi}_c(x, x') = -\delta^n(x - x'), \quad (2.48)$$

and the boundary condition $S_R(x, x') = 0$ for $\eta - \eta' < 0$. Substituting equation (2.47) into (2.45), we have
\[ \left( \frac{\delta S_0}{\delta \Theta} \right)(\tau) = - \int_{-\infty}^{\infty} d\tau' \bar{S}_R(x_0(\tau), x_0(\tau')) \left( -H \eta'(\tau') \right)^{(n-1)/2} \left( -H \eta(\tau) \right)^{(n-1)/2} \Theta(\tau'). \quad (2.49) \]

The Fourier transformation of equation (2.49) yields
\[
\left( \frac{\delta \tilde{S}_0}{\delta \tilde{\Theta}} \right)(\omega) = - \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega' \tilde{\Theta}(\omega') e^{i(\omega' \tau' - \omega \tau)} \times \bar{S}_R(x_0(\tau), x_0(\tau')) \left( -H \eta'(\tau') \right)^{(n-1)/2} \left( -H \eta(\tau) \right)^{(n-1)/2},
\]

where the Fourier coefficients \( \tilde{\Theta}(\omega') \) and \( (\delta \tilde{S}_0/\delta \tilde{\Theta})(\omega) \) are defined in similar ways to (2.16) and (2.18), respectively. The massless Green function on the flat spacetime is well-known, and we can write down \( \bar{S}_R(x, x') \) in an explicit form. With the use of the relation (2.24), we can write
\[
\bar{S}_R(x_0(\tau), x_0(\tau')) \left( -H \eta(\tau) \right)^{(n-1)/2} \left( -H \eta'(\tau') \right)^{(n-1)/2} = i \gamma^0 \frac{\Gamma(n/2)}{2 \pi^{n/2}} \left( \frac{H}{i} \right)^{n-1} \times \theta(\Delta \tau) \left\{ 2 \sinh \frac{H}{2} \Delta \tau - i \epsilon \right\}^{1-n} - (-1)^n \left\{ 2 \sinh \frac{H}{2} \Delta \tau + i \epsilon \right\}^{1-n}. \quad (2.51)
\]

Keeping this relation in mind, after integration using the variables \( T \) and \( \Delta \tau \) defined by equation (2.25), equation (2.50) yields
\[
\left( \frac{\delta \tilde{S}_0}{\delta \tilde{\Theta}} \right)(\omega) = - \int d(\Delta \tau) \bar{S}_R(x_0(\tau), x_0(\tau')) \left( -H \eta'(\tau') \right)^{(n-1)/2} \left( -H \eta(\tau) \right)^{(n-1)/2} e^{-i \omega \Delta \tau} \tilde{\Theta}(\omega). \quad (2.52)
\]

Then we obtain the effective equation of motion for the Fourier coefficient
\[
\left( \frac{\delta \tilde{S}_0}{\delta \tilde{\Theta}} \right)(\omega) + K_{1/2}(\omega) \tilde{\Theta}(\omega) = 0,
\]

where
\[
K_{1/2}(\omega) = i \gamma^0 \frac{\Gamma(n/2)}{2 \pi^{n/2}} \left( \frac{H}{i} \right)^{n-1} \int_{-\infty}^{\infty} d(\Delta \tau) e^{-i \omega \Delta \tau} \times \left\{ 2 \sinh \frac{H}{2} \Delta \tau - i \epsilon \right\}^{1-n} - (-1)^n \left\{ 2 \sinh \frac{H}{2} \Delta \tau + i \epsilon \right\}^{1-n}. \quad (2.54)
\]

In the similar way to equation (2.28), the dissipative coefficient is defined (c.f., refs. [1,3]),
\[
R_{1/2}(\omega) = - \frac{1}{\omega} \text{Im}[K_{1/2}(\omega)]. \quad (2.55)
\]

After integration, we obtain

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\[ R_{1/2}(\omega) = -\gamma^0 \frac{\omega^{n-3}}{2^{n-1}\pi^{(n-3)/2}} \frac{1}{\Gamma((n-1)/2)} \frac{\exp(2\pi\omega/H) + 1}{\exp(2\pi\omega/H) + (-1)^n}. \]  

(2.56)

A fermionic version of the FD relation is presented in [3], which is applied to the system of the Dirac field and the detector in the Rindler spacetime in [1]. The fermionic FD relation is written

\[
\text{Tr} \left[ \gamma_0 \langle \psi(x) \bar{\psi}(x) \rangle \right] = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{-\omega}{e^{-\omega/T} + 1} \text{Tr} \left[ \gamma_0 R_{1/2}(\omega) \right].
\]  

