On the (im)possibility of a supersymmetric extension of NGT

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Abstract

We investigate the possibility of constructing a locally supersymmetric extension of NGT (Nonsymmetric Gravitation Theory), based on the graded extension of the Poincaré group. In the framework of the simple model that we propose, we end up with a no-go result, namely the impossibility of cancelling some linear contribution in the gravitino field. This drawback seems to seriously undermine the construction of a supergravity based on NGT.

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1 Introduction

Nonsymmetric gravitation theory (NGT) is a theory of gravitation based on a nonsymmetric metric and affine connection, first proposed by Moffat [1]. It stemmed from the Einstein unified field theory [2], the aim of which was to unify the gravitational and electromagnetic interactions in a geometric framework, by introducing a nonsymmetric "metric" tensor $g_{\mu\nu} = g(\mu\nu) + g_{[\mu\nu]}$, with the hope of relating the antisymmetric part $g_{[\mu\nu]}$ to the Maxwell field, the symmetric part describing gravitation. In contrast, the nonsymmetric metric $g_{\mu\nu}$ fully describes gravitation in NGT. A considerable amount of work has been devoted to the investigation of the phenomenological consequences of this theory [3], as well as to its geometrical interpretation [4, 5, 6]. The early version of this theory describes, in the linear approximation [7], the propagation of a massless spin 2 graviton and a scalar particle, the so called skewon, which would be the exchange particle of a long range fifth force, in addition to the gravitational one. Damour and al. [8] have assailed NGT, claiming it to be theoretically inconsistent, especially regarding the expansion of the antisymmetric part of the metric $g_{[\mu\nu]}$ about an Einstein (symmetric) background, and suggested the addition of a cosmological term to cure this pathological behaviour, thereby giving a mass to the skewon. Later on, after having responded to these criticisms in a series of papers [9], a new version (called massive NGT) was given by Moffat [10]; it differs from the earlier massless one by additional non-derivative terms in the action that defines the theory (more details will be given below). Besides the massless graviton, the resulting theory describes the propagation of a massive spin one particle associated with the antisymmetric part of the metric [11].

Our aim in this work is to investigate the possibility of building a supergravity based on NGT, that is a locally supersymmetric field theory having NGT as its bosonic sector (or part of it). Ordinary supergravity [12] is the gauge theory of the supersymmetric extension of the Poincaré group [13], the local invariance group of general relativity. In this connection, one should use the supersymmetric extension of the local group of invariance of NGT, that is $U(3, 1, H)$, the (pseudo)unitary group of matrices with elements in the ring of hyperbolic complex numbers $H$ [5, 6]. These numbers are introduced owing to the isomorphism of $U(3, 1, H)$ and $GL(4, R)$ [5]. However, it is well known that the spinorial representations (inherent in a supersymmetric extension) of $GL(4, R)$ are infinite dimensional [14]. To circumvent this technical (as well as physical, in the interpretation in terms of particles)
difficulty, one can take advantage of the fact that the homogeneous Lorentz group is the real subgroup of $U(3,1,H)$. Thus, we restrict ourselves to the investigation of the possibility of the construction of a minimal extension of NGT invariant under the supersymmetric extension of the Poincaré group. By minimal, we mean the achievement of invariance without additional "kinetic" (that is containing derivatives) terms to the NGT Lagrangian.

In section 2, we briefly review the field structure of NGT, with emphasis on the role of the group $U(3,1,H)$ in the anholonomic (or vierbein) formulation [6] that is relevant to our work. In section 3, we set out our model by introducing the different fields that enter it and their Lagrangians. In section 4, we put forward the arguments that will lead us to our no-go result, namely the impossibility of cancelling the supersymmetric variation of the "kinetic" Lagrangian of NGT, without the addition to this Lagrangian of terms containing derivatives. We draw our conclusions in the last section.

2 Field structure and anholonomic formulation of NGT

The nonsymmetric gravitation theory [1] is based on a nonsymmetric tensor $g_{\mu\nu}$ and affine connection $\Gamma^\mu_{\nu\rho}$, defined on a non-riemannian spacetime, which may be decomposed into symmetric and antisymmetric parts:

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]},$$

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{(\nu\rho)} + \Gamma^\mu_{[\nu\rho]},$$

The contravariant tensor $g^{\mu\nu}$ is defined by:

$$g^{\mu\alpha}g_{\mu\beta} = g^{\alpha\nu}g_{\beta\mu} = \delta^\alpha_\beta.$$  

The field equations of NGT can be derived from a variational principle with a Lagrangian density that can be defined in analogy with its counterpart in general relativity:

$$\mathcal{L}_{NGT} = \sqrt{-g}R \equiv \sqrt{-g}g^{\mu\nu} R_{\mu\nu} (W),$$

where $g \equiv \det g_{\mu\nu}$ and $R_{\mu\nu} (W)$, the NGT Ricci curvature tensor, is given by:

$$R_{\mu\nu} (W) = \partial_\rho W^\rho_{\mu\nu} - \partial_\nu W^\rho_{\mu\rho} + W^\rho_{\alpha\rho} W^\alpha_{\mu\nu} - W^{\alpha}_{\mu\rho} W^\rho_{\alpha\nu},$$
in terms of the unconstrained connection coefficients $W^{\rho}_{\mu\nu}$ related to the affine connection $\Gamma^{\rho}_{\mu\nu}$ through:

$$\Gamma^{\lambda}_{\mu\nu} = W^{\lambda}_{\mu\nu} + \frac{2}{3} \delta^{\lambda}_{\mu} W_{\nu}.\quad (6)$$

