Non-commutative geometry as a realization of varying speed of light cosmology

Stephon Alexander and João Magueijo

Theoretical Physics, The Blackett Laboratory, Imperial College, Prince Consort Road, London, SW7 2BZ, U.K.

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We examine the cosmological implications of space-time non-commutativity, discovering yet another realization of the varying speed of light model. Our starting point is the well-known fact that non-commutativity leads to deformed dispersion relations, relating energy and momentum, implying a frequency dependent speed of light. A Hot Big Bang Universe therefore experiences a higher speed of light as it gets hotter. We study the statistical physics of this “deformed radiation”, recovering standard results at low temperatures, but a number of novelties at high temperatures: a deformed Planck’s spectrum, a temperature dependent equation of state $w = p/\rho$ (ranging from $1/3$ to infinity), a new Stephan-Boltzmann law, and a new entropy relation. These new photon properties closely mimic those of phonons in crystals, hardly a surprising analogy. They combine to solve the horizon and flatness problems, explaining also the large entropy of the Universe. We also show how one would find a direct imprint of non-commutativity in the spectrum of a cosmic graviton background, should it ever be detected.

I. STATEMENT OF PURPOSE

Inflation and string theory are widely perceived as the leading schools of thought in cosmology and fundamental physics, respectively. And yet finding common ground between them, say by deriving inflation from a concrete string theory model, has so far remained elusive. Such an enterprise is far from frivolous, since cosmology is probably the most realistic laboratory for testing string theory. For this reason it is disappointing that after all these years, and following a number of radical revolutions in both fields, the two theories have still failed to condense into a single construction.

Could it be that somehow we got things wrong on one side of the story? The inspirational origins of inflation are founded on a number of problems of Big Bang cosmology, namely the horizon, flatness, homogeneity, and entropy problems [1–4]. A recently proposed alternative explanation to these problems is the varying speed of light (VSL) theory [5–7], the idea that light travelled faster in the early Universe. Currently VSL is far less developed than inflation, with full model building work hardly started. But could VSL be more amenable to a direct connection with fundamental physics [8,9], more specifically with string theory?

In this paper we establish a connection between non-commutative geometry and VSL. Over the last few years non-commutativity of space-time coordinates has become part and parcel of any attempt to quantize gravity. Similar behaviour was already spelled out in Appendix I of [6] (see also [7]), where it was shown that under VSL the partial derivatives $\partial_x$ and $\partial_x$ do not commute. Furthermore non-commutative geometries are known to lead to deformed dispersion relations [20], which imply a frequency-dependent speed of light.

A deep connection is up for grabs, and such is the purpose of this paper.

II. COSMOLOGY AND NON-COMMUTATIVITY

One of the pillars of the standard Big Bang (SBB) is the theory of General Relativity. The problems of the SBB model suggest that classical General Relativity is inconsistent in the very early universe where quantum corrections to gravity become significant. Interestingly, most attempts toward a theory of quantum gravity such as string/M-theory realize that space-time itself is non-commutative. In particular, non-commutativity is concretely realized in a non-perturbative formulation of M-theory, (M)atrix theory, where the D-0 brane collective coordinates are in general non-commuting; this is interpreted as a non-commuting space-time. In the infinite momentum frame (IMF), non-commuting space-time emerges from (M)atrix theory [11]. Also, in the open string sector of string theory, non-commutative geometry arises when the NS-B field is turned on; the open string end points become “polarized” and hence non-commuting [13,12]. In the low energy limit of open string theory in the presence of a B field, these dipoles are realized as non-commutative solitons in Non-commutative Yang-Mills (NCYM) theory where they exhibit a VSL behavior [14].

The coordinate operators of a non-commuting space obey the following commutation relation.

$$[\hat{x}^\mu, \hat{x}^\nu]\equiv i \Theta^\mu\nu$$

where $\Theta^{\mu\nu}$ is in general constant, antisymmetric, and with dimension $[\text{length}]^2$. There exists a general prescription for studying quantum field theories in non-commutative spaces with flat backgrounds. By simply taking a classical field theory defined on a commuting space and replacing the product operation by a $*$ product, one obtains a non-commutative field theory. The $*$ product is defined from

$$\phi * \psi(x) = \exp(\frac{i}{2} \Theta_{\mu\nu} \partial_{x_\mu} \partial_{y_\nu}) \phi(x) \psi(y) |_{x=y}$$

\[ = \phi(x)\psi(y) + \frac{i}{2} \Theta^{\mu\nu} \partial_\mu \phi \partial_\nu \psi + \mathcal{O}(\Theta^2) \]  

(3)

Notice that the \(*\) product is non-commutative. Also non-commutative field theories are necessarily non-local since the \(*\) product yields an infinite number of derivatives in the Lagrangian. This non-locality in space-time, as we will see in the sections that follow, gives rise to a VSL due to a modification in the dispersion relation of the quantum fields in a non-commutative space-time.