(2.57)

If the dissipative coefficient satisfies \( R_{1/2}(-\omega) = -R_{1/2}(\omega) \), equation (2.57) can be rewritten

\[
\text{Tr} \left[ \gamma_0 \langle \psi(x) \bar{\psi}(x) \rangle \right] = -\frac{1}{\pi} \int_{0}^{\infty} \omega \text{Tr} \left[ \gamma_0 R_{1/2}(\omega) \right] d\omega.
\]  

(2.58)

On the other hand, quantum field theory of the Dirac field in de Sitter spacetime is well understood. In the case of the massless field, the Wightman function in the \( N \)-dimensional de Sitter spacetime is written [5,10]

\[
\langle 0| \psi(x) \bar{\psi}(x')|0 \rangle = \gamma^0 \frac{\Gamma(n/2)}{2\pi^{n/2}} \left( \frac{H}{i} \right)^{n-1} \left\{ 2 \sinh \frac{H}{2} \Delta \tau - i\epsilon \right\}^{1-n}.
\]  

(2.59)

Similar to the case of the scalar field, we write the fluctuation of the Dirac field as

\[
\text{Tr} \left[ \gamma_0 \langle 0| \psi(x) \bar{\psi}(x')|0 \rangle \right] = \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(\tau-\tau')}}{2\pi} \text{Tr} \left[ \gamma_0 \langle 0| \psi(x) \bar{\psi}(x')|0 \rangle \right]
\]

\[
\equiv \frac{\Delta_n}{2\pi} \int_{-\infty}^{\infty} F_{1/2}(\omega) d\omega,
\]  

(2.60)

where \( \Delta_n \) is the dimensions of the \( \gamma \)-matrices, and the power spectrum is defined as a scalar quantity by

\[
F_{1/2}(\omega) \equiv \frac{1}{\Delta_n} \int_{-\infty}^{\infty} d\tau' e^{-i\omega(\tau-\tau')} \text{Tr} \left[ \gamma_0 \langle 0| \psi(x) \bar{\psi}(x')|0 \rangle \right].
\]  

(2.61)

Inserting equation (2.59) into (2.61), we obtain

\[
F_{1/2}(\omega) = \frac{\omega^{n-2}}{2^{n-2}\pi^{(n-1)/2}} \frac{1}{\Gamma((n-1)/2)} \frac{1}{\exp(2\pi\omega/H) + (-1)^n}. \]

(2.62)

This means the existence of the FD relation (2.57) or (2.58) with the dissipative coefficient (2.56). From equation (2.62) we can see that the statistical inversion occurs in the fermionic case too, however, the FD-relation exists irrespective of the statistical inversion.
III. ESSENTIAL CONDITIONS FOR THE FD RELATION

A. real scalar field

In the previous section we have derived the FD relation in an explicit manner, according to the method proposed by Terashima [1]. In the previous section, however, we have considered the simple cases of massless fields in de Sitter spacetime. In this section we summarize the calculations in the previous section from a general point of view, which clarifies why the Terashima’s prescription works properly. Let us start from considering the classical equations of motion of the scalar field and the detector (2.5) and (2.6). As we have done in the previous section, the field equation (2.6) is formally solved by introducing the retarded Green function, which satisfies

\[
\left( \nabla_x^\mu \nabla_\mu + m^2 + \xi R \right) G_R(x,x') = \frac{1}{\sqrt{-g}} \delta^n(x-x'),
\]

to give

\[
\phi(x) = \int d^nx' G_R(x,x') \int d\tau q(\tau) \delta^n(x'-x_0(\tau)).
\]

Inserting this into equation (2.5), we have

\[
\left( \frac{\delta S_0}{\delta q} \right)(\tau) + \int d\tau' G_R(x_0(\tau), x_0(\tau')) q(\tau') = 0.
\]

Fourier transformation of this equation (3.3) yields the same expression as (2.26) but with

\[
K(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\omega' G_R(x(\tau), x'(\tau')) e^{-i(\omega\tau - \omega'\tau')},
\]

where the Fourier coefficients \( \tilde{q}(\omega') \) and \( (\tilde{\delta S_0}/\tilde{\delta q})(\omega) \) are defined by (2.16) and (2.18), respectively. In the case that the retarded Green function is the function of \( \Delta \tau (= \tau - \tau') \), i.e., \( G_R(x(\tau), x'(\tau')) = G_R(\Delta \tau) \), equation (3.4) reduces to

\[
K(\omega) = \int_{-\infty}^{\infty} d(\Delta \tau) \ G_R(\Delta \tau)e^{-i\omega \Delta \tau}.
\]

