The Cosmological Constant Problem
and Kaluza-Klein Theory

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Abstract
We present technical results which extend previous work and show that the cosmological constant of general relativity is an artefact of the reduction to 4D of 5D Kaluza-Klein theory (or 10D superstrings and 11D supergravity). We argue that the distinction between matter and vacuum is artificial in the context of ND field theory. The concept of a cosmological “constant” (which measures the energy density of the vacuum in 4D) should be replaced by that of a series of variable fields whose sum is determined by a solution of ND field equations in a well-defined manner.

1 Introduction
The cosmological constant problem in 4D field theory can shortly be stated as a mismatch in energy densities between those predicted for particle interactions and that measured by astrophysics and cosmology [1-4]. The former energies include those due to zero-point fields and contributions from the quantum-mechanical vacuum; while the latter is the energy density of the vacuum in general relativity, and (up to absorbable constants) is given by the cosmological constant Λ. Following earlier work [5], it was recently shown that when the 5D Kaluza-Klein field equations are reduced to the 4D Einstein equations, a cosmological constant appears which is fixed in size by a length parameter in the metric but has no more fundamental significance [6,7]. It is by now widely known that the 15 Kaluza-Klein equations for apparent vacuum can be rewritten as 10 Einstein equations with matter, plus 4 Maxwell or conservation equations, and 1 wave equation in a scalar field which augments gravity and electromagnetism [3,5,6,7]. It is in fact always possible in Riemannian geometry to locally embed an ND manifold in an (N+1)D Ricci-flat manifold [8-10]. This implies that the appearance of an artefact like Λ is generic in the reduction to 4D general relativity of 5D Kaluza-Klein theory [11], 10D superstrings [12] and 11D supergravity [13]. In what follows, we will extend previous work [6-7] to show that the sign of Λ depends on the signature of the 5D metric, and that its size depends on parameters in the metric. Our conclusion will be that when an $N \geq 5$ theory is reduced to give an $N = 4$ energy-momentum tensor, the latter in general contains matter and vacuum parts whose distinction is artificial. It therefore makes little sense to talk of a cosmological “constant”. Rather,
there are contributions to the energy density which are in general variable, but whose sum is determined by a solution to the field equations in a well-defined manner.

2 The Cosmological “Constant” in 5D

In this section, we concentrate on 5D Kaluza-Klein theory, which is commonly regarded as the low-energy limit of higher-dimensional theories and can be embedded in them. The aim is to see what can be said about the reduction of the Kaluza-Klein equations

$$ R_{AB} = 0 $$

(1)

to the Einstein equations

$$ R_{\alpha\beta} - \frac{R g_{\alpha\beta}}{2} - \Lambda g_{\alpha\beta} = T_{\alpha\beta} , $$

(2)

and especially the cosmological constant $\Lambda$. Here $R_{AB}$ is the 5D Ricci tensor ($A, B = 0, 123, 4$); and $R_{\alpha\beta}$, $R$, $g_{\alpha\beta}$ and $T_{\alpha\beta}$ are the 4D Ricci tensor, Ricci scalar, metric tensor and energy-momentum tensor ($\alpha, \beta = 0, 123$). We use units which render $c = 1$, $8\pi G = 1$. It is well known that we can move the last term on the l.h.s. of (2) to the r.h.s., where it determines an energy density and pressure for the 4D vacuum via

$$ \rho = -p = \Lambda . $$

(3)

It is also known [3,5] that (1) contain (2), provided we move $\Lambda$ and define

$$ T_{\alpha\beta} \equiv \frac{\Phi_{\alpha\beta}}{\Phi} - \frac{\varepsilon}{2\Phi^2} \left\{ \Phi g^{\alpha\beta} \Phi - g^{\alpha\beta} + g^{\lambda\mu} g_{\alpha\lambda} g_{\beta\mu} - \frac{g_{\mu\nu} g_{\lambda\sigma} g_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4} \left[ g^{\mu\nu} g_{\mu\nu} + \left( g^{\mu\nu} g_{\mu\nu} \right)^2 \right] \right\} . $$

(4)