This last relation then implies that $\Gamma^{\mu}_{\mu} \equiv \Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2} \left( \Gamma^{\alpha}_{\mu\alpha} - \Gamma^{\alpha}_{\alpha\mu} \right) = 0$. Varying independently the metric $g^{\mu\nu}$ and the connection $W^{\rho}_{\mu\nu}$ in the action principle $\delta S \equiv \delta \left[ -\frac{1}{16\pi G} \delta \int dx^4 \mathcal{L}_{NGT} \right] = 0$, one obtains the empty space field equations:

$$\partial_{\lambda} g^{\mu\nu} - \Gamma^{\nu}_{\mu\lambda} g^{\alpha\nu} - \Gamma^{\mu}_{\lambda\nu} g^{\mu\alpha} = 0,\quad (7)$$

$$\partial_{\nu} \left( \sqrt{|g|} g^{\mu\nu} \right) = 0,\quad (8)$$

$$R^{\mu\nu}_{\rho\nu} (W) - \frac{1}{2} g^{\mu\nu} R (W) = 0,\quad (9)$$

where $R (W) \equiv g^{\mu\nu} R^{\rho}_{\mu\nu} (W)$. It is worth to mention that the non trivial combination (6) is chosen so as to yield the compatibility condition (7), which entirely determines the affine connection $\Gamma^{\rho}_{\mu\nu}$ but not $W^{\rho}_{\mu\nu}$ [15]. In subsequent work on NGT, another choice $R'_{\mu\nu} (W)$ for the Ricci curvature tensor was adopted, which is:

$$R'_{\mu\nu} (W) = \partial^{\rho} W^{\mu}_{\nu \rho} - \frac{1}{2} \left( \partial_{\nu} W^{\rho}_{\mu \rho} + \partial_{\mu} W^{\rho}_{\nu \rho} \right) + W^{\rho}_{\alpha \rho} W^{\nu}_{\mu \alpha} - W^{\alpha}_{\mu \rho} W^{\nu}_{\rho \alpha}.\quad (10)$$

The field equations (7)-(9) are unchanged, except that $R^{\mu\nu}_{\rho\nu} (W)$ is replaced by $R'_{\mu\nu} (W)$. In fact this arbitrariness is due to the absence of symmetries of the Riemann curvature tensor of NGT:

$$R'^{\sigma}_{\mu\nu\rho} \equiv \partial_{\rho} W^{\sigma}_{\mu\nu} - \partial_{\nu} W^{\sigma}_{\mu\rho} + W^{\sigma}_{\alpha \rho} W^{\nu}_{\mu \alpha} - W^{\sigma}_{\alpha \nu} W^{\alpha}_{\mu \rho},\quad (11)$$

which has two independant contractions $R'^{\sigma}_{\mu\nu\rho}$ and $R'^{\sigma}_{\rho\mu\nu}$ (the latter vanishes in general relativity). The former contraction gives the Ricci tensor (5), while (10) corresponds to $R'^{\sigma}_{\mu\nu\rho} + \frac{1}{8} R^{\sigma}_{\rho\mu\nu}$ [16]. The linear approximation of massless NGT has been studied in ref. [7], where it was shown that it describes the propagation of the massless spin 2 graviton, associated with the symmetric part of the metric, and a massless scalar particle called skewon, associated with the antisymmetric part of the metric. These two particles represent 2 + 1 bosonic degrees of freedom.
The more recent massive version of NGT [10, 11] differs from the massless one by additional non derivative terms to the Lagrangian density (4). These are given by:

\[ \mathcal{L}_{\text{cosm.}} = -2\lambda \sqrt{-g}, \quad (12) \]
\[ \mathcal{L}_{\text{skew}} = -\frac{1}{4}\mu^2 \sqrt{-gg^{\mu\nu}} g_{[\mu}\rho \nu], \quad (13) \]
\[ \mathcal{L}_W = -\frac{1}{6} \sqrt{-gg^{(\mu\nu)}} W_\mu W_\nu, \quad (14) \]

where \( \lambda \) is the cosmological constant and \( \mu^2 \) an additional cosmological constant associated with \( g_{[\mu\rho \nu]} \). The main consequence of these additional terms is that, in the linear approximation, the skewon becomes massive with mass \( \mu \) and spin 1. Nevertheless, it is worth to mention, for future purposes, that the geometric (or "kinetic") part (4) of the Lagrangian is the same for both massless and massive NGT.

The anholonomic formulation of NGT is based on the hypercomplexification of the tangent space at each point \( x \) of the real space-time manifold, by allowing the functions defined on this manifold to take their values in the ring of hyperbolic complex numbers \( H \equiv \{ a + jb, (a, b) \in \mathbb{R}^2 \text{ and } j^2 = +1 \} \) (\( j^2 = -1 \) in the case of the field of ordinary complex numbers) [4]. This leads to the introduction of an hyperbolic complex vierbein \( e^a_\mu (\mu = 1, 4 ; a = 1, 4) \) [6]:

\[ e^a_\mu = a^a_\mu + jb^a_\mu, \quad (15) \]

where \( a^a_\mu \) and \( b^a_\mu \) are real valued. The hyperbolic complex conjugate (hereafter abbreviated as H.C.C.) vierbein is \( \overline{e^a_\mu} = a^a_\mu - jb^a_\mu \). One also introduces the inverse vierbein \( e^a_\mu \), orthonormal to \( e^a_\mu \) and thus: \( e^a_\mu e^b_\nu = \delta^a_\nu \) and \( e^a_\mu e^c_\alpha = \delta^\mu_\alpha \) (idem. for H.C.C.).