The dissipative coefficient defined by (2.28) becomes
\[ R(\omega) = -\frac{1}{\omega} \text{Im} \left[ \int_{-\infty}^{\infty} d(\Delta \tau) G_R(\Delta \tau) e^{-i\omega \Delta \tau} \right]. \] (3.6)

In the meanwhile, it is well-known that, in the thermal field theory, the Wightman functions satisfy a periodicity condition in the direction of imaginary time, which is called the Kubo-Martin-Schwinger (KMS) condition [12,13]. In the quantum field theory in curved spacetime, we assume that the Wightman functions, which are defined by

\[ G^+(\Delta \tau) \equiv \langle 0|\phi(x(\tau))\phi(x'(\tau'))|0 \rangle, \] (3.7)
\[ G^-(\Delta \tau) \equiv \langle 0|\phi(x'(\tau'))\phi(x(\tau))|0 \rangle, \] (3.8)
satisfy the KMS condition on the complex plane of the detector’s proper time,

\[ G^+(\Delta \tau - i\beta) = G^-(-\Delta \tau), \] (3.9)

where \( \beta \) is a period of the periodicity in the direction of imaginary time. In the thermal field theory, \( \beta = 1/T. \) For the massless (conformally coupled) scalar field, assuming a detector at rest in de Sitter spacetime, the Wightman functions are

\[ G^\pm(\Delta \tau) = \frac{\Gamma(n/2 - 1)}{4\pi^{n/2}} \frac{H^{n-2}}{(\pm i)^{n-2}} \frac{1}{2 \sinh(H \Delta \tau/2) \mp i\epsilon} \right]^{n-2}, \] (3.10)
hence we can easily check that the Wightman functions of the conformal vacuum state satisfy the KMS condition with \( \beta = 2\pi/H. \)

Following the investigations by Takagi [5] and Ooguri [6], let us introduce the Fourier coefficients, which are defined by

\[ F^\pm(\omega) \equiv \int_{-\infty}^{\infty} G^\pm(\Delta \tau) e^{-i\omega \Delta \tau} d(\Delta \tau). \] (3.11)

Because of the definition, the Wightman functions are related to each other as \( G^+(\Delta \tau) = G^-(-\Delta \tau). \) This relation is equivalent to \( F^+(\omega) = F^-(\omega). \) On the other hand, the result of the KMS condition (3.9) for the Fourier coefficients is derived as follows: By extending \( \Delta \tau \)-integration in (3.11) to the complex plane, we can deform the path of the integration from \( C_1 \) on the real axis to \( C_2 \) on the complex plane (see Figure 2),
\[ F^+(\omega) = \int_{C_2} G^+(z)e^{-i\omega z}dz. \quad (3.12) \]

This can be done unless a singular pole appears in the region between the paths \( C_1 \) and \( C_2 \) in Figure 2. Then, using the KMS condition, (3.12) is written as [5,6],

\[ F^+(\omega) = e^{-\beta\omega} \int_{-\infty}^{\infty} G^-(\Delta\tau)e^{-i\omega \Delta\tau}d(\Delta\tau) = e^{-\beta\omega}F^-(\omega). \quad (3.13) \]

The retarded Green function is related with the Wightman functions (e.g., [10]),

\[ G_R(x, x') = i\theta(t - t') \left( G^+(x, x') - G^-(x, x') \right). \quad (3.14) \]

Inserting this relation into equation (3.6), we obtain the dissipative coefficient

\[ R(\omega) = \frac{e^{\beta\omega} - 1}{2\omega}F^+(\omega), \quad (3.15) \]

where we used (3.13) and \( G^+(-\Delta\tau) = G^-(\Delta\tau) \), which is equivalent to \( F^+(-\omega) = F^-(\omega) \).

Finally by using equation (3.15) the FD relation is derived as an identity

\[ \langle 0|\phi(x)^2|0 \rangle \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} F^+(\omega) \, d\omega = \frac{2}{\pi} \int_{0}^{\infty} E(\omega, 1/\beta)R(\omega) \, d\omega. \quad (3.16) \]

