Superconformal Tensor Calculus in Five Dimensions

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Abstract

We present a full superconformal tensor calculus in five spacetime dimensions in which the Weyl multiplet has 32 Bose plus 32 Fermi degrees of freedom. It is derived using dimensional reduction from the 6D superconformal tensor calculus. We present two types of 32+32 Weyl multiplets, a vector multiplet, linear multiplet, hypermultiplet and nonlinear multiplet. Their superconformal transformation laws and the embedding and invariant action formulas are given.

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§1. Introduction

Five-dimensional supergravity, which was once extensively studied, 1) has recently received much attention again. 2)-5) This renewed interest is due partly to the study of the AdS/CFT correspondence conjecture. This conjecture suggests that gauged supergravity in a background geometry $\text{AdS}_5 \times H$ ($H = S^5$ in the original example of Ref. 6)) is related to superconformal field theory in a four-dimensional Minkowski space on the boundary of $\text{AdS}_5$.

Another line of investigation that has motivated study of five-dimensional supergravity is the search for supersymmetric brane-worlds scenarios. In particular, from both phenomenological and theoretical viewpoints, it is interesting to supersymmetrize Randall-Sundrum scenarios. 7), 8) The simplest candidate for the supersymmetric Randall-Sundrum two branes model, namely RS1, 7) is five-dimensional supergravity compactified on the $S^1/Z^2$ orbifold. In the five-dimensional bulk, there exists a minimal or nonminimal supergravity multiplet 4) that contains a graviton, gravitino and graviphoton. This multiplet is trapped on the branes, reduces to the four-dimensional minimal multiplet, and couples to the four-dimensional matter multiplets, e.g., the chiral and vector multiplets. Further, we can couple this multiplet to various matter multiplets in the bulk, for example the vector, hyper and tensor multiplets. In order to work with these models, off-shell formalisms 9)-12) rather than on-shell formalisms facilitate the analysis, because with these, we need not change the transformation laws of the supersymmetry, whichever couplings are considered. Still, it is laborious to study a large class of such models systematically. However, the gauge equivalence method using superconformal tensor calculus makes this task easy. In this formulation, to construct different off-shell formulations, we have only to add a compensating multiplet to the Weyl multiplet, so we can treat all of the above mentioned couplings in a common framework. However, unfortunately, five-dimensional conformal supergravity has not yet been studied.

Standard conformal supergravity can be described on the basis of superconformal algebra. Superconformal algebra exists only in six or fewer dimensions, 13) and its gauge theory has been constructed in the case of 16 or fewer supercharges, 14)-18) except for the case of $N = 2, d = 5$ theory, in which we are interested. (Five-dimensional conformal supergravity that is not based on superconformal algebra was constructed through dimensional reduction from ten-dimensional conformal supergravity. 19)) In this paper we fill the gap in the literature by constructing $N = 2, d = 5$ superconformal tensor calculus in a complete form.

In Ref. 9), 5D tensor calculus was derived from the known 6D superconformal tensor calculus 17) using the method of dimensional reduction. However, unfortunately, some of the superconformal symmetries ($S$ and $K$) are gauge-fixed in the process of the reduction. The
dimensional reduction is in principle straightforward and hence more convenient than the conventional trial-and-error method to find the multiplet members and their transformation laws. Therefore, here we follow essentially the dimensional reduction used in Ref. 9) to find the 5D superconformal tensor calculus from the 6D one. We keep all the 5D superconformal gauge symmetries unfixed in the reduction process. The Weyl multiplet obtained from a simple reduction contains 40 bose and 40 fermi degrees of freedom. However, it turns out that this $40 + 40$ multiplet splits into two irreducible pieces, a $32 + 32$ minimal Weyl multiplet and an $8 + 8$ ‘central charge’ vector multiplet (which contains a ‘dilaton’ $e_z^5$ and a graviphoton $\propto e^\mu_5$ as its members). This splitting is performed by inspecting and comparing the transformation laws of both the Weyl and vector multiplets. It contains a process that is somewhat trial-and-error in nature, but can be carried out relatively easily. Once this minimal Weyl multiplet is found, the other processes of finding matter multiplets and other formulas, like invariant action formulas, proceed straightforwardly and are very similar to those in the previous Poincaré supergravity case. 9)

For the reader’s convenience, we give the details of the dimensional reduction procedure in Appendix B and present the resultant transformation law of the minimal $32 + 32$ Weyl multiplet in $\S 2$. The transformation rules of the matter multiplets are given in $\S 3$; the multiplets we discuss are the vector (Yang-Mills) multiplet, linear multiplet, hypermultiplet and nonlinear multiplet. In $\S 4$, we present some embedding formulas of multiplets into multiplet and invariant action formulas. In $\S 5$, we present another $32 + 32$ Weyl multiplet that corresponds to the Nisino-Rajpoot version of Poincaré supergravity. This multiplet is expected to appear by dimensional reduction from the ‘second version’ of 6D Weyl multiplet containing a tensor $B_{\mu\nu}$ and by separating the $8 + 8$ vector multiplet. Here, however, we construct it directly in 5D by imposing a constraint on a set of vector multiplets. Section 6 is devoted to summary and discussion. The notation and some useful formulas are presented in Appendix A. In Appendix D, we explain the relation between conformal supergravity constructed in this paper and Poincaré supergravity worked out in Ref. 9).

2. Weyl multiplet

The superconformal algebra in five dimensions is $F^2(4)$. Its Bose sector is $SO(2, 5) \oplus SU(2)$. The generators of this algebra $X_A$ are

$$X_A = P_a, \ Q_i, \ M_{ab}, \ D, \ U_{ij}, \ S_i, \ K_a, \ (2.1)$$

where $a, b, \ldots$ are Lorenz indices, $i, j, \ldots (= 1, 2)$ are $SU(2)$ indices, and $Q_i$ and $S_i$ have spinor indices implicitly. $P_a$ and $M_{ab}$ are the usual Poincaré generators, $D$ is the dilatation,
\( U_{ij} \) is the \( SU(2) \) generator, \( K_a \) represents the special conformal boosts, \( Q_i \) represents the \( N = 2 \) supersymmetry, and \( S_i \) represents the conformal supersymmetry. The gauge fields \( h_{\mu}^A \) corresponding to these generators are

\[
\bar{h}_{\mu}^A = e_{\mu}^a, \quad \psi_{\mu}^i, \quad \omega_{\mu}^{a\beta}, \quad b_{\mu}, \quad V_{\mu}^{ij}, \quad \phi_{\mu}^i, \quad f_{\mu}^a, \quad (2.2)
\]

respectively, where \( \mu, \nu, \ldots \) are the world vector indices and \( \psi_{\mu}^i, \phi_{\mu}^i \) are \( SU(2) \)-Majorana spinors. (All spinors satisfy the \( SU(2) \)-Majorana condition in this calculus.) In the text we omit explicit expression of the covariant derivative \( \hat{\mathcal{D}}_{\mu} \) and the covariant curvature \( \hat{\mathcal{R}}_{\mu\nu}^A \) (field strength \( \hat{F}_{\mu\nu}^A \)). In our calculus, the definitions of these are given as follows:

\[
\hat{\mathcal{D}}_{\mu}\Phi \equiv \partial_{\mu}\Phi - h_{\mu}^AX_A\Phi, \quad (2.3)
\]

\[
\hat{\mathcal{R}}_{\mu\nu}^A = e_{\mu}^b e_{\nu}^a f_{ab}^A = 2\partial_{[\mu}h_{\nu]}^A - h_{\mu}^Ch_{\nu}^B f_{BC}^A. \quad (2.4)
\]

Here, \( X_A \) denotes the transformation operators other than \( P_a \), and \( f_{AB}^C \) is a 'structure function', defined by \( [X_A, X_B] = f_{AB}^CX_C \), which in general depends on the fields. The prime on the structure function in (2.4) indicates that the \([P_a, P_b]\) commutator part, \( f_{ab}^A \), is excluded from the sum. Note that this structure function can be read from the transformation laws of the gauge fields: \( \delta(\varepsilon)h_{\mu}^A = \delta_B(\varepsilon^B)h_{\mu}^A = \partial_{\mu}\varepsilon^A + \varepsilon^Ch_{\mu}^B f_{BC}^A \).

### 2.1. Constraints and the unsubstantial gauge fields

In the superconformal theories in 4D and 6D, the conventional constraints on the superconformal curvatures are imposed to lift the tangent-space transformation \( P_a \) to the general coordinate transformation of a Weyl multiplet. These constraints are the usual torsion-less condition,

\[
\hat{R}_{ab}^c(P) = 0, \quad (2.5)
\]

and two conditions on \( Q \) and \( M \) curvatures of the following types:

\[
\gamma^b\hat{R}_{ab}(Q) = 0, \quad \hat{R}_{ac}^{\ b}(M) = 0. \quad (2.6)
\]

The spin-connection \( \omega_{\mu}^{a\beta} \) becomes a dependent field by the constraint (2.5), and the \( S_i \) and \( K_a \) gauge fields, \( \phi_{\mu}^i \) and \( f_{\mu}^a \), also become dependent through the constraints (2.6). To this point, it has been the conventional understanding that imposing these curvature constraints is unavoidable for the purpose of obtaining a meaningful local superconformal algebra. However, it is actually possible to avoid imposing the constraints explicitly, but we can obtain an equivalent superconformal algebra. This fact is not familiar, so for a

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*Only a spinor of the hypermultiplet \( \zeta_\alpha \) is not such a spinor, but a \( USp(2n) \)-Majorana spinor.*
better understanding we illustrate this approach with the transformation laws of the well-known \( N = 1, d = 4 \) Weyl multiplet in Appendix C. We now explain how this is possible by considering an example. In 5D, the covariant derivative of the spinor \( \Omega^i \) of a vector multiplet and the \( Q \) curvature contain the \( S_i \) gauge field \( \phi^i_\mu \) in the form

\[
\hat{D} \Omega^i \big|_{\phi-\text{term}} = M^a \gamma^a \phi^i_\mu, \quad \gamma \cdot \hat{R}^i(Q) \big|_{\phi-\text{term}} = 8 \gamma^a \phi^i_\mu. \tag{2.7}
\]

However, in fact, in the supersymmetry transformation \( \delta Y^{ij} \) of the auxiliary field \( Y^{ij} \) of the vector multiplet, only the combination

\[
C \equiv 2 \hat{D} \Omega^i - \frac{1}{4} \gamma \cdot \hat{R}^i(Q) M \tag{2.8}
\]

appears, and the gauge field \( \phi^i_\mu \) is actually canceled in these two terms. Since this combination contains no \( \phi^i_\mu \), we can set \( \phi^i_\mu = \phi^{\text{sol}}_i \), the solution \( \phi_\mu \) to the conventional constraint \( \gamma^a \hat{R}_{ab}^i(Q) = 0 \). Then the \( \gamma \cdot \hat{R}^i(Q) \) term vanishes, and the combination clearly reduces to

\[
C = 2 \hat{D} \Omega^i \big|_{\phi_\mu \rightarrow \phi^{\text{sol}}_i}, \tag{2.9}
\]

reproducing the result of the conventional approach. The virtue of our approach is, however, that it is independent of the form of the constraints. If the constraints are changed into \( \gamma^a \hat{R}_{ab}^i(Q) = \gamma_b \chi^i \), with a certain spinor \( \chi^i \), then the combination takes an apparently different form,

\[
C = 2 \hat{D} \Omega^i \big|_{\phi_\mu \rightarrow \phi^{\text{sol}}_i} + \frac{5}{4} \chi^i M. \tag{2.10}
\]

Everywhere in this calculus, in the transformation laws, the algebra, the embedding formulas, the action formulas, and so on, such cancellations occur, so the gauge fields \( \phi^i_\mu \) and \( f^{\mu a} \) actually disappear completely.