The hyperbolic complex sesquilinear form \( ^c g_{\mu\nu} \) (and its inverse \( ^c g^{\mu\nu} \)) acting on the tangent space is given by:

\[ ^c g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \quad (16) \]
\[ ^c g^{\mu\nu} = e^a_\mu e^b_\nu \eta^{ab}, \quad (17) \]

where \( \eta_{ab} = \text{diag}(+1, -1, -1, -1) \) is the Minkowski flat space-time metric. As \( \eta_{ab} \) is symmetric, we have \( ^c g_{\mu\nu} = ^c g_{\nu\mu} \), where \( ^c g_{\mu\nu} \) is the H.C.C. of \( ^c g_{\mu\nu} \). This means that \( ^c g_{\mu\nu} \) can be decomposed as follows:

\[ ^c g_{\mu\nu} = g_{(\mu\nu)} + jg_{[\mu\nu]}, \quad (18) \]
where $g_{(\mu\nu)}$ and $g_{[\mu\nu]}$ are respectively the symmetric and antisymmetric parts of the real metric tensor. The group of local isometries that preserves the metric is the group of hyperbolic complex valued transformations $U^a\ b(x)\ c^b\ (x)$ acting on the tangent space such that $c\ g'_{\mu\nu}\ (x) = c\ g_{\mu\nu}\ (x)$, where:

$$c\ g'_{\mu\nu}\ (x) \equiv c^a\ (x)\ c^b\ (x)\ \eta_{ab} = U^a\ c\ (x)\ \tilde{U}^b\ d\ (x)\ \eta_{ab} c^c\ (x)\ c^d\ (x).$$  \hspace{1cm} (19)

Comparing (16) with (19), one then deduces that the transformation $U$ must satisfy the relation:

$$\eta = U^+\ \eta U,$$  \hspace{1cm} (20)

where $U^+ \equiv \tilde{U}^T$. This is the defining relation for the pseudo-unitary group (in the sens of hyperbolic complex numbers) $U(3,1,H)$, which has been shown to be isomorphic to $GL(4,R)$ [5]. It is precisely this isomorphism that selects the hyperbolic complex numbers. Had one chosen ordinary complex numbers, then one would have ended with $U(3,1,C)$ which is not isomorphic to $GL(4,R)$. The group $U(3,1,H)$ plays the same role in NGT as does the local homogeneous Lorentz group $SO(3,1)$ in general relativity.

To each of the sixteen generators of the local group of invariance $U(3,1,H)$ corresponds a compensating field, namely the hyperbolic complex spin connection $c\omega^{\mu a b}$, with the inhomogeneous transformation law:

$$c\omega^\mu_{\ a\ b} = U^a\ d\ c\omega^d_{\ b} (U^{-1})^f\ b - (\partial^a U)^a\ d\ U^d\ b.$$  \hspace{1cm} (21)

In accordance with (20), $c\omega^{\mu a b}$ satisfies the relation $c\tilde{\omega}^{\mu a b} = -c\omega^{\mu b a}$. This property then implies the decomposition into antisymmetric “real” part and symmetric “imaginary” part:

$$c\omega^{\mu a b} = \omega^{\mu [a b]} + j\omega^{\mu (a b)}.$$  \hspace{1cm} (22)

This spin connection enters the definition of the covariant derivative, which acts on hyperbolic complex vectors $V^a$ defined on the tangent space:

$$cD^\mu V^a = \partial^\mu V^a + c\omega^{\mu a\ b} V^b.$$  \hspace{1cm} (23)

The anholonomic curvature tensor can then be introduced through the commutator of two covariant derivatives:

$$[cD^\mu, cD^\nu]^a b = R^a_{\mu\nu} b.$$  \hspace{1cm} (24)

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where:

\[
R_{\mu\nu}{}^{ab} = \partial_\mu c^c \omega^a_{\nu}{}^{b} - \partial_\nu c^c \omega^a_{\mu}{}^{b} + c^c \omega^a_{\mu}{}^{d} c^d \omega^b_{\nu} - c^c \omega^a_{\nu}{}^{d} c^d \omega^b_{\mu}.
\]  (25)

To recover the Lagrangian (4) in terms of anholonomic quantities, one has to impose a compatibility condition between the connections \( W^\rho_{\mu\sigma} \) and \( c^c \omega^a_{\mu}{}^{b} \). This condition is given by [6, 16]:

\[
\partial_\sigma e^a_\mu + c^c \omega^a_{\sigma}{}^{b} e^b_\mu - c^c W^\rho_{\mu\sigma} e^a_\rho = 0,
\]  (26)

where we have introduced the hyperbolic complex valued connection:

\[
c^c W^\rho_{\mu\sigma} \equiv \Re (c^c W^\rho_{\mu\sigma}) + j \Im (c^c W^\rho_{\mu\sigma}).
\]  (27)

After solving (26) for \( c^c \omega^a_{\mu}{}^{b} (e^a_\mu, c^c W^\rho_{\mu\sigma}) \) and substituting in (25), one can relate holonomic to anholonomic tensors:

\[
R^a_\mu \omega^b_\nu = c^c R^\lambda_{\sigma\nu} (c^c W^\alpha_{\beta\rho})
\]  (28)

\[
e^a_\mu (e_{a} R^a_\nu \omega^b_\mu) = c^c R^\mu_{\nu} (c^c W^\alpha_{\beta\rho})
\]  (29)

\[
e^a_\mu R^a_\mu \equiv e^a_\mu (e_{b} R^\mu_{\nu} \omega^b_\nu) = c^c g^{\mu\nu} c^c R^\mu_{\nu} (c^c W^\alpha_{\beta\rho}) = R.
\]  (30)

The hyperbolic complex Riemann and Ricci tensors, \( c^c R^\lambda_{\sigma\nu} (c^c W^\alpha_{\beta\rho}) \) and \( c^c R^\mu_{\nu} (c^c W^\alpha_{\beta\rho}) \), have the same structure as their real counterparts (11) and (5), with the connection \( W^\rho_{\mu\sigma} \) replaced by the hyperbolic complex one defined in (27). Most important is the last relation (30) : it defines the Ricci scalar which is real and coincides with the one that enters the definition (4) of \( \mathcal{L}_{NGT} \) [16]. It follows that the real NGT Lagrangian density (4) can be written in terms of anholonomic quantities as:

\[
\mathcal{L}_{NGT} = (e\bar{e})^\frac{1}{2} e^a_\mu \bar{e}^b_\nu R^\mu_{\nu}{}^{ab} (c^c \omega),
\]  (31)

where \( e \equiv \det e^a_\mu, \bar{e} \equiv \det \bar{e}^a_\mu \) and \( (e\bar{e}) = |g_{\mu\nu}| \in R \). This is the Lagrangian that we will need for our model of the following section.