### B. Massive scalar field

Following the above argument, the derivation of the FD relation can be generalized to the real scalar field with any mass \( m \) in de Sitter spacetime. The Wightman functions for the massive field in \( N \)-dimensional de Sitter spacetime is

\[ G_{\pm}(x, x') = \frac{H^{n-2}}{(4\pi)^{n/2}} \frac{\Gamma(a_+)\Gamma(a_-)}{\Gamma(c)} \, _2F_1(a_+, a_-, c; 1 + z_{\pm}), \quad (3.17) \]

where

\[ a_{\pm} = \frac{1}{2} \left( n - 1 \pm \sqrt{(n - 1)^2 - 4m^2/H^2} \right), \quad c = \frac{n}{2}, \quad (3.18) \]

and \( z_{\pm} \) is in proportion to the geodesic distance between the two-points \( x \) and \( x' \), which is written using the coordinate of the spatially flat chart.
$$z_{\pm} = \frac{(\eta - \eta' \mp i\epsilon)^2 - |x - x'|^2}{4\eta \eta'}.$$  \hfill (3.19)

For the two-points on a geodesics of the detector, \(x(\tau)\) and \(x'(\tau')\), we have

$$z_{\pm} = \left( \sinh \frac{H}{2} \Delta \tau \mp i\epsilon \right)^2.$$  \hfill (3.20)

Hence it is apparent that the Wightman functions satisfy the periodicity condition (3.9).

A delicate condition in deriving the FD relation might be the structure of singular poles of the Wightman functions on the complex plane of the imaginary time. If \(G^+(\Delta \tau)\) has additional singular poles on the region between the contours \(C_1\) and \(C_2\) (see Fig. 2), equation (3.13 or 3.31) can not be satisfied. For a physical situation, it would be expected that the singular pole of \(G^+(x, x')\) appears only at \(x = x'\). Hence the condition for the singular poles of the Wightman functions would be satisfied. Actually in the case of the de Sitter spacetime, the Wightman functions satisfy this condition because of a property of the hypergeometric function in (3.17). Thus the FD relation holds between the dissipative coefficient of the detector’s motion and the vacuum noise of the field with any mass as long as the detector moves along a geodesics.

C. Dirac field

We next consider the case of the Dirac field. We show that the argument similar to the case of the scalar field holds. Let us start with equations of motion (2.42) and (2.43). Equation (2.42) is solved by introducing the retarded Green function, which satisfies equation

$$(i\nabla_x - m) S_R(x, x') = \frac{-1}{\det V} \delta^n(x - x'),$$  \hfill (3.21)

as

$$\psi(x) = \int d^n x' S_R(x, x') \int d\tau \Theta(\tau) \delta^n(x' - x_0(\tau)).$$  \hfill (3.22)

Inserting this expression into equation (2.42), we have
\[ \left( \frac{\delta S_0}{\delta \Theta} \right)(\tau) + \int d\tau' S_R(x_0(\tau), x'_0(\tau')) \Theta(\tau') = 0. \] (3.23)

In the similar way to the case of the scalar field, we assume that the retarded Green function is a function of only \( \Delta \tau \), i.e., \( S_R(x_0(\tau), x'_0(\tau')) = S_R(\Delta \tau) \). In this case the Fourier transformation of this equation yields the same expression as (2.53) with

\[ K_{1/2}(\omega) = \int_{-\infty}^{\infty} d(\Delta \tau) \ S_R(\Delta \tau)e^{-i\omega \Delta \tau}. \] (3.24)

Hence the dissipative coefficient defined by (2.55) is

\[ R_{1/2}(\omega) = -\frac{1}{\omega} \text{Im} \left[ \int_{-\infty}^{\infty} d(\Delta \tau) \ S_R(\Delta \tau)e^{-i\omega \Delta \tau} \right]. \] (3.25)

For the Dirac field we define

\[ S^+(\Delta \tau) \equiv \langle 0 | \psi(x(\tau)) \bar{\psi}(x'(\tau')) | 0 \rangle, \] (3.26)

\[ S^-(\Delta \tau) \equiv \langle 0 | \bar{\psi}(x'(\tau')) \psi(x(\tau)) | 0 \rangle, \] (3.27)

and assume that these functions satisfy the KMS condition,

\[ S^+(\Delta \tau - i\beta) = S^-(-\Delta \tau). \] (3.28)

By the definition of the functions \( S^\pm(\Delta \tau) \), we have

\[ S^-(\Delta \tau) = S^+(\Delta \tau) \big|_{m \rightarrow -m}, \] (3.29)

where we used the relation \( S^\pm(x, x') = \pm (i\nabla_x + m) G^\pm(x, x') \) and the assumption of the detector being at rest. The Fourier coefficients are defined by

\[ F^\pm_{1/2}(\omega) = \frac{1}{\Delta_n} \text{Tr} \left[ \gamma_0 \int_{-\infty}^{\infty} S^\pm(\Delta \tau)e^{-i\omega \Delta \tau} d(\Delta \tau) \right]. \] (3.30)