Although non-commutativity is concretely realized in string theory, space-time non-commutativity is better understood than its space-time counterpart. In both cases, however, non-commutativity is best understood for constant B field (theta parameter); while more complicated versions of non-commutativity in string theory generalized to non-constant B-field and also its gravitational analogue is currently under investigation [16,15,12]. For VSL we are concerned about space-time non-commutativity with non-constant B field (\(\Theta\)).

In light of the string-theoretic realization of non-commutativity, we shall implement a more general algebra of (1) relativistic invariance remains classical and the translations

\[ \hat{x}_\mu \to \hat{x}'_\mu = \hat{x}_\mu + a_\mu \]  

(4)

preserving the algebra of (1) are commutative. However, if \(\Theta_{\mu\nu}\) depends on \(\hat{x}_\mu\), in particular if we assume the linear \(\hat{x}_\mu\) dependence, i.e.

\[ [\hat{x}_\mu, \hat{x}_\nu] = i c_{\mu\nu}^\lambda \hat{x}_\lambda. \]  

(5)

then, the translations

\[ \hat{x}'_\mu \to \hat{x}_\mu + \hat{v}_\mu \]  

(6)

are not commutative. For particular choices of the structure constants \(c_{\mu\nu}^\lambda\) this type of non-commutative structure falls under the category of the quantum-deformed Poincaré group, with non-commutative group parameters\(^1\). In this work we consider a particular type of quantum Poincaré group; the \(\kappa\) deformation of space-time symmetries with non-commutativity of space-time coordinates described by the following solvable Lie-algebra relations:

\[ [\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0. \]  

(7)

Thus spatial translations still commute; non-commutativity affects only operations involving the time coordinate.

This type of non-commutativity has yet to be realized in string/M theory, however we expect that it will be better understood in the future, especially in the context of the non-commutative version of AdS/CFT correspondence and in formulations where the B field (\(\Theta\)) is non-constant [14,15].

In the case of the deformed Poincaré algebra, the spatial momenta remain commutative

\[ [P_i, P_j] = 0 \]  

(8)

and the rotation part of the Lorentz sector is left unchanged. For this reason we assume there is a scale \(\Lambda_{NC}\) in the Early universe such that

\[ \Theta = \frac{\kappa}{\Lambda_{NC}} \]  

(9)

where \(\kappa\) is \(\mathcal{O}(1)\); i.e. non-commutativity is strong. As a result it is safe to conjecture that since

(1) The \(\kappa\) deformation does not act on the rotations and the canonical momenta.

(2) The deformation is strong, \(\kappa \sim \mathcal{O}(1)\),

thus the non-commutativity can be described as a gas of massless non-commutative radiation propagating in a conformally flat Friedmann-Lemaître-Robertson-Walker space-time.

The radiation is in thermal equilibrium and is maximally correlated. This is a statement that the space-time is a Non-commutative FRW (NCFRW) with a gas of photons. The modified dispersion relation of the photons captures the non-commutativity of the space-time. The FRW space-time is endowed with flat spatial sections, and is given by the line element

\[ ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2). \]  

(10)

where \(\eta\) is the conformal time and \((x, y, z)\) are spatial comoving coordinates. This metric is conformally flat and is locally equivalent to the Minkowski metric \(\eta_{\mu\nu}\).

Since our space-time is non-commutative, the notion of a continuous space-time manifold is lost. In fact the separation between energy-momentum and curvature, which is manifest in the Einstein field equations is not as trivial. However, we can transcend this difficulty by deforming the dispersion relations of the photons and study their dynamics in a commuting space-time; this yields the nature of how matter behaves in a quantized space-time. So long as the photons are in equilibrium, our approximation is valid.

Therefore, our starting point for cosmology is a deformed Poincaré group whose action does not affect the conformal factor of the metric. A similar approach was taken in the context of non-commutativity and inflation [17]. This is analogous to the standard hot big bang scenario where our light degrees of freedom consist of a

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\(^1\)For a more descriptive classification of the quantum Poincaré' groups see [10]
gas of non-commutative radiation in thermal equilibrium. This radiation will exhibit a modified dispersion relation which alters its equation of state.

### III. THE THERMODYNAMICS OF NON-COMMUTATIVE RADIATION

In the rest of the paper we will study the cosmological consequences of modified dispersion relations for photons. However, it is instructive to provide a discussion of how modified dispersion relations arise in the free field regime of the deformed $\kappa$ Poincaré group. In what follows, we will provide one of a few examples of a modified frequency dependent dispersion relation.

In [19] the authors were able to define at the perturbative level a free field theory for massless bosons by identifying the propagator and constructing the intertwiners between the representations of the Poincare group acting on fields and states. These intertwiners are simply the wavefunctions of particles of definite spin. One can define these intertwiners through the appropriate wave equations describing free fields.