We note, for completeness, that the Lagrangian density (4) with the choice (10) for \( R^\mu_{\nu} (W) \) can also be written in the anholomic formalism as [16] :

\[
\mathcal{L}_{NGT} = (e\bar{e})^\frac{1}{2} \left[ e^a_\mu \bar{e}^b_\nu R^\mu_{\nu}{}^{ab} (c^c \omega) + \frac{1}{8} e^a_\mu \bar{e}^b_\nu \bar{e}^c_\alpha R^\mu_{\nu}{}^{a} (c^c \omega) \right].
\]  (32)
3 The model

Let us first state more precisely our goal. We want to investigate the possibility of constructing a minimal model of a locally supersymmetric extension of NGT, that is a field theory containing NGT and invariant under local supersymmetric transformations, a kind of nonsymmetric simple supergravity where NGT would replace general relativity. We want furthermore to work in the framework of the superPoincaré group, the representations of which are well known. In this connection, we must associate to each bosonic particle its fermionic partner. In the context of massless NGT (to which we restrict our investigation), one must have two fermionic fields: a spin $\frac{3}{2}$ gravitino $\psi_\mu \equiv e_\mu^a \psi_a$, where $e_\mu^a$ is the hyperbolic complex vierbein and $\psi_a$ a non-hyperbolic complex vectorial spinor, and a (non-hyperbolic complex) spin $\frac{1}{2}$ skewino $\chi$. As we work with Majorana spinors, these two fields represent four fermionic degrees of freedom (in $d = 4$ dimensions). However, the graviton and skewon represent only three bosonic degrees of freedom. There is thus a mismatch of bosonic and fermionic degrees of freedom. To overcome this difficulty one has to introduce a second scalar field $\varphi$, thereby completing two representations of the superPoincaré group: one contains the spin 2 graviton and the spin $\frac{3}{2}$ gravitino, the other contains the skewon and the additional field $\varphi$ (two scalar fields) and the spin $\frac{1}{2}$ skewino.

By minimal, we mean that NGT enters the model through the Lagrangian (31), without additional terms containing the derivative of the spin connection (22). Insofar as we are concerned with representations of the superPoincaré group, we introduce the $SO(3,1)$ covariant derivative $D_\nu$ acting on the spinors $\psi_\mu$ and $\chi$:

$$D_\nu \equiv \partial_\nu + \frac{1}{2} \omega_\nu^{[ab]} \sigma_{ab},$$

where $\omega_\nu^{[ab]}$ is the antisymmetric real part of the hyperbolic complex spin connection (22), and $\sigma_{ab} \equiv \frac{1}{4} [\gamma_a, \gamma_b]$, $\gamma_a$ being the constant Dirac matrices. This derivative is covariant under the restriction of the local group $U(3,1,H)$ to its real unimodular subgroup, which is precisely the homogeneous Lorentz group $SO(3,1)$. To accommodate this privileged role of $\omega_\mu^{[ab]}$, we split the Ricci tensor $R_{\mu\nu}^{ab}$ and the Lagrangian $L_{NGT}$ (31) into two parts, one of which contains only the Lorentz antisymmetric part $\omega_\mu^{[ab]}$. We thus get $R_{\mu\nu}^{ab} = R_{\mu\nu}^1 \omega^{[ab]} + R_{\mu\nu}^2 \omega^{[ab]}$ where:

$$R_{\mu\nu}^1 \omega^{[ab]} = \partial_\mu \omega_\nu^{[ab]} + \omega_\mu^{[a} \omega_\nu^{c]} - (\mu \longleftrightarrow \nu),$$

$$R_{\mu\nu}^2 \omega^{[ab]} = \omega_\mu^{[a} \omega_\nu^{b]} - (\mu \longleftrightarrow \nu).$$
\[ R_{\mu\nu}^{\alpha\beta} = jD_\mu \omega_\nu^{(\alpha\beta)} + \omega_\mu^{(a\beta)} \omega^{(cb)}_\nu - (\mu \leftrightarrow \nu), \]  
\[ \text{(35)} \]

with \( D_\mu \omega_\nu^{(ab)} = \partial_\mu \omega_\nu^{(ab)} + \omega_\mu [a] \omega_\nu^{(cb)} + \omega_\mu [b] \omega_\nu^{(ac)} \). By inserting this decomposition into (31), we get:

\[ \mathcal{L}_{NGT} = \mathcal{L}^1 + \mathcal{L}^2, \]
\[ \text{(36)} \]

where:

\[ \mathcal{L}^1 = (e \bar{e})^{1/2} e_\alpha^\mu \bar{e}_\nu^\rho R_{\mu\nu}^{1 \alpha\beta}, \]
\[ \mathcal{L}^2 = (e \bar{e})^{1/2} e_\alpha^\mu \bar{e}_\nu^\rho R_{\mu\nu}^{2 \alpha\beta}. \]
\[ \text{(37)} \]
\[ \text{and (38)} \]

The relevance of this decomposition will appear shortly.