In this 5D calculus, we adopt the usual torsion-less condition (2.5), but we do not impose constraints on the \( Q_i \) and \( M_{ab} \) curvatures, because no such constraints significantly simplify the 5D calculus, and the formulation with no constraint is convenient to reduce Poincaré supergravity calculus from this conformal one. We comment on these reductions in Appendix D. To make the expressions brief, we define the covariant quantities \( \phi^i_a(Q), f^{a b}(M), K_{ab}(Q) \) as

\[
\phi^i_a(Q) \equiv \frac{1}{8} \gamma^b \hat{R}_{ab}^i(Q) - \frac{1}{24} \gamma_a \gamma \cdot \hat{R}^i(Q),
\]

\[
f^{a b}(M) \equiv - \frac{1}{6} \hat{R}_{ab}(M) + \frac{1}{18} \eta_{ab} \hat{R}(M),
\]

\[
K_{ab}(Q) \equiv \hat{R}_{ab}(Q) + 2 \gamma_{[a} \phi_{b]}(Q)
\]

\[
= \hat{R}_{ab}(Q) + \frac{2}{3} \gamma_{[a} \gamma^c \hat{R}_{bc]}(Q) - \frac{1}{12} \gamma_{ab} \gamma \cdot \hat{R}(Q), \tag{2.11}
\]

as

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Table I. Weyl multiplet in 5D.

<table>
<thead>
<tr>
<th>field</th>
<th>type</th>
<th>remarks</th>
<th>SU(2)</th>
<th>Weyl-weight</th>
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<tr>
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<td>fünfbein</td>
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<td>-1</td>
</tr>
<tr>
<td>$\psi^i_\mu$</td>
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<td>SU(2)-Majorana</td>
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<td>$b_\mu$</td>
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<td>0</td>
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<tr>
<td>$V^{ij}_\mu$</td>
<td>boson</td>
<td>$V^{ij}<em>\mu = V^{ji}</em>\mu = (V^{\mu ij})^*$</td>
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<td>0</td>
</tr>
<tr>
<td>$v_{ab}$</td>
<td>boson</td>
<td>real, antisymmetric</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^i$</td>
<td>fermion</td>
<td>SU(2)-Majorana</td>
<td>2</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
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<td>2</td>
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<table>
<thead>
<tr>
<th>dependent (unsubstantial) gauge fields</th>
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</thead>
<tbody>
<tr>
<td>$\omega^{ab}_\mu$</td>
</tr>
<tr>
<td>$\phi^i_\mu$</td>
</tr>
<tr>
<td>$f^a_\mu$</td>
</tr>
</tbody>
</table>

where, $\hat{R}_{ab}(M) \equiv \hat{R}_{ac}^{\phantom{ac}cb}(M), \hat{R}(M) \equiv \hat{R}_{a}^{\phantom{a}a}(M)$. These quantities are defined in such a way that they contain the $S$ and $K$ gauge fields in the simple forms

$$
\phi^i_a(Q)|_{\phi,f} = \phi^i_a, \quad f_{ab}(M)|_{\phi,f} = f_{ab}, \quad K_{ab}(Q)|_{\phi,f} = 0, \quad (2.12)
$$

and $K_{ab}(Q)$ satisfies

$$
\gamma^a K_{ab}(Q) = 0. \quad (2.13)
$$

Since we impose the torsion-less constraint (2.5) in 5D too, the spin-connection is a dependent field given by

$$
\omega^{ab}_\mu = \omega^{0ab}_\mu + i(2\bar{\psi}_\mu \gamma^{[a} \psi^{b]} + \bar{\psi}^a \gamma_\mu \psi^b) - 2e^{[a}_\mu b^b],
$$

$$
\omega^{0ab}_\mu \equiv -2e^{[a}_\mu \partial_{[a} e^{b]} + e^{[a}_\mu \gamma^\sigma e_{[a} \epsilon_{\mu b]} e_{\sigma c}. \quad (2.14)
$$

Of course, it would also be possible to avoid this torsion-less constraint in a similar way, but here we follow the conventional procedure.

2.2. The transformation law and the superconformal algebra

The superconformal tensor calculus in 5D can be obtained from the known one in 6D by carrying out a simple dimensional reduction. However, the Weyl multiplet directly obtained this way contains 40 + 40 degrees of freedom. Using the procedure explained in detail in Appendix B, we can separate an 8 + 8 component vector multiplet from it and obtain an irreducible Weyl multiplet which consists of 32 Bose plus 32 Fermi fields,

$$
e^a_\mu, \quad \psi^i_\mu, \quad V^{ij}_\mu, \quad b_\mu, \quad v^{ab}, \quad \chi^i, \quad D, \quad (2.15)
$$
whose properties are summarized in Table I. The full nonlinear $Q$, $S$ and $K$ transformation laws of the Weyl multiplet are given as follows. With $\delta \equiv \varepsilon i Q_i + \bar{\eta}^i S_i + \xi^a K_a \equiv \delta Q(\varepsilon) + \delta S(\eta) + \delta K(\xi^a)$,

$$
\delta e^a_{\mu} = -2i\varepsilon \gamma^a \psi_{\mu}, \\
\delta \psi^i_{\mu} = D_{\mu} \varepsilon^i + \frac{1}{2} \epsilon^{ab} \gamma_{\muab} \varepsilon^i - \gamma_{\mu} \eta^i, \\
\delta b_{\mu} = -2i \bar{\varepsilon} \phi_{\mu} + 2i \bar{\varepsilon} \phi_{\mu} (Q) - 2i \bar{\eta} \psi_{\mu} - 2 \xi_{K\mu}, \\
\delta \omega^{ab}_{\mu} = 2i \bar{\varepsilon} \gamma^a \phi_{\mu} - 2i \bar{\varepsilon} \gamma^a [\hat{R}_b] (Q) - 2i \bar{\varepsilon} \gamma_\mu \hat{R}^{ab} (Q) + 4i \bar{\varepsilon} \phi^{[ab} (Q) \epsilon_{\mu]} \\
- 2i \bar{\varepsilon} \gamma^{abcd} \psi_{\mu} \nu_{cd} - 2i \bar{\eta} \gamma^a \psi_{\mu} - 4 \xi_{K[a} \epsilon_{\mu]} b, \\
\delta V^{ij}_{\mu} = -6i \bar{\varepsilon} (i \gamma^{i} \hat{R}_{\mu}^{j}) (Q) - \frac{i}{4} \bar{\varepsilon} (i \gamma_{\mu} \hat{J}) (Q) \\
+ 4i \bar{\varepsilon} (i \gamma \cdot \nu \psi_{\mu}) - \frac{i}{4} \bar{\varepsilon} (i \gamma_{\mu} \lambda_{j}) + 6i \bar{\eta} (i \psi_{\mu}), \\
\delta v_{ab} \equiv \frac{i}{8} \bar{\varepsilon} \gamma_{ab} \chi - \frac{i}{8} \bar{\varepsilon} \gamma_{cd} \gamma_{ab} \hat{R}_{cd} (Q) + \frac{i}{2} \bar{\varepsilon} \hat{R}_{ab} (Q), \\
\delta \chi^i = D \varepsilon^i - 2 \gamma^\varepsilon \gamma_{ab} \varepsilon_{i} \hat{D}_{a} \nu_{bc} + \gamma_{\cdot} \hat{R} (U)^{ij} \varepsilon^j \\
- 2 \gamma^a \varepsilon^i \epsilon_{abce} \nu_{de} + 4 \gamma \cdot \nu \eta^i, \\
\delta D = -i \bar{\varepsilon} \hat{D} \chi - \frac{i}{2} \bar{\varepsilon} \gamma \cdot \nu \cdot \hat{R} (Q) - 8i \bar{\varepsilon} \hat{R}_{ab} (Q) \nu^{ab} + i \bar{\eta} \chi, \tag{2.16}
$$

where the derivative $D_{\mu}$ is covariant only with respect to the homogeneous transformations $M_{ab}, D$ and $U^{ij}$ (and the $G$ transformation for non-singlet fields under the Yang-Mills group $G$). We have also written the transformation law of the spin connection for convenience. The algebra of the $Q$ and $S$ transformations takes the form

$$
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_P(2i \bar{\varepsilon}_1 \gamma_a \varepsilon_2) + \delta_M(2i \bar{\varepsilon}_1 \gamma^{abcd} \varepsilon_2 \nu_{ab}) + \delta_U(-4i \bar{\varepsilon}_1 \gamma \cdot \nu \varepsilon_2) \\
+ \delta_S \left( \begin{array}{c}
2i \bar{\varepsilon}_1 \gamma^a \varepsilon_2 \phi_{ai}(Q) + i \bar{\varepsilon}_1 (i \gamma_{ab} \varepsilon_2 j) K^{abj}(Q) \\
+ \frac{3}{32} i \bar{\varepsilon}_1 \gamma_{ab} \chi_i + \frac{1}{32} i \bar{\varepsilon}_1 \gamma^a \varepsilon_2 \gamma_{a} \chi_i - \frac{1}{32} i \bar{\varepsilon}_1 (i \gamma_{ab} \varepsilon_2 j) \gamma_{ab} \chi^j \\
\end{array} \right) \\
+ \delta_K \left( \begin{array}{c}
2i \bar{\varepsilon}_1 \gamma^b \varepsilon_2 f_{\mu}^{a}(M) + \frac{1}{12} i \bar{\varepsilon}_1 \gamma^{abc} \varepsilon_2 (\hat{R}_{bc}^{ij}(U)) \\
+ \frac{1}{36} i \bar{\varepsilon}_1 \gamma_{abcd} \varepsilon_2 \hat{D}_{b} \nu_{cd} + \frac{1}{6} i \bar{\varepsilon}_1 \hat{D}_{b} \nu_{ab} \\
+ \frac{1}{3} i \bar{\varepsilon}_1 \gamma^a \varepsilon_2 \nu_{ab} + \frac{3}{3} i \bar{\varepsilon}_1 \gamma^b \varepsilon_2 \nu_{bc} \nu_{ca} \\
- \frac{1}{6} i \bar{\varepsilon}_1 \gamma_{abcd} \varepsilon_2 \nu_{bc} \nu_{de} \\
\end{array} \right), \tag{2.17}
$$

$$
[\delta_S(\eta), \delta_Q(\varepsilon)] = \delta_D(-2i \bar{\varepsilon} \eta) + \delta_M(2i \bar{\varepsilon} \gamma^{ab} \eta) + \delta_U(-6i \bar{\varepsilon} (i \eta^j)) \\
+ \delta_K \left( - \frac{5}{6} i \bar{\varepsilon} \gamma_{ab} \eta \nu^{ab} + i \bar{\varepsilon} \gamma^b \eta \nu_{ab} \right), \tag{2.18}
$$

where the translation $\delta_P(\xi^a)$ is understood to be essentially the general coordinate transformation $\delta_G(\xi^a)$:

$$
\delta_P(\xi^a) = \delta_G(\xi^a) - \delta_A(\xi^\lambda h^A). \tag{2.19}
$$
Table II. Matter multiplets in 5D.