Therefore, the dispersion relations can be obtained from the invariant wave operator on the $\kappa$-deformed Minkowski space $\frac{\partial^2}{\partial x^2} - \frac{m^2}{c^2}$, which is expressed in momentum space as

$$C_{1_{bcp}}^{\kappa}(1 - \frac{C_{1_{bcp}}^{\kappa}}{4\kappa^2}),$$

where

$$C_{1_{bcp}}^{\kappa} = (P^2)e^{-\frac{P_0}{\kappa}} - (2\kappa \sinh(\frac{P_0}{2\kappa}))^2$$

is the first Casimir of the $\kappa$-deformed Poincaré algebra. It follows that the spectrum of the modified massless wave operator contains the deformed massless mode

$$C_{1_{bcp}}^{\kappa}\phi = 0$$

which leads to the following dispersion relation

$$k^2e^{-\omega/\kappa} - \left(2\kappa \sinh(\frac{\omega}{2\kappa})\right)^2 = 0$$

The wave equation leading to the above dispersion relations is non-local in time but can be reexpressed so that the dispersion relations corresponds to an operator of second order in time derivatives and non-local in space.

$$\omega^2 = \left(k \log(1 + \frac{k}{\kappa})\right)^2$$

Let us now consider the following generalization of 15 for massless particles, for which:

$$E^2 - p^2c^2f^2 = 0$$

Here $f(E)$ gives rise to a frequency dependent speed of light, whereas $c$ is a possibly space-time dependent, but frequency independent, speed. We thus factorize two different VSL effects previously considered in the literature - a frequency dependent $c$ (as studied in [20]), and a space-time dependent $c$ (as examined in [5–7]). Most likely, the final theory will contain both. Naturally the most general deformation need not factorize these two effects, or even produce a quadratic invariant as in Eqn. (16), but we shall retain this assumption.

We shall consider two proposals for $f$, previously considered in the literature. Amelino-Camelia and collaborators (eg. [21]) have considered

$$f = 1 + \lambda E$$

associated with the $\kappa$-Poincaré group, whereas following the quantum deformation of the Poincaré group as studied by Majid one finds [20]:

$$f = \frac{2\lambda E}{1 - e^{-2\lambda E}}$$

Here $\lambda$ is the deformation parameter and we have set $\hbar = 1$. The last deformation has also been proposed by Kowalsky-Glikman. We will also consider a generalization of (17), of the form

$$f = (1 + \lambda E)^\gamma$$

in particular for $2/3 < \gamma < 1$. The case $\gamma > 1$ will be studied in detail elsewhere [32]: it leads to a realization of inflation, but not VSL. We shall call these models 1, 2, 3, respectively.

At once we find two possible definitions for the speed of light. One may use the definition proposed in [20]:

$$\hat{c} = \frac{dE}{dp} = \frac{cf}{1 - \frac{E}{E_f}}$$

(20)

(where ‘ denotes a derivative with respect to $E$) for which model 1 gives:

$$\hat{c} = c(1 + \lambda E)^2$$

(21)

wheras model 2 gives

$$\hat{c} = ce^{2\lambda E}.$$  

(22)

Alternatively we may define the speed of light as

$$\hat{c} = \frac{E}{p} = cf.$$  

(23)

Both definitions, $\hat{c}$ and $\check{c}$, play a role in what follows.

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2We have applied $\lambda \to -2\lambda$ to the convention used in [20] so that the 2 dispersion relations to be considered agree to first order.
A. The partition function and phase space densities

We now proceed to examine the statistical physics of this “deformed” radiation. We first note that the form of the partition function

$$Z = \sum_r e^{-\beta E_r}$$  \hspace{1cm} (24)

does not depend on the dispersion relations; indeed it amounts to a definition of temperature based upon the fact that the thermal reservoir has a much larger number of states than the system under examination. Hence, for a boson gas, the average number density of states with a given momentum still satisfies the Bose-Einstein result:

$$n(p) = \frac{1}{e^{\beta E(p)} - 1}$$  \hspace{1cm} (25)

since this formula depends only on the partition function and the rules of counting associated with bosons. A similar undeformed expression applies to fermions.

The density of momentum states for point particles is again unchanged, as it simply reflects the use of periodic boundary conditions ($p = (2\pi \hbar/L)n$ where $n$ is a triplet of numbers, and $L$ is the side of a given cubic volume). All that changes is the relation between $E$ and $p$ and therefore the density of states $\Omega(E)$ with a given energy $E$ per unit of volume. We find:

$$\Omega(E) = \frac{E^2}{\pi^2 \hbar^3 c^2} \frac{1}{\pi^2 (\hbar c)^3} \left(1 - \frac{f'E}{f}\right).$$  \hspace{1cm} (26)

simply from $\Omega(E)dE = \Omega(p)dp$.

B. The deformed Planck spectrum and the graviton background

From $\rho(E) = n(p(E))\Omega(E)$ we thus obtain the deformed Planck’s spectrum:

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{E^3}{e^{\beta E} - 1} \left(1 - \frac{f'E}{f}\right).$$  \hspace{1cm} (27)

One can easily show that only deformations of the form:

$$f = [1 + (\lambda E)^{-3}]^{-1/3}$$  \hspace{1cm} (28)

leave the Planck spectrum unchanged. These are not very physical in a cooling Universe, where the speed of light would initially be a constant, then start to drop, with $c \propto 1/a$, until nowadays.