Here the 5D metric is $dS^2 = g_{AB} dx^A dx^B$ and contains the 4D metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. We have $g_{\alpha\beta} = g_{\alpha\beta}(x^A)$, $g_{4\alpha} = 0$ and $g_{44} = \varepsilon \Phi^2(x^A)$, where $\varepsilon = \pm 1$. We have only used 4 of the 5 available coordinate degrees of freedom to set the electromagnetic potentials ($g_{4\alpha}$) to zero, so the metric is
still general. The effective 4D energy-momentum tensor (4) is also general, and is the basis of induced-matter theory (for reviews see refs. 3 and 11). There is now a large literature on this, but its essence is clear from (4): the source of the 4D gravitational field in (2) can be derived from the vacuum equations (1), provided the latter are not restricted by arbitrary symmetry conditions. [In (4), $\Phi^\alpha \equiv \partial \Phi / \partial x^\alpha$, $\Phi_{\alpha\beta}$ is the 4D covariant derivative of $\Phi$, and $\Phi \equiv \partial \Phi / \partial \ell$ with $x^4 = \ell$ etc., so $T_{\alpha\beta}$ depends on $g_{44} = \varepsilon \Phi^2$ and $g_{\alpha\beta} = g_{\alpha\beta}(x^\alpha, \ell)$.] Many exact solutions of (1) are known, which have been applied with (4) to systems ranging from cosmological fluids [14] to elementary particles [15]. It is clear from (4) that 4D matter as derived from 5D geometry is a sum of contributions that depend on the scalar field ($\Phi$), the 4D metric ($g_{\alpha\beta}$) and the signature of the 5D metric ($\varepsilon = \pm 1$). Although it is already apparent that the exercise is somewhat artificial, let us proceed to try to isolate the energy density of the vacuum as measured by $\Lambda$.

In what are termed canonical coordinates we write $g_{\alpha\beta} = (\ell^2 / L^2) \tilde{g}_{\alpha\beta}(x^\alpha, \ell)$ and $g_{44} = \varepsilon \Phi^2 = -1$ [6]. The latter condition uses up the last degree of freedom allowed by the metric, so the problem is still general. But if we now impose also $\partial \tilde{g}_{\alpha\beta} / \partial \ell = 0$, we find that (2) are satisfied with a cosmological constant

$$\tilde{\Lambda} = \frac{3}{L^2}.$$  \hspace{1cm} (5)

Here $L$ is a constant length, introduced to the metric for dimensional consistency. If $L$ is large, then $\tilde{\Lambda}$ is small as required by cosmology [3,4]. However, $\tilde{\Lambda}$ cannot be zero if we require that the 4D part of the 5D metric be finite in the solar system [3,7]. For then (1) are satisfied with $\partial \tilde{g}_{\alpha\beta} / \partial \ell = 0$ and $\tilde{\Lambda} = 3 / L^2$ by

$$dS^2 = \frac{\tilde{\Lambda} \ell^2}{3} \left\{ 1 - \frac{2M}{r} - \frac{\tilde{\Lambda} r^2}{3} \right\} dt^2 - \left[ 1 - \frac{2M}{r} - \frac{\tilde{\Lambda} r^2}{3} \right]^{-1} dr^2 - r^2 d\Omega^2 \right\} - d\ell^2.$$ \hspace{1cm} (6)

Here $M$ is the mass and $d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2)$, so this is a 5D embedding for the 4D Schwarzschild solution. It is known that geodesic motion for the 5D metric (6) reproduces that for the embedded 4D metric [3,6] so there is no way to differentiate them using the classical tests of relativity [4]. We
Note in passing that the imaginary transformation \( t \rightarrow it, \ r \rightarrow ir, \ \ell \rightarrow i\ell, \ M \rightarrow iM \) reproduces (6) with the opposite signs for \( \tilde{\Lambda} \) and the last term in the metric. We will return to the signature of the metric below, where we will inquire whether as in (5) we need to have \( \tilde{\Lambda} > 0 \) or are allowed to have \( \tilde{\Lambda} < 0 \).