<table>
<thead>
<tr>
<th>field</th>
<th>type</th>
<th>remarks</th>
<th>$SU(2)$</th>
<th>Weyl-weight</th>
</tr>
</thead>
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<tr>
<td>$W_\mu$</td>
<td>boson</td>
<td>real gauge field</td>
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<td>0</td>
</tr>
<tr>
<td>$M$</td>
<td>boson</td>
<td>real</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Omega^i$</td>
<td>fermion</td>
<td>$SU(2)$-Majorana</td>
<td>2</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$Y_{ij}$</td>
<td>boson</td>
<td>$Y^{ij} = Y^{ji} = (Y_{ij})^*$</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

**Hypermultiplet**

| $\mathcal{A}_i^\alpha$ | boson | $\mathcal{A}_i^\alpha = \epsilon^{ij} A_j^\beta \rho^\alpha = -(\mathcal{A}_i^\alpha)^*$ | 2       | $\frac{3}{2}$ |
| $\zeta^\alpha$         | fermion | $\tilde{\zeta}^\alpha \equiv (\zeta^\alpha)^\dagger \gamma_0 = \zeta^\alpha T_C$ | 1       | 2           |
| $\mathcal{F}_i^\alpha$ | boson | $\mathcal{F}_i^\alpha \equiv \alpha Z A_i^\alpha$, $\mathcal{F}_i^\alpha = -(\mathcal{F}_i^\alpha)^*$ | 2       | $\frac{5}{2}$ |

**Linear multiplet**

| $L_{ij}$ | boson | $L_{ij} = L_{ji} = (L_{ij})^*$ | 3       | 3           |
| $\varphi^i$ | fermion | $SU(2)$-Majorana | 2       | $\frac{7}{2}$ |
| $E_a$ | boson | real, constrained by (3.15) | 1       | 4           |
| $N$ | boson | real | 1       | 4           |

**Nonlinear multiplet**

| $\Phi^i_\alpha$ | boson | $SU(2)$-valued | 2       | 0           |
| $\lambda^i$ | fermion | $SU(2)$-Majorana | 2       | $\frac{1}{2}$ |
| $V^a$ | boson | real | 1       | 1           |
| $V^5$ | boson | real | 1       | 1           |

On a covariant quantity $\Phi$ with only flat indices, $\delta_P(\xi^a)$ acts as the full covariant derivative:

$$\delta_P(\xi^a)\Phi = \xi^a \left( \partial_a - \delta_A(h^4_a) \right) \Phi \equiv \xi^a \hat{D}_a \Phi.$$  

(2.20)

Note the consistency that the quantities $\phi^i_\alpha(Q)$ and $f_a^b(M)$ on the right-hand side of the algebra (2.17) cancel out the $S$ and $K$ gauge fields contained in $\delta_P(\xi^a)$.

§3. Transformation laws of matter multiplets

In 5D there are four kinds of multiplets: a vector multiplet, hypermultiplet, linear multiplet and nonlinear multiplet. The components of the matter multiplets and their properties are listed in Table II. The tensor multiplet in 6D reduces to a vector multiplet in 5D with constraints, and solving these constraints gives rise to an alternative type of the Weyl multiplet containing the two-form gauge field $B_{\mu\nu}$ in the same way as in 6D. We discuss $B_{\mu\nu}$
in §5.

The supersymmetry transformation laws of the matter multiplets are almost identical to those obtained in Ref. 9 in the Poincaré supergravity case if the ‘central charge vector multiplet’ components are omitted in the latter.

3.1. Vector multiplet

An important difference between the vector multiplets in 5D and in 6D is the existence of the scalar component $M$ in 5D, which allows for the introduction of the ‘very special geometry’

$$c_{IJK} M^I M^J M^K = 1$$

in the Poincaré supergravity theory. All the component fields of this multiplet are Lie-algebra valued, e.g., $M$ is a matrix $M^{\alpha\beta} = M^I(t_I)^{\alpha\beta}$, where the $t_I$ are (anti-hermitian) generators of the gauge group $G$. The $Q$ and $S$ transformation laws of the vector multiplet are given by

$$\delta W_\mu = -2i\bar{\varepsilon}_\mu \Omega + 2i\bar{\varepsilon}_\mu M,$$

$$\delta M = 2i\bar{\varepsilon}\Omega,$$

$$\delta \Omega^i = -\frac{1}{4}\gamma\cdot\hat{F}(W)\varepsilon^i - \frac{1}{2}\hat{D}M\varepsilon^i + Y^i_j\varepsilon^j - M\eta^i,$$

$$\delta Y^{ij} = 2i\bar{\varepsilon}^{(i}\hat{D}\Omega^{j)} - i\bar{\varepsilon}^{(i}\gamma\cdot\hat{F}(Q)\varepsilon^{j)} - \frac{i}{4}\bar{\varepsilon}^{(i}\chi^{j)M} - \frac{i}{4}\bar{\varepsilon}^{(i}\gamma\cdot\hat{R}^{j)}(Q)M + 2i\bar{\varepsilon}^{(i\hat{R}^{j)}M} - 2i\bar{\varepsilon}(M,\Omega^{j)}] - 2i\bar{\eta}^{(i\Omega^{j)}].$$

(3.1)

The gauge group $G$ can be regarded as a subgroup of the superconformal group, and the above transformation law of the gauge field $W_\mu$ provides us with the additional structure functions, $f_{PQ}^G$ and $f_{QQ}^G$. For instance, the commutator of the two $Q$ transformations becomes

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = (\text{R.H.S. of (2.17)}) + \delta_G(-2i\bar{\varepsilon}_1\varepsilon_2 M).$$

(3.2)

For the reader’s convenience, we give here the transformation laws of the covariant derivative of the scalar $\hat{D}_a M$ and the field strength $\hat{F}_{ab}(W)$:

$$\delta \hat{D}_a M = 2i\bar{\varepsilon}\hat{D}_a \Omega - 2i\bar{\varepsilon}\phi_a(Q)M + i\bar{\varepsilon}\gamma_{abc}\Omega v^{bc} + 2i\bar{\varepsilon}\gamma_{a}[\Omega, M] + 2i\bar{\eta}\gamma_a \Omega + 2\xi_{Ka}\Omega,$$

$$\delta \hat{F}_{ab}(W) = 4i\bar{\varepsilon}\gamma_{[a}\hat{D}_{b]}\Omega - 2i\bar{\varepsilon}\gamma_{cd[a}\gamma_{b]}\Omega v^{cd} + 2i\bar{\varepsilon}\hat{R}_{ab}(Q)M - 4i\bar{\eta}\gamma_{ab}\Omega.$$  

(3.3)

The transformation laws of a matter field acted on by a covariant derivative and the supercovariant curvature (field strength) are derived easily using the simple fact that the transformation of any covariant quantity also gives a covariant quantity and hence cannot contain gauge fields explicitly; that is, gauge fields can appear only implicitly in the covariant derivative or in the form of supercovariant curvatures, as long as the algebra closes. Similarly, the Bianchi identities can be computed by discarding the naked gauge fields with
no derivative, because both sides of the identity are, of course, covariant. For example, we have

\[ \mathcal{D}_{[a} \hat{F}_{bc]}(W) = -2i \bar{\Omega}_{[a} \hat{R}_{bc]}(Q). \]  

(3.4)

### 3.2. Hypermultiplet

The hypermultiplet in 5D consists of scalars \( A^i_\alpha \), spinors \( \zeta_\alpha \) and auxiliary fields \( F^i_\alpha \). They carry the index \( \alpha = (1, 2, \ldots, 2r) \) of the representation of a subgroup \( G' \) of the gauge group \( G \), which is raised (or lowered) with a \( G' \) invariant tensor \( \rho_{\alpha \beta} \) (and \( \rho^{\alpha \beta} \) with \( \rho^{\alpha \gamma} \rho_{\gamma \beta} = \delta_\beta^\alpha \)) like \( A^i_\alpha = A^{i\beta} \rho_{\beta \alpha} \). This multiplet gives an infinite dimensional representation of a central charge gauge group \( U_Z(1) \), which we regard as a subgroup of the group \( G \). The scalar fields \( A^i_\alpha \) satisfy the reality condition

\[ A^i_\alpha = \epsilon^{ij} A^j_\beta \rho_{\beta \alpha} = -(A^i_\alpha)^*, \quad A_{i\alpha} = (A^{i\alpha})^*, \]  

(3.5)

and the tensor \( \rho_{\alpha \beta} \) can generally be brought into the standard form \( \rho_{\alpha \beta} \) by a suitable field redefinition. Therefore \( A^i_\alpha \) can be identified with \( r \) quaternions. Thus the group \( G' \) acting linearly on the hypermultiplet should be a subgroup of \( GL(r; H) \):

\[ \delta_{G'}(t) A^i_\alpha = g t^{\alpha \beta} A^j_\beta, \quad \delta_{G'}(t) A^i_{i\alpha} = g(t^\alpha_\beta)^* A^j_{j\beta} = -g t^{\alpha \beta} A^j_{j\beta}, \]

\[ t^{\alpha \beta} \equiv \rho_{\alpha \gamma} t^{\gamma \delta} \rho^{\delta \beta} = -(t^\alpha_\beta)^*. \]  

(3.6)

Note that the spinors \( \zeta_\alpha \) do not satisfy the \( SU(2) \)-Majorana condition explicitly, but rather the \( USp(2r) \)-Majorana condition,

\[ \bar{\zeta}^\alpha \equiv (\zeta_\alpha)^T \gamma^0 = \rho^{\alpha \beta} (\zeta_\beta)^T C = (\zeta^\alpha)^T C. \]

(3.7)

The \( Q \) and \( S \) transformations of the \( A^i_\alpha \) and \( \zeta_\alpha \) are given by

\[ \delta A^i_\alpha = 2i \bar{\varepsilon}^j \zeta_\alpha, \]

\[ \delta \zeta_\alpha = \bar{\mathcal{D}} A^i_\alpha \varepsilon^j - \gamma \cdot v \varepsilon^j A^\alpha_j - M_s A^\alpha_j \varepsilon^j + 3 A^\alpha_j \eta^j, \]

(3.8)

and with these rules, to realize the superconformal algebra on the hypermultiplet requires the following two \( Q \) and \( S \) invariant constraints:

\[ 0 = \bar{\mathcal{D}} \zeta^\alpha + \frac{1}{2} \gamma \cdot v \zeta^\alpha - \frac{1}{8} \chi^i A^\alpha_i + \frac{3}{8} \gamma \cdot \hat{R}(Q) A^\alpha_i + M_s \zeta^\alpha - 2 \Omega^i_s A^\alpha_i, \]

\[ 0 = -\bar{\mathcal{D}}^s \hat{D}_a A^\alpha_i + M_s M^a_i A^\alpha_i + 4i \hat{\Omega}_{is} \zeta^\alpha - 2 Y_{ij} A^{\alpha j} - \frac{i}{4} \bar{\gamma}^\alpha \left( \chi + \gamma \cdot \hat{R}(Q) \right) \left( \frac{3}{16} \hat{R}(M) + \frac{1}{8} D - \frac{1}{4} v^2 \right) A^\alpha_i, \]

(3.9)
where \( \theta_* = M_*, \Omega_*, \ldots \) represents the Yang-Mills transformations with parameters \( \theta \), including the central charge transformation, that is \( \delta G(\theta) = \delta G'(\theta) + \delta Z(\theta^0) \). \( Z \) denotes the generator of the \( U_z(1) \) transformation and \( V^0 = (V^0_\mu \equiv A_\mu, \Omega^0, Y^{0ij}) \) denotes the \( U_z(1) \) vector multiplet. For example, acting on the scalar \( A^\alpha_i \), we have

\[
M_* A^\alpha_i = g M^\alpha_\beta A^\beta_i + \alpha Z A^\alpha_i. \tag{3.10}
\]

The hypermultiplet in 6D exists only as an on-shell multiplet, since constraints similar to (3.9) are equations of motion there. Here in 5D, however, it becomes an off-shell multiplet, as explained in Ref. 9).