All other dispersion relations affect Planck’s law. This does not conflict with experiment if $f \approx 1$ for $T \ll 1/\lambda$. However even then there may be an observational imprint. Gravitons decouple around Planck time, and should constitute a thermal background similar to the cosmic microwave background (with a temperature of the same order). However their spectrum will be a deformed Planck spectrum, since it mimics the spectrum they had at the time they decoupled. Hence if and when a graviton background is discovered its spectrum will supply a direct measurement of the dispersion relations.

More specifically model 1 leads to the deformed spectrum:

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{E^3}{e^{\beta E} - 1} \left(1 + \lambda E\right)^4$$  \hspace{1cm} (29)

For $T \ll 1/\lambda$ distortions to this spectrum are negligible. For high temperatures, though, the peak in $\rho(E)$ is not at $T$, but at $E = 1/\lambda$ for all temperatures. In other words the color temperature saturates at $T_{\text{max}} \approx 1/\lambda$.

The general form of the spectrum in this regime is

$$\rho(E) = \frac{E^2 T}{\pi^2 (1 + \lambda E)^4}$$  \hspace{1cm} (30)

Model two leads to:

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{E}{\lambda^2 E} \left(1 - e^{-\lambda E} \sinh^2 (\lambda E)\right)$$  \hspace{1cm} (31)

which for $T \gg 1/\lambda$ leads to

$$\rho(E) = \frac{T}{\lambda^2 \pi^2} e^{-\lambda E} \sinh^2 (\lambda E)$$  \hspace{1cm} (32)

In both cases the spectrum is super-black (as opposed to gray); that is, the overall amplitude of the spectrum with respect to black at $T \approx 1/\lambda$ is enhanced by a factor of $T\lambda$. In Fig. 1 we plot spectra for model 1 and 2, as well as the undeformed Planck spectrum.

Model 3 is unique in that it never saturates the color temperature, i.e. for all temperatures hotter radiation means a peak in the photons distribution at higher energies. It’s thermal spectrum is given by

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{E^3}{e^{\beta E} - 1} \left(1 + (1 - \gamma)(\lambda E)^{-\gamma}\right)$$  \hspace{1cm} (33)

For $\gamma < 2/3$ the peak of $\rho(E)$ scales like $T$. For $2/3 < \gamma < 1$ the peak becomes very wide and covers energies from $E \approx \lambda^{-1}$ to $E \approx T$. We illustrate this feature in Fig. 2.

C. The energy and entropy densities

Integrating $\rho(E)$ leads to a modified Stephan-Boltzmann law. For low temperatures $\rho \propto T^4$ as usual, but for $\lambda T \gg 1$ we find that $\rho \propto T$. This dependence can be expressed in terms of the function

$$n(E) = \frac{d \log \rho}{d \log T}$$  \hspace{1cm} (34)

which we plot in Fig. 3. At low temperatures the pro-
FIG. 1. Deformed thermal spectra at high temperature \((T \lambda \gg 1)\), divided by \(T^2\) (we have assumed \(\lambda = 1\)). For reference we plotted a Planck spectrum at temperature \(T = 1/\lambda\). For model 1 and 2 the color temperature saturates, so that above \(T = 1/\lambda\) the peak in \(\rho(E)\) does not shift to higher energies.

For model 3 the situation is a bit different. The high energy Stephan-Boltzmann equation takes the form \(\rho \propto T^\zeta\) with a power \(\zeta\) in the range \(1 < \zeta < 4\). An approximate high temperature formula for \(\zeta\) may be obtained by noting that in Eqn. 33 the last factor is approximately \(1/((\lambda T)^{3\gamma})\) over the most relevant parts of the integrand. Hence we may expect at high energies \(\rho \propto T^{4-3\gamma}\) that is \(\zeta \approx 4 - 3\gamma\). We have verified numerically that this analytical approximation works extremely well. In Fig. 4 we describe the transition from the low to the high temperature behaviour for various values of \(\gamma\).

The entropy density then follows from

\[
s = \int \frac{dT}{T} \frac{\partial \rho}{\partial T}
\]

For low temperatures we recover the usual expression

\[
s = \frac{4}{3} \sigma T^3
\]

whereas for high temperatures we now have

\[
s \approx \sigma^+ \log \lambda T
\]

FIG. 2. Deformed thermal spectra for model 3, with \(\gamma = 1/2\), as the temperature is increased. We see that for this model at all temperatures, hotter radiation implies higher mean energies for photons, and a shift to the higher energies of \(\rho(E)\).

FIG. 3. The transition from \(\rho \propto T^4\) to \(\rho \propto T\) at very high temperatures for non-commutative radiation.
D. The equation of state

The pressure formula (and so the equation of state) is also modified. As before it is given by a sum over all states

$$p = \sum_s n_s \left( -\frac{\partial E_s}{\partial V} \right)$$

but since now we have for each state

$$E = \frac{2\pi \hbar c f}{V^{1/3}}$$

we have

$$p = \frac{1}{3V} \sum_s n_s \frac{E_s}{1 - \frac{E}{T}} = \frac{1}{3} \int \frac{\rho(E)dE}{1 - \frac{E}{T}} \neq \frac{1}{3}\rho$$

and so the equation of state of radiation is modified.