For the gravitino field \( \psi_\mu \), we choose to generalize the minimal coupling of the Rarita-Schwinger field to ordinary gravity [12], by merely replacing the real vierbein of general relativity by its NGT hyperbolic complex counterpart. We thus put:

\[ \mathcal{L}_\psi = k \Omega^{\mu\nu\rho} e^\mu_\alpha \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\alpha \psi_\rho + H.C.C., \]
\[ \text{(39)} \]

where \( k \) is a numerical factor to be fixed later. \( \Omega^{\mu\nu\rho} \) is some linear combination of terms of the form \( (e \bar{e})^{1/2} \eta^{abcd} a_\mu^a a_\nu^b a_\rho^c a_\sigma^d \), where \( \eta^{abcd} \) is the completely antisymmetric symbol and each factor \( a_i^\sigma \) (\( \sigma = \mu, \nu, \alpha, \rho \); \( i = a, b, c, d \)) is either \( e_i^\sigma \) or \( \bar{e}_i^\sigma \). The matrices \( \gamma_\nu \) and \( \gamma_5 \) are defined by \( \gamma_\nu \equiv e_\nu^\alpha \gamma_a (\gamma_a \text{ constant Dirac matrix}) \) and \( \gamma_5 \equiv i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) (\( i^2 = -1 \)). The covariant derivative \( D_\alpha \psi_\rho \) is given by:

\[ D_\alpha \psi_\rho = \partial_\alpha \psi_\rho + \frac{1}{2} \omega_\alpha^{[ab]} \sigma_{ab} \psi_\rho, \]
\[ \text{(40)} \]

and \( \bar{\psi}_\mu \equiv e_\mu^a \psi_\mu^T C, C \) being the charge conjugation matrix \( (C \gamma_a C^{-1} = -\gamma_a^T, C^T = -C) \). It must be stressed that the use of the superPoincaré representations entails the introduction of this covariant derivative, due to the antisymmetry of the \( SO(3,1) \) generators \( \sigma_{ab} \). Furthermore, the commutator of two derivatives generates the tensor \( R_{\mu\nu}^{1 \alpha\beta} \) (34):

\[ [D_\mu, D_\nu] \psi_\rho = \frac{1}{2} R_{\mu\nu}^{1 \alpha\beta} \sigma_{ab} \psi_\rho, \]
\[ \text{(41)} \]

hence the decomposition (37)-(38).
We likewise take the minimally coupled Lagrangians for the skewino $\chi$ and the scalar field $\varphi$, namely:

$$L_{\chi} = \frac{1}{2} \chi \overline{D} \chi,$$

$$L_{\varphi} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = \frac{1}{2} g^{(\mu\nu)} \partial_\mu \varphi \partial_\nu \varphi,$$

where:

$$\overline{D} \chi \equiv \gamma^\mu \overline{D}_\mu \chi = \gamma^\mu \left( \partial_\mu \chi + \frac{1}{2} \omega^\mu_{[ab]} \sigma_{ab} \chi \right),$$

$$g^{(\mu\nu)} \equiv \frac{1}{2} \left( e^a_\mu \tilde{e}_b^\nu + e^b_\mu \tilde{e}_a^\nu \right) \eta^{ab}.$$

Although not necessary for our line of argument, we give them for completeness.

Having postulated the Lagrangians of the fields that enter our model, we will show in the following section that, with the simplest choice of supersymmetric variation of the vierbein $e^a_\mu$ and gravitino $\psi_\mu$, it is impossible to fix the tensor $\Omega^{\mu\nu\rho}$ so that the linear contributions in $\psi_\mu$ coming from variations of $L_{NGT}$ and $L_{\psi}$ cancel.

We finally note that, in order to simplify matters, we will use the 1.5 order formalism [17]. The spin connection $\omega^a_\mu$ is considered as an independent field obeying an algebraic equation which gives its expression in terms of the vierbein and the spinorial fields. This equation is obtained by varying the spin connection $\omega^a_\mu$ in the action, as an independent field (or the real and imaginary parts $\omega^a_\mu$ and $\omega^a_\mu$ as independent fields) and is given by:

$$D_\mu \left[ (\epsilon \overline{\epsilon})^\frac{1}{2} \left( e^a_\mu \tilde{e}_b^\nu - \tilde{e}_b^\mu e^a_\nu \right) \right] = j (\epsilon \overline{\epsilon})^\frac{1}{2} \left[ \omega^a_\mu \right. \left. \left( e^a_\mu \tilde{e}_b^\nu - \tilde{e}_b^\mu e^a_\nu \right) + \omega^a_\nu \right. \left. \left( e^a_\nu \tilde{e}_b^\mu - \tilde{e}_b^\nu e^a_\mu \right) \right] - S^{\nu}_{[ab]},$$

where $S^{\nu}_{[ab]}$ is defined by $\delta (L_{\psi} + L_{\chi}) = S^{\nu}_{[ab]} \delta \omega^a_\mu$. Most important in the 1.5 order formalism is that the spin connection is not varied when we come to the local supersymmetric variation of the Lagrangians.

**4 Terms linear in the gravitino field**

Our strategy is to first investigate the piece of the supersymmetric variation of $L_{NGT}$ and $L_{\psi}$ which is linear in the gravitino field $\psi_\mu$. To clearly state
our line of argument, it is convenient to first write down the variation of $\mathcal{L}^1$ (37), under an infinitesimal variation of the hyperbolic complex vierbein $\delta e_\mu^a$. This variation can be put into the form:

$$\delta \mathcal{L}^1 = \frac{1}{2}(\bar{e}\tilde{e})^{\frac{3}{2}} \left[ e^i_\rho R^1 - 2e^\mu_i e_a R^1_{\mu a} \right] \delta e_\rho^i + H.C.C., \quad (47)$$