First, there appears no constraint on the first \( U_z(1) \) transformation of \( A^\alpha_i \), so it defines the auxiliary field

\[
F^\alpha_i \equiv \alpha Z A^\alpha_i, \tag{3.11}
\]

which is necessary for closing the algebra off-shell and balancing the numbers of boson and fermion degrees of freedom. Next, there are the undefined \( U_z(1) \) transformations \( Z \zeta^\alpha, Z(Z A^\alpha_i) \) (= \( \alpha^{-1} Z F^\alpha_i \)) in the constraints (3.9), and therefore we do not interpret the constraints as the equation of motion but as definitions of these \( U_z(1) \) transformations. The first constraint of (3.9), for example, gives the \( U_z(1) \) transformation of the spinor \( \zeta^\alpha \) as

\[
Z \zeta^\alpha = -\frac{\alpha + \gamma^a A^a}{\alpha^2} \frac{\partial^f \zeta^\alpha}{A^a A^a} \left( \frac{1}{2} \gamma \cdot v \zeta^\alpha - \frac{1}{8} \chi^i A^\alpha_i + \frac{3}{8} \gamma \cdot \hat{R} (Q) A^\alpha_i \right) + g M^\alpha_\beta A^\beta_i + 2 \bar{i} \epsilon^i \zeta^\alpha + 2 \bar{i} \alpha \bar{\epsilon} \Omega^0_i F^\alpha_i. \tag{3.12}
\]

Note that \( \hat{D} \zeta^\alpha \) contains the \( U_z(1) \) covariantization \(-\delta Z(A_\alpha) \zeta^\alpha \) and \( \hat{D}'_a \) denotes a covariant derivative with the \(-\delta Z(A_\alpha) \) term omitted. Also, the second constraint gives the \( U_z(1) \) transformation of \( F^\alpha_i \), which we do not show explicitly here. Finally, the \( Q \) and \( S \) transformations of the auxiliary field \( F^\alpha_i \) are given by requiring that the \( U_z(1) \) transformation commute with the \( Q \) and \( S \) transformations on \( A^\alpha_i \):

\[
\delta F^\alpha_i = \delta (\delta (\alpha) A^\alpha_i) = (\delta Z(\alpha) \delta + \delta Z(\delta \alpha)) A^\alpha_i = 2i \bar{\epsilon}^i (\alpha Z \zeta^\alpha) + \frac{2i}{\alpha} \bar{\epsilon} \Omega^0_i F^\alpha_i. \tag{3.13}
\]

3.3. **Linear multiplet**

The linear multiplet consists of the components listed in Table II and may generally carry a non-Abelian charge of the gauge group \( G \). This multiplet, apparently, contains 9 Bose and 8 Fermi fields, so that the closure of the algebra on this multiplet requires the constraint
which can be solved in terms of a three-form gauge field $E_{\mu\nu\lambda}$. A four-form gauge field $H_{\mu\nu\rho\sigma}$ can also be introduced for rewriting the scalar component of this multiplet.

The $Q$ and $S$ transformation laws of the linear multiplet are given by

\[
\begin{align*}
\delta L^{ij} &= 2i\bar{e}^{(i} \varphi^{j)}, \\
\delta \varphi^i &= -\hat{\mathcal{D}} L^{ij} \varepsilon_j + \frac{1}{2} \gamma^a \varepsilon^i E_a + \frac{1}{2} \varepsilon^i N \\
&\quad + 2\gamma \cdot \varepsilon_j L^{ij} + g M L^{ij} \varepsilon_j - 6L^{ij} \eta_j, \\
\delta E^a &= 2i\bar{e} \gamma^a \hat{\mathcal{D}} \varphi - 2i\bar{e} \gamma^{abc} \varphi v_{bc} + 6i\bar{e} \gamma_b \varphi v^{ab} + 2i\bar{e}^{(ij} \gamma^{abc} \hat{R}_{ij}^a(Q) L_{ij} \\
&\quad + 2ig \bar{e} \gamma^a M \varphi - 4ig \bar{e} \gamma^a \Omega^j L_{ij} - 8i\bar{\eta} \gamma^a \varphi, \\
\delta N &= -2i\bar{e} \hat{\mathcal{D}} \varphi - 3i\varepsilon \gamma \cdot \varphi + \frac{1}{2} i\bar{e} \cdot \gamma L_{ij} - \frac{3}{2} i\bar{e} (i \gamma \cdot \hat{R}^i)(Q) L_{ij} \\
&\quad + 4ig \bar{e} (i \Omega^i) L_{ij} - 6i\bar{\eta} \varphi.
\end{align*}
\]  

(3.14)

The algebra closes if $E^a$ satisfies the following $Q$ and $S$ invariant constraint:

\[
\hat{\mathcal{D}}_a E^a + i \bar{\varphi} \gamma^a \hat{R}(Q) + g M N + 4i \bar{\varphi} \Omega + 2g Y^{ij} L_{ij} = 0.
\]  

(3.15)

This constraint can be separated into two parts, a total derivative part and the part proportional to the Yang-Mills coupling $g$:

\[
e^{-1} \partial_\lambda (e^\lambda) + 2ge^{-1} \mathcal{H}_{VL} = 0,
\]  

(3.16)

where, $\mathcal{V}^a$ and $\mathcal{H}_{VL}$ are given by

\[
\mathcal{V}^a = E^a - 2i \bar{\psi}_b \gamma^{ba} \varphi + 2i \bar{\psi}_b \gamma^{abc} L \psi_c,
\]

\[
e^{-1} \mathcal{H}_{VL} = Y^{ij} L_{ij} + 2i \bar{\varphi} \varphi + 2i \bar{\psi}_i^a \gamma_a \Omega_j L_{ij} - \frac{1}{2} W_a \mathcal{V}^a
\]

\[
+ \frac{1}{2} M \left(N - 2i \bar{\psi}_b \gamma^b \varphi - 2i \bar{\psi}_i (i \gamma^{ab} \psi^b_j) L_{ij}\right).
\]  

(3.17)

When the linear multiplet is inert under the $G$ transformation, that is $g = 0$, this constraint can be solved in terms of a three-form gauge field $E_{\mu\nu\lambda}$ as $\mathcal{V}^a = e^{-1} \epsilon^{\lambda\mu\nu\rho} \partial_\mu E_{\nu\rho\sigma}/6$, which possesses the additional gauge symmetry $\delta E(\Lambda) E_{\mu\nu\lambda} = 3 \partial_\mu (\Lambda_{\nu\lambda})$. Hence the linear multiplet becomes an unconstrained multiplet $(E_{\mu\nu\lambda}, L^{ij}, \varphi^i, N)$.

It should be noted that in 6D, the linear multiplet requires a similar constraint on the 6D vector $E^a$, and this constraint can be solved in terms of the four-form gauge field $E_{\mu\nu\rho\sigma}$ in a similar manner. This 6D four-form field yields a three-form field $E_{\mu\nu\lambda}$ and a four-form field $H_{\mu\nu\rho\sigma}$ through the simple reduction, while the 6D vector $E^a$ reduces to the 5D vector $E^a$ and the scalar $N$. Thus we expect that the scalar field $N$ can be rewritten in terms of a four-form field $H_{\mu\nu\rho\sigma}$ in this 5D linear multiplet. (Note the number of degrees of freedom of the $H_{\mu\nu\rho\sigma}$ is 1 in 5D.) The quantity $\mathcal{H}_{VL}$ contains $N$, and any transformation of this
quantity becomes a total derivative, because the constraint (3.16) is invariant under the full transformations. Thus it can be rewritten with $H_{\mu \nu \rho \sigma}$ in the form

$$2 \mathcal{H}_{V L} = - \frac{1}{4!} \epsilon^{\lambda \mu \nu \rho \sigma} \partial_\lambda (H_{\mu \nu \rho \sigma} - 4W_\mu E_{\nu \rho \sigma}), \quad (3.18)$$

where the extra term $W_{[\mu} E_{\nu \rho \sigma]}$ on the right hand-side is inserted for later convenience. With this rewriting, the constraint (3.16) can be solved even for the case that the linear multiplet carries a charge of the gauge group $G$. Indeed, since the r.h.s. of (3.18) is a total derivative, we have

$$e^{-1} \partial_\lambda \left\{ e^{\nu} - \frac{9}{4!} \epsilon^{\lambda \mu \nu \rho \sigma} (H_{\mu \nu \rho \sigma} - 4W_\mu E_{\nu \rho \sigma}) \right\} = 0$$

$$\rightarrow \quad \nu = \frac{1}{4!} e^{-1} \epsilon^{\lambda \mu \nu \rho \sigma} \left( 4 \partial_\mu E_{\nu \rho \sigma} - 4gW_\mu E_{\nu \rho \sigma} + gH_{\mu \nu \rho \sigma} \right). \quad (3.19)$$

The transformation laws of $E_{\mu \nu \lambda}$ and $H_{\mu \nu \rho \sigma}$ must be determined up to the additional gauge symmetry $\delta_H(A) H_{\mu \nu \rho \sigma} = 4(\partial_\mu - gW_{[\mu}) A_{\nu \rho \sigma]}, \quad \delta_H(A) E_{\mu \nu \lambda} = -gA_{\mu \nu \lambda}$, so that the all transformation laws of both sides of (3.19) are the same for consistency. Also the transformation laws of the tensor gauge fields defined in this way are consistent with the replacement equation (3.18), because this equation is satisfied automatically due to the invariant equations (3.16) and (3.19). Now, let us rewrite the replacement equation of $N$, (3.18), and the solution of $E^a$, (3.19), into the following two invariant equations:

$$E^a = \frac{1}{4!} \epsilon^{abcde} \tilde{F}_{bcde}(E),$$

$$MN + 2Y_{ij} L^{ij} + 4i \tilde{\Omega} \varphi = - \frac{1}{4!} \epsilon^{abcde} \tilde{F}_{abcde}(H). \quad (3.20)$$

The quantities $\tilde{F}_{abcde}(E)$ and $\tilde{F}_{abcde}(H)$ are the field strengths given by

\begin{align*}
\tilde{F}_{\mu \nu \rho \sigma}(E) &= 4D_{[\mu} E_{\nu \rho \sigma]} + gH_{\mu \nu \rho \sigma} + 8i \bar{\psi}_{[\mu} \gamma_{\nu \rho \sigma]} \varphi + 24i \bar{\psi}_{[\mu} \gamma_{\nu \rho \sigma]} j_{\lambda]} L_{ij}, \\
\tilde{F}_{\lambda \mu \nu \rho \sigma}(H) &= 5D_{[\lambda} H_{\mu \nu \rho \sigma]} - 10F_{[\lambda} (W) E_{\mu \nu \rho \sigma]} \\
&\quad - 10i \bar{\psi}_{[\lambda} \gamma_{\mu \nu \rho \sigma]} M \varphi + 20i \bar{\psi}_{[\lambda} \gamma_{\mu \nu \rho \sigma]} j_{\lambda]} L_{ij} - 40i \bar{\psi}_{[\lambda} j_{\mu \nu \rho \sigma]} j_{\lambda]} M L_{ij},
\end{align*}