Then we have \( F^+_{1/2}(-\omega) = F^-_{1/2}(\omega) \) from (3.29). We repeat the argument to derive (3.13), and obtain

\[ F^+_{1/2}(\omega) = e^{-\omega \beta} F^-_{1/2}(\omega). \] (3.31)

For the Dirac field the retarded Green function should be related with \( S^\pm \) as
\[ S_R(x, x') = i\theta(t - t') \left\{ S^+(x, x') + S^-(x, x') \right\}. \]  

(3.32)

Inserting this relation into (3.25), we have the dissipative coefficient

\[ \frac{1}{\Delta_n} \text{Tr}[\gamma_0 R_{1/2}(\omega)] = - \frac{1 + e^{\beta\omega}}{2\omega} F_{1/2}^+(\omega). \]  

(3.33)

This implies the existence of the FD relation,

\[ \text{Tr}[\gamma_0 \langle 0 | \psi(x)\bar{\psi}(x) | 0 \rangle] = \frac{\Delta_n}{2\pi} \int_{-\infty}^{\infty} F_{1/2}^+(\omega) \; d\omega = -\frac{1}{\pi} \int_{0}^{\infty} \omega \text{Tr}[\gamma_0 R_{1/2}(\omega)] \; d\omega. \]  

(3.34)

\[ \text{IV. SUMMARY AND DISCUSSIONS} \]

In summary we have investigated the FD relation between the dissipative coefficients of a detector and the vacuum noise of the scalar and the Dirac fields which are linearly coupled to each other in the \( N \)-dimensional de Sitter spacetime. We have derived the dissipative coefficient from the classical equations of motion, then the existence of the FD relation has been shown in an explicit manner by inserting the dissipative coefficient into the FD relation of Callen and Welton [2]. Though our result might depend on the model of the couplings between the detector and fields, this investigation is interesting because information of quantum fluctuations can be obtained by observing the classical motion of the detector in principle. As is shown in § 3, the periodicity in the Wightman functions (the KMS condition) is essential for the existence of the FD relation. Hence the FD relation exists irrelevant to the statistical inversion on the contrary to naive expectation associated with the inversion phenomenon.

Some equations in the present paper are well-recognized so far. One is the fact that the FD relation of the vacuum noise in de Sitter spacetime has been well-known in connection with the the KMS condition of the Wightman functions (see e.g., [5] and references therein). However, our explicit derivation of the FD relation according to the Terashima prescription is instructive to understand why the Terashima’s method works properly and might give a hint as to the origin of the statistical inversion phenomenon in odd dimensions of de Sitter.
Our investigation is useful to show the key conditions so that the Terashima’s prescription works properly in general curved spacetime. Our calculation also demonstrates the validity of the use of the retarded Green function to obtain a dissipative coefficient from classical equations of motion.

It is apparent that the vacuum state of the field in curved spacetime and the boundary condition of the retarded Green function should be carefully chosen for the FD relation existing. For example if we adopt a vacuum state in de Sitter spacetime other than the Bunch-Davies vacuum, the existence of the FD relation is not guaranteed. Furthermore it should be noted that we assumed that the retarded Green function is related with the Wightman functions by equation (3.14) or (3.32) that depend on the choice of the vacuum state.

Finally we mention a connection of the FD relation with the Euclidean path integral approach. It has been pointed out that the thermal property of the Hawking radiation is traced back to the periodicity in time of the Euclidean section of the black hole spacetime [14]. That is, if a curved spacetime possesses the periodicity in its Euclidean section, then the thermal property of a field on the Lorentzian spacetime can be expected. The Bunch-Davies vacuum corresponds to the Euclidean vacuum of the de Sitter spacetime, hence the periodicity of the Wightman functions can be regarded as the result of the periodicity of Euclidean section of the de Sitter spacetime. As long as the retarded Green function relevant to the Euclidean vacuum is adopted, the prescription proposed by Terashima is supposed to work properly in the black hole spacetime.

ACKNOWLEDGMENTS

We would like to thank Misao Sasaki, Takahiro Tanaka, Masafumi Seriu, and Yasufumi Kojima for useful conversations on this topics. This research was supported by the Inamori Foundation and by the Grants-in-Aid by the Ministry of Education, Science, Sports and Culture of Japan (11640280).
REFERENCES


FIG. 1. Contour of integration on the complex plane of $p^0$ to obtain the retarded Green function.
FIG. 2. Structure of $G^+(\Delta \tau)$ on the complex plane of $\Delta \tau$. The black circles show the positions of the singular poles of $G^+(\Delta \tau)$. $C_1$ denotes the contour of the integration in (3.11), and $C_2$ denotes the contour in (3.12).