In the preceding paragraph, we took the 4D part of the 5D metric as \( g_{\alpha\beta} = (\ell^2/L^2) \tilde{g}_{\alpha\beta}(x^\alpha) \) and effectively used \( \tilde{g}_{\alpha\beta} \) to define \( \tilde{\Lambda} = 3/L^2 \) in (5). However, if we use \( g_{\alpha\beta} \) instead, we find \( R_{\alpha\beta} = -3g_{\alpha\beta}/\ell^2 \). This describes an empty spacetime with a cosmological constant

\[ \Lambda = \frac{3}{\ell^2} . \tag{7} \]

Here, \( \ell \) is the fifth coordinate, and in the static limit the correspondence between the energy of a test particle in 4D and 5D requires the identification \( \ell = m \) where \( m \) is the rest mass [3,7]. Thus (7) implies that each particle of mass \( m \) determines its own \( \Lambda \). This is Machian [16,17]; but does not qualify \( \Lambda \) of (7) to be called the cosmological “constant”. The ambiguity between \( \tilde{\Lambda} \) of (5) and \( \Lambda \) of (7) is connected technically with whether we use \( \tilde{g}_{\alpha\beta} \) or \( g_{\alpha\beta} \) to raise and lower indices, and practically with whether a 4D observer experiences the “pure” (\( \ell \)-independent) 4D metric or the “mixed” (\( \ell \)-dependent) 4D part of a 5D metric. We defer a consideration of this question, because it will be seen by what follows to become moot.

The signature of the 5D metric in Kaluza-Klein theory has important implications for the sign of the cosmological constant. In older work, the signature was often taken to be (\(+ - - - -\)). However, in modern work it is frequently left general via (\(+ - - - \varepsilon\)); and there are well-behaved solutions with good physical properties which describe waves in vacuum [18] or galaxies in clusters [19] which have signature (\(+ - - - +\)). Let us consider a situation similar to those above, but now with a 5D metric \( dS^2 = (\ell^2/L^2) \tilde{g}_{\alpha\beta}(x^\alpha, \ell) \ dx^\alpha dx^\beta + \varepsilon d\ell^2 \) which contains a 4D metric \( ds^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta \) which we restrict as before via \( \partial \tilde{g}_{\alpha\beta}/\partial \ell = 0 \). For this, the non-vanishing 5D Christoffel symbols of the second kind in terms of their counterparts in 4D may be shown to be

\[
(5) \Gamma^\alpha_{\beta\gamma} = (4) \Gamma^\alpha_{\beta\gamma}, \quad (5) \Gamma^\alpha_{\beta4} = \ell^{-1} (4) \delta^\alpha_{\beta}, \quad (5) \Gamma^4_{\alpha\beta} = -\varepsilon \ell L^{-2} \tilde{g}_{\alpha\beta} . \tag{8}
\]

Using these, we calculate the components of the Ricci tensor as

\[
(5) R_{\alpha\beta} = (4) R_{\alpha\beta} - 3\varepsilon L^{-2} \tilde{g}_{\alpha\beta}, \quad (5) R_{4\alpha} = 0, \quad (5) R_{44} = 0 . \tag{9}
\]

\[ 5 \]
These with the field equations (1) give $R_{\alpha\beta} = -\tilde{\Lambda}\tilde{g}_{\alpha\beta}$ with a cosmological constant

$$\tilde{\Lambda} = \frac{-3\varepsilon}{L^2}. \quad (10)$$

We see that if $\varepsilon = -1$ and the fifth dimension is spacelike, then as above in (5) $\Lambda > 0$. While if $\varepsilon = +1$ and the fifth dimension is timelike, then $\Lambda < 0$. This raises the intriguing possibility that we could determine the signature of the (higher-dimensional) world if we could determine the sign of the cosmological constant.

Let us now turn our attention from the field equations to the equations of motion. The latter are commonly derived from the variational principle, which in 5D is writeable symbolically as $\delta \int dS = 0$, and leads to the 5D geodesic equation

$$\frac{d^2x^A}{dS^2} + (5)\Gamma^A_{BC} \frac{dx^B}{dS} \frac{dx^C}{dS} = 0. \quad (11)$$

This has been much studied (see ref. 3 for a bibliography); but here we follow a new method using as above the metric $dS^2 = (\ell^2/L^2) ds^2 + \varepsilon d\ell^2$ with $ds^2 = \tilde{g}_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta$. The first of these relations can be written as

$$\frac{\ell^2}{L^2} \left( \frac{ds}{dS} \right)^2 + \varepsilon \left( \frac{d\ell}{dS} \right)^2 = 1. \quad (12)$$