(3.21)

where the derivative $D_\mu$ is covariant with respect to the $G$ transformation $D_\mu \equiv \partial_\mu - gW_\mu$. The transformation laws of $E_{\mu \nu \lambda}$ and $H_{\mu \nu \rho \sigma}$ can be understand from the fact that the left-hand sides of the equations in (3.20) are covariant under the full transformation, and so the field strengths on the r.h.s. must also be fully covariant. With $\delta \equiv \delta_Q(\varepsilon) + \delta_S(\eta) + \delta_G(A) + \delta_E(A_{\mu \nu}) + \delta_H(A_{\mu \nu \lambda})$, we have

\begin{align*}
\delta E_{\mu \nu \lambda} &= 3D_{[\mu} A_{\nu \lambda]} + gA E_{\mu \nu \lambda} - gA_{\mu \nu \lambda} - 2i \bar{\varepsilon} \gamma_{\mu \nu \lambda} \varphi - 12i \bar{\varepsilon} \gamma_{[\mu} j_{\nu \lambda]} L_{ij}, \\
\delta H_{\mu \nu \rho \sigma} &= 4D_{[\mu} A_{\nu \rho \sigma]} + gA H_{\mu \nu \rho \sigma} + 6F_{[\mu} (W) A_{\rho \sigma]} \\
&\quad + 2i \bar{\varepsilon} \gamma_{[\mu \nu} M \varphi - 4i \bar{\varepsilon} \gamma_{[\mu \nu} \varphi_{j]} O^j L_{ij} \\
&\quad + 16i \bar{\varepsilon} \gamma_{[\mu \nu} j_{\rho \sigma]} M L_{ij} + 4(\delta_Q(\varepsilon) W_{[\mu} E_{\nu \rho \sigma]}).
\end{align*}

(3.22)
These transformation laws are truly consistent with (3.20), and thus with (3.19). With these laws, the following modified algebra closes on the tensor gauge fields $E_{\mu\nu\lambda}$ and $H_{\mu\nu\rho\sigma}$:

$$
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = (\text{R.H.S. of (3.2)}) + \delta_E(4i\varepsilon^1_1\gamma_{\mu\nu}\varepsilon^2_2 L_{ij}) + \delta_H \left( 4i\varepsilon^1_1\gamma_{\mu\nu\lambda}\varepsilon^2_2 M L_{ij} - 2i\varepsilon_1\varepsilon_2 ME_{\mu\nu\lambda} \right),
$$

$$
[\delta_Q(\varepsilon), \delta_E(A_{\mu\nu})] = \delta_H \left( 3\delta_Q(\varepsilon)W_{[\mu}A_{\nu]} \right),
$$

$$
[\delta_G(A), \delta_E(A_{\mu\nu})] = \delta_E(-g\Lambda A_{\mu\nu}), \quad [\delta_G(A), \delta_H(A_{\mu\nu\lambda})] = \delta_H(-g\Lambda A_{\mu\nu\lambda}).
$$

(3.23)

This fact also justifies the replacement (3.18) algebraically. Nevertheless, in order to actually claim that $(E_{\mu\nu\lambda}, H_{\mu\nu\rho\sigma}, L^j, \varphi^i)$ gives a new version of the linear multiplet, we must show that the component $N$ can be expressed in terms of $H_{\mu\nu\rho\sigma}$ by solving (3.18). The point is that the left hand side of (3.18) $\mathcal{H}_{VL}$ contains $N$ in the form $MN$, but the Lie-algebra valued scalar $M$ is, of course, not always invertible. In some particular cases, the matrix $M$ can be invertible. For example, the determinant of the $SU(2)$-valued matrix $M^a(\sigma_a/2)$ does not vanish in the domain $\sum_{a=1}^3 M^aM^a \neq 0$. Therefor, the linear multiplet can take the doublet representation of $SU(2)$ as a subgroup of $G$ in this domain.

### 3.4. Nonlinear multiplet

A nonlinear multiplet is a multiplet whose component fields are transformed nonlinearly. The first component, $\Phi^i_\alpha$, carries an additional gauge-group $SU(2)$ index $\alpha (= 1, 2)$, as well as the superconformal $SU(2)$ index $i$. The index $\alpha$ is also raised (and lowered) by using the invariant tensor $\epsilon^{\alpha\beta}$ (and $\epsilon_{\alpha\beta}$) as $\Phi^i_\alpha = \Phi^{i\beta} \epsilon_{\beta\alpha}$. The field $\Phi^i_\alpha$ takes values in $SU(2)$ and hence satisfies

$$
\Phi^i_\alpha \Phi^\alpha_j = \delta^i_j, \quad \Phi^\alpha_i \Phi^i_\beta = \delta^\alpha_\beta.
$$

(3.24)

The $Q$, $S$ and $K$ transformation laws of this multiplet are given by

$$
\delta \Phi^i_\alpha = 2i\tilde{\varepsilon}^{(i}\lambda_j)\Phi^{j\alpha},
$$

$$
\delta \lambda^i = -\Phi^i_j \hat{\nabla} \Phi^{j\alpha} \varepsilon^\alpha_j + M_{a\beta} \Phi^{a\alpha} \Phi^{\beta j} \varepsilon^\beta_j + \gamma^i_\alpha \nu \varepsilon^\alpha
$$

$$
+ \frac{1}{2} \gamma^a V_{a\alpha} \varepsilon^\alpha_i + \frac{1}{2} V^5 \varepsilon^i + 2i\varepsilon^\alpha_\lambda \lambda_j - 3\eta^\alpha_i,
$$

$$
\delta V_a = 2i\tilde{\varepsilon}^\alpha_{ab} \hat{D}^b \lambda - i\varepsilon_{a\alpha} \gamma^i V^\alpha + i\varepsilon_\alpha \lambda V^5
$$

$$
+ 2i\varepsilon^\alpha_\beta \nu_{ab} + \frac{1}{4} i\varepsilon_{a\alpha} \chi + 2i\varepsilon^\alpha \hat{R}_{ab}(Q) - i\varepsilon_{a\alpha} \gamma^i \tilde{R}(Q)
$$

$$
+ 2i\varepsilon^\alpha_\gamma \Phi^iapersim \Phi^\alpha_j - 4i\varepsilon^\alpha_\gamma \Omega_{a\alpha\beta} \Phi^\beta \Phi^\alpha_j - 2i\varepsilon^\alpha_\gamma \lambda^i M_{a\beta} \Phi^\beta \Phi^\alpha_j
$$

$$
- 2i\tilde{\eta} \gamma^\alpha \lambda - 6\xi \kappa_a.
$$

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\[ \delta V^5 = -2i\varepsilon \hat{\Phi} \lambda + i\varepsilon \gamma \cdot V \lambda - i\varepsilon \lambda V^5 - i\varepsilon \gamma \cdot v \lambda - \frac{1}{4}i\varepsilon \chi + \frac{3}{4}i\varepsilon \gamma \cdot \hat{R}(Q) \]
\[ -2i\varepsilon \Phi_{ai} \hat{\Phi} \alpha \lambda^i + 4ig\varepsilon \Omega_{\alpha\beta} \Phi_{ai}^\alpha \Phi_{bj}^\beta + 2ig\varepsilon \lambda^j M_{ij} \Phi_{ai}^\alpha \Phi_{bj}^\beta. \]

(3.25)

As in the linear multiplet case, the nonlinear multiplet also needs the following \(Q, S\) and \(K\) invariant constraint for the closure of the algebra:

\[ \hat{D}^a V_a - \frac{1}{2} V_a V^a + \frac{1}{4}(V^5)^2 + \hat{D}^a \Phi_i \hat{D}_a \Phi_i^a + 2i\lambda \hat{\Phi} \lambda \]
\[ +2i\lambda \Phi_{ai} \hat{\Phi} \alpha \lambda^i + i\lambda \gamma \cdot v \lambda \]
\[ + \frac{2}{3} \hat{R}(M) + \frac{1}{2} D - \frac{1}{2} v^2 + \frac{i}{2} \lambda \chi - \frac{i}{2} \lambda \gamma \cdot \hat{R}(Q) \]
\[ +2gY_{ij} \Phi_{ai}^\alpha \Phi_{bj}^\beta - 8ig\lambda \Omega_{\alpha\beta} \Phi_{ai}^\alpha \Phi_{bj}^\beta - 2ig\lambda^i \lambda^j M_{ij} \Phi_{ai}^\alpha \Phi_{bj}^\beta \]
\[ +g^2 M_{ij} M_{ij}^\beta = 0. \]

(3.26)

This constraint can be solved for the scalar of the Weyl multiplet \(D\), and this solution presents us with a new (40+40) Weyl multiplet, which possesses the unconstrained nonlinear multiplet instead of \(D\).

§4. Embedding and invariant action formulas

4.1. Embedding formulas

We now give some embedding formulas that give a known type of multiplet using a (set of) multiplet(s).

To determine embedding formulas that give the linear multiplet \(L\) by means of other multiplets is not difficult for the following reason. When the transformation of the lowest component \(L_{ij}\) of a multiplet takes the form \(2i\varepsilon(i, \varphi^j)\), the superconformal algebra consisting of (2-17) and (2-18) demands that all the other higher components must uniquely transform in the form given in Eq. (3.14) and that the constraint (3.15) should hold. Therefore, in order to identify all the components of the linear multiplet, we have only to examine the transformation law up to the second component \(\varphi^i\), as long as the algebra closes on the embedded multiplets.

The vector multiplets can be embedded into the linear multiplet with arbitrary quadratic homogeneous polynomials \(f(M)\) of the first components \(M^I\) of the vector multiplets. The index \(I\) labels the generators \(t_I\) of the gauge group \(G\), which is generally non-simple. These embedding formulas \(L(V)\) are

\[ L_{ij}(V) = Y_{ij} f_I - i\tilde{\Omega}_i^I \Omega_j^J f_{IJ}, \]

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\[\varphi_i(V) = -\frac{1}{4} (\chi_i + \gamma \cdot \hat{R}_i(Q)) f + \left( \hat{\mathcal{D}}^b (4v_{ab} f + \hat{F}_{ab}^i (W) f_I + i\Omega^i \gamma_{ab} \Omega^I f_{IJ} \right) \\
E_a(V) = \hat{\mathcal{D}}^b \left( 4v_{ab} f + \hat{F}_{ab}^i (W) f_I + i\Omega^i \gamma_{ab} \Omega^I f_{IJ} \right) + \left( -i\bar{\Omega}^I \gamma_{abc} \hat{R}^{bc}(Q) - 2ig[\Omega, \gamma_a \Omega]^I + g[M, \hat{\mathcal{D}}_a M]^I \right) f_I \\
N(V) = -\hat{\mathcal{D}}^a \hat{\mathcal{D}}_a f \left(-\frac{1}{2} D + \frac{1}{4} \hat{R}(M) - 3v \cdot v \right) f + \left( -2\hat{F}_{ab}(W) v_{ab} + \frac{1}{2} i\bar{\Omega}^I \gamma_{ab} \Omega^J + 2i\Omega^I \gamma_{ab} \Omega^J \right) f_{IJ} \]

where the commutator \([X, Y]^I\) represents \([X, Y]^I t_I \equiv X^I Y^J [t_I, t_J],\) and

\[f \equiv f(M), \quad f_I \equiv \frac{\partial f}{\partial M^I}, \quad f_{IJ} \equiv \frac{\partial^2 f}{\partial M^I \partial M^J}.\]

Here, it is easy to see that we cannot generalize the function \(f(M)\) further. For example, the lowest component \(L_{ij} = S\) is invariant. Thus the right-hand side of the first equation of this formula has to be \(S\) invariant, and this fact requires \(M^I f_I = 2f\). Therefore \(f(M)\) must be a homogeneous quadratic function of the scalar field \(M\): \(f(M) = f_{IJ} M^I M^J / 2\).