To find the modified equations of state we therefore have to compute the integral:

$$p(T) = \frac{1}{3} \int \frac{\rho(E,T)dE}{1 - \frac{E}{T}}$$

Combined with $\rho(T)$ this leads to a modified equation of state $p = w(\rho)\rho$. It is not difficult to guess its form. At low temperatures $w \approx 1/3$. At high temperatures we have

$$\rho = \sigma^+ \frac{T}{\lambda^3}$$

$$p = \frac{\sigma^+ T}{3\lambda^3} (A + \log(T\lambda))$$

where $A$ is order 1. So, for $T\lambda \gg 1$, we have

$$w(\rho) \approx A + \log(T\lambda) \approx B + \log(\rho\lambda^4)$$

that is $w$ grows logarithmically with $\rho$ (and therefore with $T$). This approximate argument is confirmed by a numerical integration as shown in Figure 5.

Not surprisingly one may violate all of Hawking’s energy conditions apart from the weak one (the energy density is still always positive). Indeed a sign of non-commutativity appears to be that it generates equations of state with $|w| > 1$. The fact that the usually sensible energy conditions are badly violated should not deter us.

Model 3 leads to a simpler equation of state. At low energies $w = 1/3$; at high energies the equation of state becomes again a constant. Indeed for this model $(f'E)/f' = \gamma$ so that the denominator of Eqn. 44 becomes approximately a constant at high temperatures, over the peak of $\rho(E)$. Hence we may expect:

$$w_\infty = w(\rho \to \infty) \approx \frac{1}{3(1 - \gamma)}$$

A numerical integration reveals that this is only an approximate formula, albeit with the correct qualitative behaviour: in Fig. 6 we show the asymptotic equation of state as predicted by our analytical fit and by a numerical integration. In [32] we shall explore the dramatically different range $\gamma > 1$, to find that it is possible to generate an inflationary equation of state.

Notice that the results in this section lead to a closed form expression for the entropy:

$$s = \frac{p + \rho}{T} = \frac{\rho}{T}(1 + w(\rho))$$
E. The phonon analogy

We conclude this section by noting that to some extent the thermodynamics of deformed radiation is very similar to that of phonons in crystals. Thermal phonons satisfy Bose-Einstein statistics; however, unlike photons, they are subjected to very complicated dispersion relations. Hence the density of states for phonons $\Omega(E)$ (cf. Eqn. (26)) at high energies is very different from that of photons (see for instance Fig. 8-4, pp 217 of [22]), giving rise to a multitude of thermodynamical novelties similar to the ones we have just derived for deformed radiation. At low energies the phonons cannot see the discrete structure of the crystal and behave like ordinary photons. At high energies, on the contrary, they become highly sensitive to crystal properties and exhibit exotic behaviour. The borderline between these two regimes, for thermal phonons, is determined by the Debye temperature, which is therefore a solid state physics counterpart to the Planck temperature.

The analogy we have spelled out is far from surprising. After all space-time non-commutativity is nothing but a method for quantising space-time. In some sense this means introducing a discrete space-time structure not dissimilar to that of a crystal (another interpretation is space-time uncertainty [29]). As photon frequencies get higher and higher (or their temperature approaches the Planck temperature) their dispersion relations, and all derived thermodynamical properties, start being sensitive to the discreteness of the space-time structure supporting them.

IV. THE TRANS-PLANCKIAN COSMOLOGICAL EVOLUTION

We now proceed to integrate the Friedmann equations with the $w = w(\rho)$ peculiar to non-commutative radiation. We first consider the case of zero spatial curvature $K = 0$, leaving for Section VI the cases $K = \pm 1$. The equations may be written:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \rho$$

(50)

$$\frac{\ddot{a}}{a} = -\frac{1}{6} \rho (1 + 3w(\rho))$$

(51)

where $a$ is the scale factor and dots represent derivatives with respect to proper time. Instead of the second equation one may use the conservation equation

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (1 + w(\rho)) \rho = 0$$

(52)

From the equations above it is also possible to find an equation for the speed of sound $c_s$, defined by:

$$c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\dot{p}}{\dot{\rho}}$$

(53)

Computing $\dot{w}$ and using equation (52) leads to [24]:

$$\dot{w} = -3 \frac{\dot{a}}{a} (1 + w(\rho))(c_s^2 - w)$$

(54)

and so

$$c_s^2 = w + \rho \frac{dw}{d\rho}$$

(55)

an equation which shall be of great relevance in a future publication, in which we discuss density fluctuations in these scenarios [23].