where $R^1_{\mu a} \equiv \tilde{e}_b^\nu R^1_{\mu \nu ab}$ and $R^1 \equiv e^a_\mu R^1_{\mu a}$. We have used the relation $\delta (\bar{e}\tilde{e})^{\frac{3}{2}} = \frac{1}{2}(\bar{e}\tilde{e})^{\frac{3}{2}} e^\mu_a \delta e_\mu^a + H.C.C.$ and the symmetries of $R^1_{\mu \nu ab}$ ($R^1_{\mu \nu ab} = -R^1_{\mu \nu ba} = R^1_{\nu \mu ba}$). Generalizing the supersymmetric variation of the tetrad of general relativity in the context of ordinary supergravity, to the hyperbolic complex vierbein of NGT, we put:

$$\delta_0 e_\mu^a = \bar{\varepsilon} \gamma^a \psi_\mu \equiv \bar{\varepsilon} \gamma^a \psi_b e^b_\mu, \quad (48)$$
$$\delta_0 \tilde{e}_\mu^a = \bar{\varepsilon} \gamma^a \tilde{\psi}_\mu \equiv \bar{\varepsilon} \gamma^a \psi_b \tilde{e}^b_\mu, \quad (49)$$

where $\varepsilon (x)$ is an infinitesimal spin $\frac{1}{2}$ Majorana spinor, the parameter of the local supersymmetric transformation. Upon substitution in (47), one gets the term linear in $\psi_\mu$:

$$\delta \mathcal{L}^1_0 = \frac{1}{2}(\bar{e}\tilde{e})^{\frac{3}{2}} \left[ e^i_\rho R^1 - 2e^\mu_i e_a R^1_{\mu a} \right] \left( \bar{\varepsilon} \gamma^i \psi_j \right) e^j_\rho + H.C.C. = \frac{1}{2}(\bar{e}\tilde{e})^{\frac{3}{2}} \left[ R^1 \left( \bar{\varepsilon} \gamma^i \psi_i \right) - 2R^1_{\mu a} \left( \bar{\varepsilon} \gamma^\mu \psi_a \right) \right] + H.C.C. \quad (50)$$

The particularity of this term is first that it is linear in $\psi_\mu$, and second that it contains contractions of the tensor $R^1_{\mu \nu ab}$ with $e^a_\mu$ and/or $\tilde{e}_\mu^a$. It is a common feature, if not a rule, of supergravity theories that this term is cancelled by the variation of the gravitino Lagrangian generated by the part of the infinitesimal variation:

$$\delta_0 \psi_\mu \sim D_\mu \varepsilon = \partial_\mu \varepsilon + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} \varepsilon. \quad (51)$$

This situation is due to the fact that the Rarita-Schwinger-like Lagrangian is the only term that can yield, upon the variation $\delta_0 \psi_\mu$ a term linear in $\psi_\mu$ and containing the Ricci tensor, which appears via the commutator of two covariant derivatives. Similarly, in our case, the only term that might cancel the contribution (50) is the contribution linear in $\psi_\mu$ coming from the variation of $\mathcal{L}_\psi$ (see (39)), with $\delta_0 \psi_\mu = D_\mu \varepsilon$ (a possible numerical factor.
can be absorbed in the constant $k$ appearing in (39)). Thus, the problem amounts, at the first stage, to find, if it exists, the form of the tensor $\Omega^{\mu\nu\rho\sigma}$, so as to cancel the contribution (50).

Taking $\delta_0\psi_\mu = D_\mu \varepsilon$, the variation $\delta \mathcal{L}_0^0$ of $\mathcal{L}_\psi$ that follows is:

\[
\delta \mathcal{L}_0^0 = \left\{ k \Omega^{\mu\nu\rho\sigma} \left( \delta_0 \psi_\mu \right)^\gamma_5 \gamma_\nu D_\alpha \psi_\rho \\
+ k \Omega^{\mu\nu\rho\sigma} \left( D_\mu \varepsilon \right)^\gamma_5 \gamma_\nu D_\alpha \psi_\rho \right\} + H.C.C.
\]

\[
= \left\{ k \Omega^{\mu\nu\rho\sigma} \left( D_\mu \varepsilon \right)^\gamma_5 \gamma_\nu D_\alpha \psi_\rho \\
+ k \Omega^{\mu\nu\rho\sigma} \left( D_\mu \varepsilon \right)^\gamma_5 \gamma_\nu D_\alpha \psi_\rho \right\} + H.C.C.
\] (52)

After partially integrating the first term, and omitting the terms containing $D_\mu (\Omega^{\mu\nu\rho\sigma})$ and $D_\mu \gamma_\nu$, we obtain two terms that contain a double covariant derivative (from which one can generate $R^{1}_{\mu\nu}^{ab}$) and are linear in $\psi_\mu$:

\[
\delta \mathcal{L}_0^0 = \left\{ -k \Omega^{\mu\nu\rho\sigma} \left( e_\rho e_\mu \right)^{1/2} \left( \eta^{\rho\alpha\bar{\mu}} + \eta^{\rho\bar{\alpha}\mu} \right) \gamma_5 \gamma_\nu D_\mu D_\alpha \psi_\rho + k \Omega^{\mu\nu\rho\sigma} \left( D_\mu \varepsilon \right)^\gamma_5 \gamma_\nu D_\alpha \psi_\rho \right\} + H.C.C.
\] (53)

These two terms are the only ones that might cancel the contribution (50), whatever the complete laws of transformation of the fields may be. Indeed, the terms that one can add to the transformation law (51) are at least linear in $\psi_\mu$, yielding quadratic terms in the variation of $\mathcal{L}_\psi$, and the linear part of the transformation laws of the fields $\chi$ and $\varphi$:

\[
\delta \chi = a_1 \gamma^\mu D_\mu (\varphi \varepsilon) + a_2 (D_\mu \gamma^\mu) \varphi \varepsilon,
\]

\[
\delta \varphi = b_1 \varepsilon \chi + b_2 \varepsilon \gamma^\mu \psi_\mu,
\]

(a1, a2, b1 and b2 are constants) cannot contribute to the cancellation of (50).