The product of the two hypermultiplets \(H = (A^i_\alpha, \zeta_\alpha, \mathcal{F}^i_\alpha)\) and \(H' = (A'^i_\alpha, \zeta'_\alpha, \mathcal{F}'^i_\alpha)\) can also compose a linear multiplet \(L(H, H')\) as follows:

\[L_{ij}^{\alpha\beta}(H, H') = A^i_\alpha A'^j_\beta, \quad \varphi_{ij}^{\alpha\beta}(H, H') = \zeta_\alpha A'^i_\beta + \zeta'_\beta A^i_\alpha, \quad E_{ij}^{\alpha\beta}(H, H') = A^i_\alpha \hat{\mathcal{D}}_\beta A'^j_\alpha + A'^i_\alpha \hat{\mathcal{D}}_\beta A^j_\alpha - 2i\bar{\zeta}_\alpha \gamma^\alpha \zeta'_\beta, \quad N_{ij}^{\alpha\beta}(H, H') = -A^i_\alpha M_\beta A'^j_\alpha - A'^i_\alpha M_\beta A^j_\alpha - 2i\bar{\zeta}_\alpha \zeta'_\beta.\]

Here, this linear multiplet transforms non-trivially under the \(U(1)\) transformation, in addition to the transformations that are self-evident from the index structure; e.g., \(\delta_Z(\alpha) L_{ij}^{\alpha\beta} = \mathcal{F}_i^{(ij)} A'^j_\beta + A'^i_\beta \mathcal{F}_j^{(ij)}\). For this multiplet, therefore, the ‘group action terms’, like \(gML^{ij}\) appearing in the \(Q\) transformation law (3.14), and the action formula, which we discuss in the next subsection, should be understood to contain not only the usual gauge group action but also the \(U(1)\) action: \(gM \rightarrow M_\star = \delta_G(M) + \delta_Z(\alpha)\). Also note that \(Z^a H\) with the arbitrary number \(n\) can be substituted for \(H\) and \(H'\) in the above formulas, because \(ZH\) also transforms as a hypermultiplet.
Conversely, we can also embed the linear multiplet into the vector multiplet. The following combination of the components of the linear multiplet is $S$ invariant and carries Weyl-weight 1:

$$M(L) = NL^{-1} + i\bar{\varphi}^i\varphi^j L_{ij} L^{-3},$$

(4.4)

with $L = \sqrt{L_{ij} L^{ij}}$. This embedding formula is a non-polynomial function of the field, and for this reason, the embedding formulas for the higher components becomes quite complicated. Though we have not confirmed that the embedding (4.4) is consistent with the transformation laws of the higher components, this formula agrees with the formula in the Poincaré supergravity in 5D presented by Zucker\(^{12}\) up to the components of the ‘central charge vector multiplet’. Thus it must be a correct form.

4.2. Invariant action formula

The quantity $\mathcal{H}_{VL}$ appearing in (3.17) transforms into a total derivative under all of the gauge transformations and has Weyl-weight 5. It therefor represents a possibility as the invariant action formula. However (3.16) implies that $\mathcal{H}_{VL}$ itself is a total derivative and so cannot give an action formula. Fortunately, the invariant action formula can be found in the following way with a simple modification of the expression of $\mathcal{H}_{VL}$. Let us consider the action formula

$$e^{-1} \mathcal{L}(V \cdot L) \equiv Y^{ij} \cdot L_{ij} + 2i\bar{\Omega} \cdot \varphi + 2i\bar{\psi}^a \gamma_a \Omega_j \cdot L^{ij}$$

$$- \frac{1}{2} W_a \cdot \left\{ E^a - 2i\bar{\psi} \gamma^a \varphi + 2i\bar{\psi}^b \gamma^{ab} \psi^j L_{ij} \right\}$$

$$+ \frac{1}{2} M \cdot \left\{ N - 2i\bar{\psi}^a \gamma^a \varphi - 2i\bar{\psi} \gamma^{ab} \psi^j L_{ij} \right\},$$

(4.5)

where the dot (e.g. that in $V \cdot L$) indicates a certain suitable operation. If this dot represents the $G$ transformation * defined by $gV \ast L \equiv \delta_G(V) L$, then this formula reduces to the original $\mathcal{H}_{VL} \equiv \mathcal{L}(V \ast L)$. The Q and G transformation law of $\mathcal{L}(L \cdot V)$ may be different from that of $\mathcal{H}_{VL}$ only in the terms proportional to $g$. For the Q transformation, for instance, we have

$$\delta_Q(\varepsilon) \mathcal{L}(V \cdot L) = \text{(total derivative)}$$

$$+ 2g i\bar{\varepsilon}^i \gamma^a [W_a, \Omega^j; L_{ij}] + 2g i\bar{\varepsilon}^i [M, \Omega^j; L_{ij}]$$

$$+ g i\bar{\varepsilon} \gamma^a [M, W_a; \varphi] + \frac{g i}{2} \bar{\varepsilon} \gamma^{ab} [W_a, W_b; \varphi]$$

$$+ g i\bar{\varepsilon} \gamma^{abc} [W_a, W_b; L] \psi_c + 2g i\bar{\varepsilon} \gamma^{ab} [M, W_a; L] \psi_b,$$

(4.6)

where $[A, B; C]$ denotes the following Jacobi-like operation:

$$[A, B; C] \equiv A \cdot (B \ast C) - B \cdot (A \ast C) - (A \ast B) \cdot C.$$  

(4.7)
The $G$ transformation of (4.5) also takes a similar form. Hence if we find a dot operation $(\cdot)$ for which (4.7) identically vanishes, then the action formula (4.5) will be invariant up to the total derivative under the $Q$ and $G$ transformation, in addition to the $S$ transformation. (Of course if we choose $\cdot$ as $\ast$, the operation (4.7) vanishes, as the Jacobi identity.)

For instance, we can see from (4.6) that the action formula (4.5) gives an invariant by taking the dot operation to be a simple product, if the vector multiplet is Abelian and the linear multiplet carries no gauge group charge or is charged only under the Abelian group of this vector multiplet, like the central charge transformation $\delta Z$. When the linear multiplet carries no charges at all, the constrained vector field $E^a$ can be replaced by the three-form gauge field $E_{\mu\nu\lambda}$. Using this, the third line of the above action (4.5) can be rewritten, up to total derivative, as

$$-\frac{1}{2}W_a \left( E^a - 2i\tilde{\psi}_b \gamma^{ba} \varphi + 2i \tilde{\psi}_b (\gamma^{abc} \psi^c) L_{ij} \right) \rightarrow -\frac{1}{4!} \epsilon^{\mu\nu\lambda\rho\sigma} F_{\mu\nu}(W) E_{\lambda\rho\sigma}. \quad (4.8)$$

Similarly replacing $\ast$ by $\cdot$ in Eq. (3.18), we can obtain another invariant action formula written in terms of the four-form gauge field $H_{\mu\nu\rho\sigma}$. This gives an off-shell formulation of the SUSY-singlet ‘coupling field’ introduced in Ref. 20, as will be discussed in a forthcoming paper.\textsuperscript{21)}

The action formula (4.5) can be used to write the action for a general matter-Yang-Mills system coupled to supergravity. If we use the above embedding formula, (4.1), of the vector multiplets into a linear multiplet and apply the action formula (4.5) and (4.8), $L_V = \mathcal{L} \left( V^A L_A(V) \right)$, then we obtain a general Yang-Mills-supergravity action. Although the action formula can be applied only to the Abelian vector multiplets $V^A$, interestingly $V^A$ can be extended to include the non-Abelian vector multiplets $V^I$ in this Yang-Mills action; that is, the quadratic function $f_A(M)$ multiplied by $M^A$ can be extended to a cubic function $N(M)$ as $-\frac{1}{6} f_{A,I,J} M^A M^I M^J \rightarrow N = c_{IJK} M^I M^J M^K$. Also, the action for a general hypermultiplet matter system can be obtained similarly. The kinetic term for the hypermultiplet is given by $\mathcal{L}_{H\text{kinetic}} = \mathcal{L} \left( A \; d^\alpha\beta L_{\alpha\beta}(H, Z H) \right)$, with the central charge vector multiplet $A = V^0$ and an antisymmetric $G$ invariant tensor $d^\alpha\beta$. The mass term for the hypermultiplet are given by $\mathcal{L}_{H\text{mass}} = \mathcal{L} \left( A \; \eta^\alpha\beta L_{\alpha\beta}(H, H) \right)$, with a symmetric $G$ invariant tensor $\eta^\alpha\beta$. Finally, the action for the unconstrained linear multiplet is given by $\mathcal{L}_L = \mathcal{L} \left( V(L) L \right)$, which may contain a kinetic term for the four-form field $H_{\mu\nu\rho\sigma}$ in addition to that for $E_{\mu\nu\lambda}$. These actions in the superconformal tensor calculus must be identified with those in the Poincaré supergravity tensor calculus.

In the case that the linear multiplet carries a non-Abelian charge, the invariant action cannot be obtained in a simple way: If we assume that the linear multiplet is Lie-algebra valued and interpret the dot operation as a trace, then the Jacobi-like operation (4.7) does
not vanish. However, it is not impossible to obtain this action. For example, one can consider Abelian vector multiplets \( V^\alpha \) carrying a non-Abelian charge. That is,

\[
[G_I, G_\alpha] = f_{I\alpha}^\beta G_\beta = -\{\rho(G_I)\}_{\alpha}^\beta G_\beta, \quad [G_\alpha, G_\beta] = 0,
\]

(4.9)

where \( G_I \) and \( G_\alpha \) are generators of non-Abelian and Abelian generators, respectively. The linear multiplet is also assumed to be a joint representation of this group, \((L^I, L^\alpha)\). Then the Jacobi-like operation (4.7) vanishes if we take the dot operation to be given by

\[
A \cdot B = A_\alpha B^\alpha = \rho_{\alpha\beta} A^\alpha B^\beta = -B \cdot A, \quad \rho_{\alpha\beta} = -\rho_{\beta\alpha},
\]

(4.10)

and \( L(V^\alpha L^\beta \rho_{\alpha\beta}) \) gives an invariant action, while the linear multiplet carries a non-Abelian charge.

§5. The two-form gauge field and Nishino-Rajpoot formulation

To this point in the text, the three-form gauge field \( E_{\mu\nu\lambda} \) and the four-form gauge fields \( H_{\mu\nu\rho\sigma} \) have appeared, in addition to the one-form gauge fields \( W_\mu \). A two-form gauge field \( B_{\mu\nu} \) can also be introduced in the process by solving the constraint \( L_0(V) = 0 \), which we impose on a set of Vector multiplets \( V^I \) using an embedding quadratic formulation \( f_0(M) \). The solution leads to another type of the Weyl multiplet that contains \( B_{\mu\nu} \). The formulation with this new multiplet gives the alternative supergravity presented by Nishino and Rajpoot\(^4\) after suitable \( S \) and \( K \) gauge fixing. (Thus we will call this the ‘N-R formulation’.)