For model 3 it is easy to obtain an analytical solution in the asymptotic trans-Planckian regime:

$$a \propto t^{\frac{2}{3(1+w)}}$$

(56)

A numerical integration (with a $w(\rho)$ also obtained numerically) is presented in Fig. 7. We see that after Planck’s time the universe expands as a normal radiation dominated universe, but inside the Planck epoch it expands slower, the more so the higher the value of $\gamma$. This is because the deceleration of the universe is higher:

$$q_\infty = \frac{\ddot{a}}{a^2} = \frac{12 - \gamma}{21 - \gamma}$$

(57)

\[3\]In what follows we shall set $8\pi G = 1$
Nowhere, in the transition between one regime and the other, do we have accelerated expansion. This is to be contrasted with the case $\gamma > 1$, explored elsewhere [32]. Models 1 and 2 asymptotic trans-Planckian behaviour may be inferred from (47) and (52). For $\rho \lambda^4 \gg 1$:

$$\rho = \frac{\exp \left( \frac{C}{\lambda^4} \right)}{\lambda^4}$$

(58)

where $C$ is a constant. Integrating (50) then leads to

$$E_i \left( \frac{C}{2\lambda^3} \right) \propto t$$

(59)

The qualitative behaviour is similar to the one described. A numerical integration leads to a result equivalent to model 3 with $\gamma \rightarrow 1$.

V. A SOLUTION TO THE HORIZON PROBLEM

Naively it may seem self-evident that the horizon problem has been solved in this model. As the Universe gets hotter, the photons' average frequency and energy get higher, just like in the standard Big Bang model. However in our theory, unlike in the standard model, this results in a larger “ambient” speed of light. Hence our model realises the usual VSL solution to the horizon problem [6], not by a direct time-dependence of $c$, but indirectly, via the chain linking time, temperature, average photon frequency, and $c$.

However, closer scrutiny shows that it is not that simple; in fact not all deformations $f$ lead to a solution to the horizon problem. On the one hand our analysis confirms that as $t$ decreases below Planck’s time, the temperature keeps increasing. This is obvious for model 3 but not for model 1 or 2. However we see that $a$ keeps decreasing (cf. Fig 7 and Eqn (59)), and from Eqns. (58) and (35):

$$T = \frac{1}{\lambda c^2} \exp \left( \frac{C}{a^3} \right)$$

(60)

implying that the temperature $T$ does diverge in the early Universe for all models considered.

However, as $T$ grows above $\lambda^{-1}$, we find that for model 1 and 2 (but not for model 3, with $\gamma < 1$) the color temperature $T_c$ (defining the peak of $\rho(E)$) saturates at $T_c = \lambda^{-1}$. Hence a hotter plasma does not contain more energetic photons at the peak of the distribution; it only contains more photons (proportionally to $T^2$) at a peak located at the same energy. Hence the most abundant photons in pre-Planck times will experience a roughly constant speed of light. Given that we do not have accelerated expansion, at first it looks as if these two VSL scenarios do not actually solve the horizon problem.

Fortunately a further subtlety comes into play. Recall that for $T \gg \lambda^{-1}$ we have:

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3 \lambda^4} \frac{E^2 T}{f^3} \left( 1 - \frac{f'E}{f} \right)$$

(61)

up to $E \approx T$. For $E > T$ the distribution is then dominated by the exponential cut off $e^{-\beta E}$:

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3 \lambda^4} \frac{E^3 e^{-\beta E}}{f^3} \left( 1 - \frac{f'E}{f} \right)$$

(62)

For model 2, $\rho(E)$ falls off exponentially away from the peak at $T_c \approx \lambda^{-1}$, but for model 1 the fall-off is merely power-law. More concretely for model 1 there is a non-negligible density of photons with $E \approx T$, the fastest photons in the Universe, given by

$$\rho_{\text{max}} \approx \rho(T) T \approx \frac{T^4}{1 + (\lambda T)^4} \approx \lambda^{-4}$$

(63)

The conclusion is that even though the color temperature saturates for $T \gg \lambda^{-1}$, there is still a constant density (of the order of the Planck density) of photons with energy of order $E \approx T$. These are the fastest photons in the Universe, experiencing a speed:

$$\dot{c} = c (1 + \lambda T) = c \left( 1 + \frac{\lambda^4 \rho}{\sigma^+} \right) \propto \exp \left( \frac{C}{a^3} \right)$$

(64)

or a similar expression for $\dot{c}$. Fast photons at the tail at $E = T$ ensure causal contact at a given time. It is therefore essential that interactions exist between photons at $E = T$ and at $E = \lambda^{-1}$.

A similar calculation for model 2, on the other hand, leads to:

$$\rho_{\text{max}} \approx \rho(T) T \approx T e^{-2\lambda T}$$

(65)
Let us concentrate on model 1 and 3. the horizon problem in this VSL scenario; but some do. Nonetheless the conclusion remains that not all deformations lead to a solution to the horizon problem. Model 3 on the other hand does not suffer from any of these problems. However, the decreasing \( \dot{c} \) forces \( r_h \) to decrease, thereby solving the horizon problem.