On the other hand, to cancel (50) $\delta \mathcal{L}_0^0$ must be of the form:

\[
\delta \mathcal{L}_0^0 = \left\{ -k \left( e_\rho e_\mu \right)^{1/2} \left( \eta^{\rho\alpha\bar{\mu}} + \eta^{\rho\bar{\alpha}\mu} \right) \gamma_5 \gamma_\nu D_\mu D_\alpha \psi_\rho + k \left( e_\rho e_\mu \right)^{1/2} \left( \eta^{\rho\alpha\bar{\mu}} + \eta^{\rho\bar{\alpha}\mu} \right) \psi_\mu \gamma_5 \gamma_\nu D_\alpha D_\rho \varepsilon \right\} + H.C.C.,
\] (56)

where

\[
\eta^{\rho\alpha\bar{\mu}} = \eta^{abcd} e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d,
\]

\[
\eta^{\rho\alpha\bar{\mu}} = \eta^{abcd} e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d,
\]

\[
\eta^{\rho\mu\bar{\alpha}} = \eta^{abcd} e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d,
\]

\[
\eta^{\rho\mu\bar{\alpha}} = \eta^{abcd} e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d,
\]

\[
\eta^{\rho\bar{\alpha}\mu} = \eta^{abcd} e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d.
\]

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(\eta^{abcd} \text{ is the completely antisymmetric symbol}). The form (56) of \(\delta L_\psi^0\) is dictated by two requirements: i) the \(\eta\) factors must have the right (anti)symmetry properties as to yield the commutator of two covariant derivatives, and hence \(R_{\mu\nu}^{1\,ab}\), and ii) once the Ricci tensor \(R_{\mu\nu}^{1\,ab}\) generated, it must have \(e_\mu^a e_\nu^b\) (or \(\bar{\epsilon}_\mu^a \epsilon_\nu^b\)) in front of it, in order to get the same contractions as in (50) and cancel this term.

Let us briefly outline how (56) cancels (50). By relabeling indices of the \(\eta\) terms in (56), and taking into account the expression (41), one gets:

\[
\begin{align*}
\delta L_\psi^0 &= -k (\vec{e}e)^{\frac{1}{2}} \left\{ \eta^{\nu\rho\alpha\bar{\mu}} \epsilon_\gamma^\nu \gamma^\nu [D_{\mu} \eta_\alpha] \psi_\rho - \eta^{\nu\mu\alpha\bar{\mu}} \epsilon_\gamma^\nu \gamma^\nu [D_\alpha \eta_\mu] \xi \right\} + H.C.C. \\
&= -\frac{k}{2} (\vec{e}e)^{\frac{1}{2}} \eta^{\nu\rho\alpha\bar{\mu}} [\epsilon_\gamma^\nu \gamma^\nu \sigma_{ab} \psi_\rho] - (\bar{\psi}_\rho \epsilon_\gamma^\nu \gamma^\nu \sigma_{ab} \xi) R_{\mu\alpha}^{1\,ab} + H.C.C.. \quad (61)
\end{align*}
\]

Using the properties of the charge conjugation matrix \(C\) and Dirac matrices (especially that of the anticommutator \(\{\sigma_{ab}, \gamma_\epsilon\} = \eta_{abcd} \gamma_5 \gamma^d\)), the last expression transforms to:

\[
\begin{align*}
\delta L_\psi^0 &= -\frac{k}{2} (\vec{e}e)^{\frac{1}{2}} \eta^{\nu\rho\alpha\bar{\mu}} \epsilon_\gamma^\nu \epsilon_{\bar{\mu}}^\rho \epsilon_\gamma^\nu \gamma^\nu (\bar{\psi} \gamma^\nu \psi_\rho) R_{\mu\alpha}^{1\,ab} + H.C.C. \\
&= \frac{k}{2} (\vec{e}e)^{\frac{1}{2}} \delta_{[abcd]} \epsilon_\gamma^\nu \epsilon_{\mu}^\rho \epsilon_\gamma^\nu \gamma^\nu (\bar{\psi} \gamma^\nu \psi_\rho) R_{\mu\alpha}^{1\,ab} + H.C.C., \quad (62)
\end{align*}
\]

with

\[
\delta_{[abcd]} = \delta_a \delta_b \delta_c \delta_d - \delta_a \delta_b \delta_c \delta_d + \delta_a \delta_b \delta_c \delta_d - \delta_a \delta_b \delta_c \delta_d.
\]

Using the symmetry properties of \(R_{\mu\nu}^{1\,ab}\) (\(R_{\mu\nu}^{1\,ab} = -R_{\nu\mu}^{1\,ab} = -R_{\mu\nu}^{1\,ba} = R_{\nu\mu}^{1\,ba}\)) along with the definitions \(R_{\mu\nu}^{1\,ab} = \bar{e}_\mu^a R_{\mu\nu}^{1\,ab}, \bar{R}_{\nu\mu}^{1\,ba} = \epsilon_\alpha^\mu R_{\mu\nu}^{1\,ab}\) and \(R^1 = \epsilon_\alpha^\mu \epsilon_\mu^a R_{\mu\nu}^{1\,ab} = \bar{R}_1\), we end up with the following expression:

\[
\begin{align*}
\delta L_\psi^0 &= \left\{ -\frac{k}{2} (\vec{e}e)^{\frac{1}{2}} \left[ R^1 (\bar{\psi} \gamma^\nu \psi_\rho) - 2 R_{\mu\epsilon} {(\bar{\psi} \gamma^\nu \psi_\rho)} \right] \right\} + H.C.C.. \\
&= -\frac{k}{2} (\vec{e}e)^{\frac{1}{2}} \left[ R^1 (\bar{\psi} \gamma^\nu \psi_\rho) - 2 \bar{R}_{\alpha\mu} (\bar{\psi} \gamma^\nu \psi_\rho) \right] + H.C.C.. \quad (64)
\end{align*}
\]

Noting that the two terms in the curly brackets are the H.C.C. of one another, we finally get:

\[
\delta L_\psi^0 = -ik (\vec{e}e)^{\frac{1}{2}} \left[ R^1 (\bar{\psi} \gamma^\nu \psi_\rho) - 2 R_{\mu\epsilon} (\bar{\psi} \gamma^\nu \psi_\rho) \right] + H.C.C.. \quad (65)
\]
This term effectively cancels (50) if the constant \( k \) is chosen to be \( k = -\frac{i}{2} \).