Here, we choose the quadratic function \( f_0(M) \) to be \( G \) inert:

\[
[A, B]^I J f_{0,1J} + B^I [A, C]^J f_{0,1J} = 0.
\]

Then, we solve the equation \( L_0(V) = 0 \). The equation \( L^{ij}(V) = 0 \) sets one of the auxiliary fields \( Y^I_{ij} \) of the vector multiplets \( V^I \) equal to zero. The equation \( \varphi^i(V) = 0 \) makes the auxiliary spinor field \( \chi^i \) of the Weyl multiplet a dependent field, and similarly the equation \( N(V) = 0 \) is solved with respect to the auxiliary scalar field \( D \) of the Weyl multiplet. Then, the equation \( E^a(V) = 0 \) becomes a total derivative in this gauge-invariant case, as mentioned above:

\[
E^\mu(V) = e^{-1} \partial_\nu \left( \frac{1}{6} \epsilon^{\mu\nu\lambda\rho\sigma} E_{\lambda\rho\sigma}(V) \right)
= e^{-1} \partial_\nu \left\{ e \left( 4\nu^\mu f + \bar{F}^{\mu\nu I}(W)f_I + i\bar{Q}^J \gamma^{\mu\nu} \Omega^J f_{1J} 
+ i\bar{\psi}_\rho \gamma^{\mu\nu\rho\sigma} \psi_{J} - 2i\bar{\chi}_{\lambda} \gamma^{\mu\nu} \Omega^J f_{1J} \right) 
+ \frac{1}{2} \epsilon_{\mu\nu\lambda\rho\sigma} \left( W^I_{\lambda} \partial_{\nu} W^J_{\rho} - \frac{1}{3} g W^I_{\lambda} [W^J_{\rho}, W^J_{\sigma}] f_{1J} \right) \right\}
= 0.
\]

(5.1)

Thus this equation is also solved by making the auxiliary tensor field \( v_{ab} \) dependent. However, the last equation here does not fix all components of \( v_{ab} \), because \( E^a \) and \( v_{ab} \) have 4 and 10
degrees of freedom, respectively. Of course the equation can be solved by means of adding the two-form field $B_{\mu \nu}$, which has 6 degrees of freedom: $0 = E_{\mu \nu \lambda}(V) + 3 \partial_{[\mu} B_{\nu \lambda]}$. This can also be rewritten as

$$0 = \hat{F}_{\mu \nu \lambda}(B) - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} (4 v^{\rho \sigma} f + \hat{F}^{\rho \sigma I}(W) f_I) + i \bar{\Omega}^I \gamma_{\mu \nu \lambda} \Omega^J f_{IJ},$$

(5.2)

where $\hat{F}_{\mu \nu \lambda}(B)$ is the covariant field strength of $B_{\mu \nu}$, which is given by

$$\hat{F}_{\mu \nu \lambda}(B) = 3 \partial_{[\mu} B_{\nu \lambda]} - 6 i \bar{\psi}_{[\mu} \gamma_{\nu \lambda]} \Omega^I f_I + 6 i \bar{\psi}_{[\mu} \gamma_{\nu \lambda]} \psi_{\lambda]} f$$

$$-3 \partial_{[\mu} W^I_{\nu]} W^I_{\lambda]} f_{IJ} + W^I_{[\mu} [W_{\nu}, W_{\lambda}]^J f_{IJ}. \tag{5.3}$$

A $Q$ and $G$ transformation law of $B_{\mu \nu}$ can be easily found from the $Q$ and $G$ covariance of (5.3) in the same way as that of $H_{\mu \nu \rho \sigma}$ and $E_{\mu \nu \lambda}$ in the linear multiplet case. The gauge field $B_{\mu \nu}$ is $S$ invariant and transforms under $\delta = \delta_Q(\varepsilon) + \delta_B(A^B_\mu) + \delta_G(A)$ as

$$\delta B_{\mu \nu} = 2 i \bar{\varepsilon} \gamma_{\mu \nu} \Omega^I f_I - 4 i \bar{\varepsilon} \gamma_{\mu |\nu \mid |} \psi_i f + (\delta_Q(\varepsilon) W^I_{\mu \nu}) W^I_{\nu i} f_{IJ} + 2 \partial_{[\mu} A^B_{\nu]} + \partial_{[\mu} W^I_{\nu]} A^I f_{IJ}. \tag{5.4}$$

The algebra on $B_{\mu \nu}$, of course, closes, although the algebra is modified as follows:

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = (\text{R.H.S. of (3.2)}) + \delta_B \left( -2 i \bar{\varepsilon}_1 \gamma_{\mu \nu} \varepsilon_2 f - i \bar{\varepsilon}_1 \varepsilon_2 W^I_{\mu \nu} f_I \right),$$

$$[\delta_Q(\varepsilon), \delta_G(A)] = \delta_B \left( \delta_Q(\varepsilon) W^I_{\mu \nu} A^I f_{IJ} \right),$$

$$[\delta_G(A_1), \delta_G(A_2)] = \delta_B \left( - \frac{1}{2} W^I_{\mu \nu} \left[ A_1, A_2 \right]^J f_{IJ} \right) + \delta_G (\left[ A_1, A_2 \right]). \tag{5.5}$$

Now the constraint equations $\mathbf{L}_0(V) = 0$ has replaced the ‘matter’ sub-multiplet $(v_{ab}, \chi^i, D)$ of the Weyl multiplet by the tensor field $B_{\mu \nu}$ and a linear combination of the vector multiplets with one component of the auxiliary field $Y^I_{ij}$ eliminated. We thus have obtained the N-R formulation with an alternative Weyl multiplet. If we take only a single vector multiplet, say the central vector multiplet, $V^0 = (\alpha, A_\mu, \Omega^0, Y^{0ij})$, and set $f_0(M) = \alpha^2$, then the conventional Weyl multiplet $(e_\mu^a, \psi_i^\mu, b_\mu, V^i_{\mu}, v_{ab}, \chi^i, D)$ is replaced by a new 32 + 32 multiplet consisting of $(e_\mu^a, \psi_i^\mu, b_\mu, V^i_{\mu}, B_{\mu \nu}, A_\mu, \alpha, Q^0)$. There are two known different types of formulations of the on-shell supergravity in 5D, the conventional one with no $B_{\mu \nu}$ and the above N-R formulation. These different formulations give different physics of course. From the point of view of off-shell formulation, this difference is only a difference in the cubic function $\mathcal{N}(M)$ that characterizes super Yang-Mills systems in 5D. If there is an Abelian vector multiplet $V_{\text{NR}}$ that appears only in the form of a Lagrange multiplier, that is, if $\mathcal{N}$ takes the form

$$\mathcal{N} = c_{IJK} M^I M^J M^K + M_{\text{NR}} f_{IJ} M^I M^J, \tag{5.6}$$

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and if matter multiplets carry no charge of this vector multiplet, then the equations of motion take the form $L_0(V) = 0$ by the variation of $V_{NR}$. Therefore, after integrating out $V_{NR}$, the N-R formulation appears. Conversely, if there is no such Abelian vector multiplet, the formulation gives the conventional one.

§6. Summary and comments

In this paper, we have presented a superconformal tensor calculus in five dimensions. This work extends a previous work,\textsuperscript{9} which presents Poincaré supergravity tensor calculus that is almost completely derived using dimensional reduction and decomposition from the known superconformal tensor calculus in six dimensions. The significant difference between the superconformal tensor calculus presented in this paper and Poincaré one is that the minimal Weyl multiplet in the superconformal case has 32 Bose plus 32 Fermi degrees of freedom, while that in the Poincaré case has $(40+40)$ degrees of freedom.

In a previous paper,\textsuperscript{10} we constructed off-shell $d = 5$ supergravity coupled to a matter-Yang-Mills system by using the Poincaré supergravity tensor calculus.\textsuperscript{9} There, intricate and tedious computations were necessary (owing to a lack of $S$ symmetry) to rewrite the Einstein and Rarita-Schwinger terms into canonical form. However, now we can write down the same action with little work, thanks to the full superconformal symmetry. Actually, we can show readily that this superconformal calculus is equivalent to two Poincaré calculuses with two different $S$ gauge choices, and thus the two Poincaré calculuses are equivalent (see Appendix D). Also, it is easy to show the equivalence to other Poincaré supergravities.\textsuperscript{2) - 4)

There appeared several by-products in the text. In this calculus, we have not imposed constraints on the $Q$ and $M$ curvatures. Though this is a purely technical point and unimportant from the viewpoint of physics, it could be interesting to pursue, as it is different from usual situation in superconformal gravities. This formulation with no constraints makes clear that these constraints are completely unsubstantial. It is thus seen that superconformal gravity with various forms of constraints can describe the same physics.

Moreover, the four-form gauge field $H_{\mu\nu\rho\sigma}$ and the two-form gauge field $B_{\mu\nu}$ have appeared in this off-shell formulation in addition to the three-form gauge field $E_{\mu\nu\lambda}$.

In recent studies of the brane world scenario, the four-form gauge field $H_{\mu\nu\rho\sigma}$ plays an important role in connection with the $Q$ singlet scalar ‘coupling field’ $G$. Now we can construct an off-shell formulation of $H_{\mu\nu\rho\sigma}$ and $G$, though only an on-shell formulation is known to this time. This extension may allow for the extraction of general properties from the brane world scenario without going into the details of the models. This will be discussed in a forthcoming paper.\textsuperscript{21)} Also $\mathcal{L}(V(L)L)$ may contain a kinetic term for $H_{\mu\nu\rho\sigma}$ and lead
to interesting physics in the brane world scenario.

Introducing the two-form gauge field $B_{\mu\nu}$ implies a new non-minimal Weyl multiplet, which should be equivalent to that presented by Nishino and Rajpoot. $^4$ From the viewpoint of the off-shell formalism, this system is unified with the general matter Yang-Mills system. $^2,^3$

The tensor multiplet as a matter multiplet, containing a two-form gauge field $B_{\mu\nu}$, is known in on-shell formulation. But we have not yet understood it in the present tensor calculus. Excluding this problem, however, this superconformal tensor calculus should produce all types of supergravity in 5D. Superconformal tensor calculus will provide powerful tools for the brane world scenario from a more unified viewpoint.

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Appendix A

— Conventions and Useful Identities —

We employ the notation of Ref. 9). The gamma matrices $\gamma^a$ satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $(\gamma_a)^\dagger = \eta_{ab}\gamma^b$, where $\eta^{ab} = \text{diag}(+, -, -, -, -)$. $\gamma_{a\cdots b}$ represents an antisymmetrized product of gamma matrices:

$$\gamma_{a\cdots b} = \gamma[a \cdots \gamma b], \quad (A.1)$$

where the square brackets denote complete antisymmetrization with weight 1. Similarly $(\cdots)$ denote complete symmetrization with weight 1. We chose the Dirac matrices to satisfy

$$\gamma^{a_1\cdots a_5} = \epsilon^{a_1\cdots a_5} \quad (A.2)$$

where $\epsilon^{a_1\cdots a_5}$ is a totally antisymmetric tensor with $\epsilon^{01234} = 1$. With this choice, the duality relation reads

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{(n-1)/2}}{(5-n)!} \epsilon^{a_1\cdots a_nb_1\cdots b_{5-n}}\gamma^{b_1\cdots b_{5-n}}. \quad (A.3)$$

The $SU(2)$ index $i$ ($i=1,2$) is raised and lowered with $\epsilon_{ij}$, where $\epsilon_{12} = \epsilon^{12} = 1$, in the northwest-southeast (NW-SE) convention:

$$A^i = \epsilon^{ij}A_j, \quad A_i = A^j\epsilon_{ji}. \quad (A.4)$$
A useful formula in treating these indices is $A^i B^j C_j = -A^i B_j C^i - A_j B^i C^j$.