Taking $d/dS$ of this we get

$$\frac{\ell^2}{L^2} \frac{ds}{dS} \frac{d^2s}{dS^2} + \frac{\ell}{L^2} \frac{d\ell}{dS} \left( \frac{ds}{dS} \right)^2 + \varepsilon \frac{d\ell}{dS} \frac{d^2\ell}{dS^2} = 0. \quad (13)$$

However, the $A = 4$ component of (11) gives with (8) the motion in the extra dimension as

$$\frac{d^2\ell}{dS^2} = -(5)\Gamma^4_{BC} \frac{dx^B}{dS} \frac{dx^C}{dS} = -(5)\Gamma^4_{\beta\gamma} \frac{dx^\beta}{dS} \frac{dx^\gamma}{dS}$$

$$= \frac{\varepsilon\ell}{L^2} \tilde{g}_{\alpha\beta} \frac{dx^\alpha}{dS} \frac{dx^\beta}{dS} = \frac{\varepsilon\ell}{L^2} \left( \frac{ds}{dS} \right)^2. \quad (14)$$

Substituting this into (13) gives
\[
\frac{d^2 s}{dS^2} = -2 \frac{d\ell}{\ell} \left( \frac{ds}{dS} \right)^2 .
\] (15)

This is a convenient relation. We note that it is invariant under \( S \to iS \),
which connects with the possibility that an acausal 5D manifold may contain
a 4D causal one [20]. To proceed, we note that we can manipulate derivatives
and use (15) to write

\[
\frac{d^2 x^A}{dS^2} = \left( \frac{ds}{dS} \right)^2 \left[ \frac{d^2 x^A}{ds^2} - \frac{2}{\ell} \frac{d\ell}{ds} \frac{dx^A}{ds} \right] .
\] (16)

We can use this with (14) to rewrite the fourth component of the geodesic
equation (11) as

\[
\frac{d^2 x^A}{dS^2} + (5) \Gamma^4_{BC} \frac{dx^B}{dS} \frac{dx^C}{dS} = \left( \frac{ds}{dS} \right)^2 \left[ \frac{d^2 \ell}{dS^2} - \frac{2}{\ell} \left( \frac{d\ell}{dS} \right)^2 - \frac{\varepsilon\ell}{L^2} \right] .
\] (17)

The motion is geodesic if

\[
\frac{d^2 \ell}{ds^2} - \frac{2}{\ell} \left( \frac{d\ell}{ds} \right)^2 - \frac{\varepsilon\ell}{L^2} = 0 .
\] (18)

This can be rewritten as

\[
\frac{d^2}{ds^2} \left( \frac{1}{\ell} \right) + \frac{\varepsilon}{L^2} \left( \frac{1}{\ell} \right) = 0 .
\] (19)

Solutions of this will depend on two arbitrary constants, which we take to
be special values \( \ell_* \) and \( s_* \) of the fifth coordinate and the 4D interval. Then
for the choices of \( \varepsilon \) we can write the solutions of (19) as

\[
\ell = \frac{\ell_*}{\cosh \left[ \frac{(s-s_*)}{L} \right]}, \quad \varepsilon = -1
\]

\[
\ell = \frac{\ell_*}{\cos \left[ \frac{(s-s_*)}{L} \right]}, \quad \varepsilon = +1 .
\] (20)

We see that the motion in the fifth dimension depends on the signature.
And if \( L \) is related to \( \Lambda \) via a relation like (5) or (6), it also depends on the
cosmological constant. Further, if \( \ell \) is related to the rest mass \( m \) of a particle.
then the latter may either increase or decrease with cosmological time depending on the signs of $\varepsilon$ and $\Lambda$. These possibilities naturally lead us to wonder if the motion in 4D spacetime is geodesic or not. To answer this, we use (8) and (16) in the expanded version of (11), which then gives

$$\frac{d^2 x^\alpha}{dS^2} + \Gamma^\alpha_{BC} \frac{dx^B}{dS} \frac{dx^C}{dS} = \left( \frac{ds}{dS} \right)^2 \left[ \frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} \right].$$ (21)

Clearly, if the 5D motion is geodesic then so is the 4D motion. This result agrees with others in the literature [3, 6, 7, 20, 21] and is remarkable: questions to do with $\varepsilon$, $\Lambda$, and $m$ are confined to the fifth dimension and the motion is the standard kind in the four dimensions of spacetime.