Now the expression (53) is identical to (56), and then cancels (50), if and only if the two following identifications are simultaneously satisfied:

\[
\Omega^{\mu \nu \alpha \rho} = (e \tilde{e})^{\frac{1}{2}} \left( \eta^{\mu \rho \alpha \mu} + \eta^{\mu \rho \tilde{\alpha} \tilde{\mu}} \right), \quad (66)
\]

\[
\Omega^{\mu \nu \alpha \rho} = (e \tilde{e})^{\frac{1}{2}} \left( \eta^{\mu \nu \alpha \rho} + \eta^{\mu \nu \tilde{\alpha} \tilde{\rho}} \right). \quad (67)
\]

However the two expressions are not compatible as one can readily verify. Indeed, choosing one of the preceding expressions, we inevitably end up with either an \( \eta \) term that does effectively have the right antisymmetry properties but gives rise to contractions of \( R_{\mu \nu}^{1 \ ab} \) with two vierbeins \( (e_\alpha^a e_\rho^b) \), whereas a vierbein and a conjugate vierbein are required \( (e_\mu^a \tilde{e}_\nu^b) \), or with an \( \eta \) term that does not have the right antisymmetry property that can lead to \( R_{\mu \nu}^{1 \ ab} \) via the commutator of two covariant derivatives. Thus, there is no linear combination \( \Omega^{\mu \nu \alpha \rho} \) that can give rise, through the variation (51), to the linear variation in \( \psi_\mu \) that might cancel (50). It is worth emphasizing, once more, that neither the variation of \( \mathcal{L}_{\text{NGT}}^2 \) (38) nor that of \( \mathcal{L}_\chi \) and \( \mathcal{L}_\varphi \) (42)-(43) can give rise to contributions to the cancellation of (50).

One may wonder if the massive version of NGT would make things work better. This is not the case because i) the massless and the massive versions of NGT have the same ”kinetic” part (4) (or (31)) in their Lagrangians and ii) both versions describe a massless graviton, to which one associates a massless gravitino, and thus one can adopt the generalization of the Rarita-Schwinger Lagrangian (39). We thus end up with the same problem(s) as with the massless version.

Finally, we note that the undesired extra-terms generated by (66) or (67) in the variation \( \delta \mathcal{L}_\psi \) linear in \( \psi_\mu \) suggest non-minimal models in which the linear variation in \( \psi_\mu \) vanishes, but at the cost of adding terms to the Lagrangian (36) and slightly modifying the infinitesimal supersymmetric law of transformation (51). One example, among others, is given by the following choices:

\[
\Omega^{\mu \nu \alpha \rho} = (e \tilde{e})^{\frac{1}{2}} \left( \eta^{\mu \nu \alpha \rho} + \eta^{\mu \nu \tilde{\alpha} \tilde{\rho}} \right)
\]

\[
= (e \tilde{e})^{\frac{1}{2}} \eta^{abcd} (e_a^\mu e_b^\nu e_c^\alpha e_d^\rho + e_a^\mu e_b^\nu \tilde{e}_c^\alpha e_d^\rho), \quad (68)
\]

and

\[
\delta_0 \psi_\mu = D_\mu \varepsilon + e_\mu^\alpha \tilde{e}_\alpha^\beta D_\beta \varepsilon, \quad (69)
\]
along with the addition of the following term to the NGT Lagrangian (36):

\[ \mathcal{L}'_{\text{NGT}} = -\frac{1}{4} (e\tilde{\epsilon})^2 \epsilon^a_b e^c_b R_{\mu\nu}^{1 \ ab} + H.C.C. \]  

(70)

We have not pursued the investigation of the full invariance of such a model because it does not seem reliable; its drawback, to say the least, is that the interpretation, in the linear approximation, in terms of particles is spoiled by the addition of extra terms like (70).

5 Conclusion

In this paper, we have tried to build a locally supersymmetric minimal model based on nonsymmetric gravitation theory. We have adopted the most straightforward generalizations from ordinary supergravity, to the case of NGT. This trial led us to a no-go result, namely the impossibility of cancelling the terms linear in the gravitino field, coming from the variation of the graviton and gravitino Lagrangians. This no-go result relies first on the choices we made in our model, and second on the hyperbolic complex structure introduced to accommodate the vierbein formalism of NGT. Nevertheless, it points out the serious difficulties facing the construction of a supersymmetric theory of NGT.

Another alternative is to use the supersymmetric extension of the full local invariance group \( GL(4, R) \sim U(3, 1, H) \) of NGT, instead of the superPoincaré group, and then to look for some symmetry breaking mechanism to reduce the former to the latter. This would entail the use of infinite component fields [18]. In addition to the mathematical difficulties that one would face in such an approach, its physical interpretation is not obvious.

Finally, we note that beyond the mathematical difficulties raised by the introduction of the nonsymmetric metric tensor \( g_{\mu\nu} \), lies the lack of physical interpretation of the antisymmetric part of this tensor, which seem to be at the heart of the problem(s) confronting NGT.

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