This picture, however, will not provide a solution to the flatness problem. Effectively we may model this effect by replacing the curvature term into a cosmological constant term, leading to a non-commutative (pseudo-)sphere, namely the fuzzy three-sphere \( \tilde{S}^3 \). For a derivation of \( \tilde{S}^3 \) see [18].

We know that such a fuzzy sphere has the usual properties of ordinary spheres as long as the curvature radius is much larger than the Planck length. However as one tries to curve them beyond Planck curvature, their curvature radius saturates. Hence we avoid a singularity.

VI. A SOLUTION TO THE FLATNESS PROBLEM

The discussion in Section II leading to the establishment of a non-commutative Friedmann metric, and equations, breaks down if we introduce spatial curvature \( K = \pm 1 \). In that case our space-time is conformally related not to non-commutative Minkowski space-time, but to a non-commutative (pseudo-)sphere, namely the fuzzy three-sphere \( \tilde{S}^3 \). For a derivation of \( \tilde{S}^3 \) see [18].

We therefore adopt a different approach: we model the fuzzy sphere’s unusual curvature by means of a direct coupling between \( K \) and the matter density \( \rho \). The idea is that the curvature of spatial surfaces depends upon the energy scale of the matter probing it. This creates an intertwining between matter and geometry, which is to be expected above Planck scale. Indeed one does not expect the usual division between matter and curvature to survive a quantum gravity epoch (see discussion in the second to last paragraph of Section II). Hence we
replace the curvature term \( K/a^2 \) by \( Kg^2/a^2 \), where now \( g = g(\rho) \) describes a deformation of the curvature of the sphere as the matter filling up the Universe approaches Planck scale. We choose:

\[
g(\rho) = (1 + \rho \lambda^4)^\alpha
\]  

(72)

As it turns out this approach provides a direct realization of the solution to the flatness problem described in [6] (just replace the \( c \) in Friedmann equations by \( c_K = cg(\rho) \)). The Friedmann equations are now:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \rho - \frac{Kg^2c^2}{a^2} 
\]

(73)

\[
\frac{\ddot{a}}{a} = -\frac{1}{6} \rho(1 + 3w(\rho))
\]

(74)

which lead to the integrability condition:

\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (1 + w(\rho)) \rho = \frac{6Kc^2g^2g' \rho}{a^2} \frac{\dot{g} \rho}{g} \]

(75)

(where \( ' \) means a derivative with respect to \( \rho \)) or instead:

\[
\dot{\rho} \left( 1 - \frac{6Kc^2g^2g'}{a^2} \right) = -3 \frac{\dot{a}}{a} (1 + w(\rho)) \rho
\]

(76)

We find therefore a coupling between the curvature \( K \) and the matter density. In order for flatness to be stable, \( g' \) should be positive: then supercritical models \( (K = 1) \) have matter removed from them, sub-critical models \( (K = -1) \) have matter put into them, thereby creating a flatness attractor.

Let \( \rho_c \) be the critical density of the Universe:

\[
\rho_c = \frac{3}{8\pi G} \left( \frac{\dot{a}}{a} \right)^2
\]

(77)

that is, the mass density corresponding to \( K = 0 \) for a given value of \( \dot{a}/a \). Let us define \( \epsilon = \Omega - 1 \) with \( \Omega = \rho/\rho_c \). A numerical integration of equations (73) and (75), expressing the result in terms of \( \epsilon(a) \), is plotted in Fig. 9. We see that at late times, when the effects of non-commutativity have switched off, \( \epsilon \propto a^2 \) (the flatness problem), but early on \( \epsilon \) decays like a power of \( \rho \).

For illustration purposes in Fig. 9 we started the integration at \( T \approx 10^5 T_P \), and with \( \epsilon \approx 1 \). In our scenario, however, there is no starting time, subjected to “natural” initial conditions (we recall that in the Big Bang model one likes to impose \( \epsilon \approx 1 \) at \( t = t_P \)). Our model plunges as deep as we like inside the Planck epoch, and as we have shown, we found that during this period flatness then becomes an attractor. Should this regime be valid all the way up to \( T = \infty \) we may conclude that when Planck time is finally reached \( \epsilon \) equals precisely zero.

### VII. THE ENTROPY OF THE UNIVERSE

The flatness problem is often rephrased as an entropy problem, both in its inflationary and VSL formulations. As explained in [1,27,6] adiabatic solutions to the flatness problem lead to a seemingly very flat Universe, but containing only one particle within each curvature radius’ volume. This problem afflicts Brans-Dicke based solutions to the flatness problem [26,27].

All our work assumes thermal equilibrium, so it would seem that we have an adiabatic scenario - that is, no entropy is produced. However this is not the case: we have introduced source terms in the energy conservation equation (cf. Eqn. (76)), and this necessarily modifies the first law of thermodynamics, a fact which, as we now show, leads to entropy generation even without leaving thermal equilibrium.