To here, we have concentrated on elucidating the nature of the cosmological constant by examining the field equations and the equations of motion for metrics of canonical form. We have seen that there are, from the 5D perspective, several different ways to define this parameter. From the field equations, and particularly $R_{\alpha\beta}$, we can obtain relations like (5) and (7), which modulo a conformal factor in the 4D metric are equivalent mathematically. However, they are different physically. From the equations of motion, we can obtain relations like (20) which implicate the cosmological constant in the fifth component of the geodesic but leave the four spacetime components (21) the same as in general relativity, which means by (6) that there is a cosmological force $\tilde{\Lambda} r/3$ that acts in the solar system and other 1-body systems. However, unlike in general relativity, the 4D part of the 5D metric is only finite if $\tilde{\Lambda}$ is finite. In addition to geometrical and dynamical ways to define the cosmological constant must be added that which embodies the equation of state, which in Einstein theory is (3). This brings us back to (4), the general expression for the induced energy-momentum tensor which is obtained when Einstein’s equations (2) are embedded in the Kaluza-Klein equations (1). In the above, we have focused for algebraic reasons on how to derive the cosmological constant for metrics in canonical coordinates (6). However, modern Kaluza-Klein theory is fully covariant [3] so the question arises of whether it is possible to identify a contribution to the $T_{\alpha\beta}$ of (4) that can be uniquely attributed to a cosmological constant. We believe that the answer to this question is No. Some comments to support this are in order. (a) While the last term in (4) is proportional to $g_{\alpha\beta}$, the coefficient is essentially the Ricci scalar [5], and so cannot be identified with a cosmological constant. (b) There is no way to tell which if any terms in (4) may
become proportional to $g_{\alpha\beta}$ after substitution of an exact solution of the field equations. (c) Most physical interpretations of (4) are made by comparison with a single perfect fluid, but multi-fluid models (e.g. matter and radiation) with possibly imperfect fluids (e.g. with viscosity) are more realistic, and it is difficult to see how to extract a component due to a cosmological constant from there. (d) The $T_{\alpha\beta}$ of (4) satisfies Einstein’s equations (2) in a formal manner, but it contains terms that depend on a scalar field which lies outside general relativity, so the “matter” in (4) can be different to what is conventional, and the equation of state of the “vacuum” can also be different. (e) Indeed, it is clear from an inspection of (4) that there is no unique way to separate what are conventionally called “matter” and “vacuum”, and while there is currently some discussion about whether the effects normally attributable to dark matter may be due to a cosmological constant with the fluid characteristics of (3), the 5D view as formalized by (4) is that “matter” and “vacuum” contributions to the energy density are mixed, the distinction having more to do with history than physics. (f) There are exact solutions of the field equations (1) known which with (4) can be interpreted as involving ordinary matter and a cosmological “constant” which is time-dependent [22], but while these may help resolve well-known problems with the matching of cosmological data [23], they indicate that a cosmological “constant” if it is defineable at all can be a function of the coordinates.

3 Conclusion

We have presented a series of technical results involving the embedding of 4D Einstein theory in 5D Kaluza-Klein theory, which by extension can be applied to 10D superstrings and 11D supergravity. In the 5D case, it is arguable that there is no logical distinction between “matter” and “vacuum” contributions to the energy density; and since the vacuum in general relativity can be related to the cosmological constant, we are lead to seriously doubt if the latter parameter has any real meaning. What does have meaning is an exact solution of the ND field equations, which when reduced to 4D defines an induced energy-momentum tensor whose various and variable terms determine the energy density in a well-defined manner. This implies that the mismatch in energy densities derived from quantum field theory and general relativity is merely a consequence of restricting the physics to 4D. Put another way, we believe that the cosmological constant problem does not
exist in $N \geq 5D$.

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**References**