Taking the equilibrium expression

\[
Td(sa^3) = d(\rho a^3) + pda^3 = a^3 d\rho + (\rho + p)da^3
\]

(80)
one may read off from (75) that
\[ T(\dot{a}s^3) = a^3 \frac{6Kc^2g^2}{a^2} \frac{\dot{g}}{g} \] (81)
that is:
\[ \dot{s} + \frac{\dot{a}}{a}s = \frac{6Kc^2g}{T a^2} \frac{\dot{g}}{g} \] (82)
We see that if \( K = 0 \) entropy is conserved, a fact which may be checked directly from Eqns. (40) and (60). However, if \( K = -1 \) and \( \alpha > 0 \), then entropy is produced even though we have only used equilibrium thermodynamics. The case \( K = 1 \) and \( \alpha > 0 \) seems to lead to entropy reduction, and thus it may be argued that it is inconsistent with the second law of thermodynamics (see [28] for such an argument in a different context).

We may now define the entropy (or number of particles) inside a volume with the curvature radius as
\[ \sigma_K = s \left( \frac{a}{g} \right)^3 \] (83)
and from (82) we have:
\[ \frac{\dot{\sigma}_K}{\sigma_K} = \frac{\dot{g}}{g} \left( \frac{6a}{gT \sigma_K} - 3 \right) \] (84)
As long as \( K = -1 \) and \( \alpha > 0 \) we see that \( \sigma_K \) increases unboundedly, under general conditions, before Planck time. After Planck time it stabilises to a constant. As in the discussion at the end of Section VI we therefore conclude that the natural state at the end of the Planck epoch is \( \sigma_K = \infty \), explaining the current bound that \( \sigma_K \) must be bigger than 10^{98}.

The entropy problem is sometimes referred to as the need to explain why the Universe is so big, or why it contains so much entropy. Usually one needs to invoke an entropy production mechanism, such as reheating at the end of inflation, in order to solve this problem. In our scenario the explanation is related to the fact that in the non-commutative phase of the Universe we break Poincaré invariance. We know that energy conservation follows from time-translation invariance - a property which is now deformed. Hence we expect violations of energy conservation, and these lead to entropy production even if we remain in thermal equilibrium.

**VIII. CONCLUSIONS**

There are many avenues which lead to the possibility that space-time at the Planck scale is non-commutative. In the context of the Early Universe, VSL is able to solve the outstanding problems of SBB. In this paper we demonstrate that non-commutativity, in the context of the early universe, is equivalent to VSL. We provide a concrete cosmological model of a non-commutative early universe scenario which exhibits VSL phenomena. In this model the speed of light does not depend explicitly on time; rather it depends on the photon’s frequency, and so as the Universe gets hotter the “ambient” speed of light increases. We showed that in this model the horizon and flatness problems are resolved.

Perhaps the most important result of all, however, is the discovery that our solution to the horizon and flatness problems is related to the emergence of Lorentz invariance in the Universe. We found that photons in non-commutative space-times are very similar to phonons in crystals, with Planck’s temperature playing the role of the Debye temperature (see Section III E). At low energies the photons perceive a Lorentz invariant continuum, and have a constant speed. This corresponds to the standard Big Bang phase of the Universe. At high frequencies, however, the photon’s dispersion relation reflects the structure and properties of the crystal. The crystal breaks Lorentz invariance, and like in the case of phonons, the photon’s speed becomes frequency dependent. It is in this phase that the horizon and flatness problems are solved. The fact that the Universe is pushed towards a colder phase, for which non-commutativity and violations of Lorentz invariance become negligible, can be seen as an explanation for the emergence of the continuum and symmetries we perceive today.

In a future publication [23] we shall go further and prove that thermal fluctuations in deformed radiation may translate into a Harrison-Zeldovich spectrum of initial conditions for structure formation. Crucially this rules out all but a well-defined class of quantum deformations. This important result provides a thermal alternative to the usual quantum origin of the structure of the Universe (which may or may not have its problems [30,31]). It also, for the first time, allows VSL to become a proper model of structure formation. Another avenue to be studied further concerns the cosmological constant problem. Moffat has recently [25] shown how a resolution may emerge from non-local field theories. Non-commutativity imposes just such a non-locality.

In a somewhat different direction in [32] we describe our findings for model 3, with \( \gamma > 1 \), a range left untouched in this paper. We find that ordinary thermal radiation subject to this type of non-commutativity may drive inflation. We thus realize the inflationary scenario without the aid of an inflaton field. As the radiation cools down below Planck’s temperature, inflation gracefully exits into a standard Big Bang universe, dispensing with a period of reheating. Curiously in this regime there is no VSL, as the color temperature is found to saturate.

Recently, progress in String/M theory and Spin Networks have realized a non-commutative phase. We have conjectured a non-commutative version of the Friedmann-Walker space-time in our realization of the equivalence of VSL and \( \kappa \) deformed non-commutativity in order study the dynamics the early universe. We expect that this type of non-commutativity to come from a non-constant B field in a curved background in String-
We appreciate that this is beyond the state of the art in string-theory. However, it would be important to find a more concrete realization of NCFRW from string/M theory.

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