The charge conjugation matrix $C$ in 5D has the properties

$$C^T = -C, \quad C^i C = 1, \quad C \gamma_a C^{-1} = \gamma_a^T. \quad (A.5)$$

Our five-dimensional spinors satisfy the $SU(2)$-Majorana condition

$$\bar{\psi}^j \equiv \psi_i^\dagger \gamma^0 = \psi_i^T C,$$  \quad (A.6)

where spinor indices are omitted. When $SU(2)$ indices are suppressed in bilinear terms of spinors, NW-SE contraction is understood, e.g. $\bar{\psi} \gamma^{a_1 \ldots a_n} \lambda = \bar{\psi}^i \gamma^{a_1 \ldots a_n} \lambda_i$. Changing the order of spinors in a bilinear leads to the following signs:

$$\bar{\psi} \gamma^{a_1 \ldots a_n} \lambda = (-1)^{(n+1)(n+2)/2} \lambda \gamma^{a_1 \ldots a_n} \psi. \quad (A.7)$$

If the $SU(2)$ indices are not contracted, the sign becomes opposite. We often use the Fierz identity, which in 5D reads

$$\psi^j \bar{\chi}^j = -\frac{1}{4} (\bar{\chi}^j \psi^j) - \frac{1}{4} (\bar{\chi}^j \gamma^a \psi^j) \gamma_a + \frac{1}{8} (\bar{\chi}^j \gamma^{ab} \psi^j) \gamma_{ab} \quad (A.8)$$

Appendix B

--- Dimensional Reduction to 5D from 6D ---

5D conformal supergravity can be obtained from 6D conformal supergravity\cite{17} through dimensional reduction. Upon reduction to five dimensions, the 6D Weyl multiplet ($40+40$) become reducible, and thus there is a need to decompose this multiplet into the 5D Weyl multiplet ($32+32$) and the central charge vector multiplet ($8+8$).

Basically, we follow the dimensional reduction procedure explained in Ref. 9, to which we refer the reader for the details. The standard form for the sechsbein $e^A_M$ is

$$e^A_M = \begin{pmatrix} e^a_\mu & e^5_\mu \\ e^a_z & e^5_z \end{pmatrix} = \begin{pmatrix} e^a_\mu & \alpha^{-1} A_\mu \\ 0 & \alpha^{-1} \end{pmatrix}. \quad (B.1)$$

Here $M, N, \ldots$ are six dimensional space-time indices and, $z$ denotes fifth spatial direction whereas $A, B, \ldots$ denote six-dimensional local Lorentz indices. The underlined fields are the components of the six-dimensional Weyl or matter multiplet. $\alpha$ and $A_\mu$ are identified with the scalar and the vector components of the central charge vector multiplet. The relation between tensors in 6D and 5D is given by the following rule: **Tensors with flat indices only are the same in 6D and 5D.** Thus, for a vector, for example, we have $\underline{v}_\mu = v_\mu$ (so we need not use an underbar for flat indices), but

$$v_\mu = e^a_\mu v_a + e^5_\mu v_5 = v_\mu + A_\mu v_z. \quad (B.2)$$

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We decompose the six-dimensional gamma matrices $\Gamma^M$ and the charge conjugation matrix $C$ as
\[
\Gamma^a = \gamma^a \otimes \sigma_1, \quad \Gamma^5 = 1 \otimes i\sigma_2, \\
C = C \otimes i\sigma_2. \tag{B.3}
\]
The six-dimensional chirality operator $\Gamma_7$ is written $\Gamma_7 = 1 \otimes i\sigma_2$. The six dimensional $SU(2)$-Majorana-Weyl spinor $\psi^i_\pm$, which satisfies the $SU(2)$-Majorana condition $\bar{\psi}^i_\pm \equiv \psi^i_\pm \Gamma^0 = \psi^T_\pm C$ and the Weyl condition $\Gamma^7 \psi^i_\pm = \pm \psi^i_\pm$, is decomposed as
\[
\psi^i_+ = \psi^i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi^i_- = i\psi^i \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{B.4}
\]
where $\psi^i$ is a five-dimensional $SU(2)$-Majorana spinor.

The generators of 6D superconformal algebra, which is $Osp(8^*|2)$, are labeled $P_A, M_{AB}, K_A, D, U_{ij}, Q^i_\alpha, S^i_\alpha$. \tag{B.5}

Of these, the generators $M_{a5}$ and $K_5$ are redundant in 5D and are used to fix redundant gauge fields, as described below. The independent gauge fields of the $N = 2, d = 6$ Weyl multiplet, which realize this algebra are
\[
e^A_M, \phi^i_M, b^A_M, V^i_{AB}, T_{ABC}, \chi^i, D. \tag{B.6}
\]
The first four are the gauge fields corresponding to the generators $P_A, Q^i_\alpha, D$ and $U_{ij}$. The last three are the additive matter fields. The other gauge fields, $\omega^A_{M\beta}, \phi^i_M$ and $f^A_M$, which correspond to the generators $M_{AB}, S^i_\alpha$ and $K_A$ can be expressed in terms of the above independent gauge fields by imposing the curvatures constraints
\[
\hat{R}^{A}_MN(P) = 0, \\
\hat{R}^{AB}_MN(M) e^N_B + T_{ABC}^R T_{ABC}^R + \frac{1}{12} e^A_M D = 0, \\
\Gamma^N \hat{R}^{i}_MN(Q) = -\frac{1}{12} \Gamma^{MN}_{AB} \chi^i, \tag{B.7}
\]
where $\hat{R}^{A}_MN(P), \hat{R}^{AB}_MN(M)$ and $\hat{R}^{i}_MN(Q)$ are the $P, M$ and $Q$ curvatures, respectively. (For more details, see Ref. 17), but note that we use the notation of Ref. 9).)

First, we decompose the six-dimensional Weyl multiplet into three classes,
\[
(e^a_\mu, \psi^i_\mu, b^a_\mu, V^i_{\mu}, T_{a5}, \chi^i, D), \\
(e^5_\mu, e^5_5, \psi^i_\mu, V^i_{ij}), \\
(e^a_\mu, b^a_\mu). \tag{B.8}
\]
Roughly speaking, the first class gives the five-dimensional Weyl multiplet and the second the central charge vector multiplet. The last class consists of redundant gauge fields. Redundant gauge fields can be set equal to zero as a gauge-fixing choice for the redundant $M_{\alpha 5}$ and $K_5$ symmetries. However, the condition $e_{\zeta}^a = b_\zeta = 0$ is not invariant under $Q$ and $S$ transformations. Thus, we have to add a suitable gauge transformation to the original $Q$ and $S$ transformations. Explicitly, the original $Q$ transformation of $e_{\zeta}^a$ is

$$
\delta_6^{6D}(\varepsilon) e_{\zeta}^a = -2i \varepsilon a \gamma^a \psi_{\zeta}.
$$

Adding a $M_{\alpha 5}$ transformation with parameter $\theta_{\alpha 5}^M = -2i \alpha \varepsilon a \gamma^a \psi_{\zeta}$ to the original $Q$ transformation, we obtain a $Q$ transformation, under which the constraint $e_{\zeta}^a = 0$ remains invariant. Similarly, in order to keep $b_\zeta = 0$ invariant under $Q$ and $S$ transformations, we should add a $K_5$ transformation with parameter $\theta_{K}^5(\varepsilon) = \varepsilon \gamma^5/24 + i \varepsilon \partial_5$ to the original $Q$ transformation and a $K_5$ transformation with parameter $\zeta_5^K(\eta) = i \eta \psi_{\zeta}$ to the original $S$ transformation, $\delta_6^{6D}(\eta)$.

In the original gauge transformation law for fields of the first class, the central charge vector multiplet components do not decouple. For example, the $Q$ transformation of $\psi_i^\mu$ is

$$
\delta_6^{6D}(\varepsilon) \psi_i^\mu = D_\mu \varepsilon^i - \frac{1}{4} \Gamma_{\rho\sigma\tau}^5 \gamma_{\mu\rho\sigma} \varepsilon^i + \frac{1}{2\alpha} \partial_\mu A_\nu \gamma^\nu \varepsilon^i - 2i (\varepsilon \gamma_\mu \psi_{\zeta}) \psi_i^\mu + \ldots.
$$

This transformation includes $\alpha = \varepsilon 5$, $A_\mu = \alpha \varepsilon_\mu$, and $\psi_{\zeta}$, which are the fields of the central charge vector multiplet. To get rid of these fields from the transformation law of the five-dimensional Weyl multiplet, we need to redefine the gauge fields and the $Q$ and $S$ transformations. The proper identification of the central charge vector multiplet components turns out to be

$$
\alpha = \varepsilon 5, \quad A_\mu = \alpha \varepsilon_\mu, \quad \Omega_0^i = -\alpha^2 \psi_i^\mu, \quad Y_0^{ij} = \alpha^2 V_0^{ij} - \frac{3i}{\alpha} \Omega_0^i \Omega_0^j.
$$

The field $\alpha$, whose Weyl weight is 1, is used to adjust the Weyl weight of the redefined field. For example, $A_\mu$ should carry Weyl weight 0 as any gauge field, but the Weyl weight of $\varepsilon_\mu$ is $-1$, so we identify $\alpha \varepsilon_\mu$ with the gauge field $A_\mu$. The correction term $3i \Omega_0^i \Omega_0^j / \alpha$ in the redefinition of $Y_0^{ij}$ is needed to remove the central charge vector multiplet from the algebra.

Similarly, the irreducible $(32+32)$ Weyl multiplet in 5D is identified as

$$
e_{\mu}^a = \varepsilon_{\mu}^a, \quad \psi_{\mu}^i = \psi_i^\mu, \quad b_\mu = b_\mu, \quad V_\mu^{ij} = V_i^{\mu \mu} + \frac{2i}{\alpha} \psi_i^\mu \Omega_0^j, \quad \frac{i}{\alpha^2} \Omega_0^i \gamma_\mu \Omega_0^j, \quad \Omega_0^i = -\frac{1}{4\alpha} \hat{F}_{ab}^\mu(A) + \frac{i}{2\alpha^2} \Omega_0^i \gamma_{\mu ab} \Omega_0^j, \quad v_{ab} = -\Omega_0^{a b 5} - \frac{1}{4\alpha} \hat{F}_{ab}^\mu(A) + \frac{i}{2\alpha^2} \Omega_0^i \gamma_{\mu ab} \Omega_0^j.
$$
where $\chi^i = \frac{16}{15}\chi^i + \frac{8}{5\alpha} \left( \hat{\Phi}_i \Omega_0 + \frac{1}{2\alpha}(\hat{\Phi}_i)\Omega_0^i + \frac{3}{2}g \cdot v\Omega_0^i \right)$

$$-\frac{1}{5} \left( \gamma \cdot \hat{R}(Q) - rac{6}{\alpha^2} \gamma \cdot \hat{F}(A)\Omega_0^i \right) = \frac{8}{\alpha^2} v_{ij} \Omega_{ij} - \frac{2i}{\alpha^3} \gamma_{ab} \Omega_0^i \Omega_0 \gamma^{ab} \Omega_0,$$

$$D = \frac{8}{15} \mathcal{D} - \frac{1}{10} \hat{R}_{ab}(M) + 2v_{ab}$$

$$+ \frac{1}{\alpha} \left( i\Omega_0 \chi + \frac{4i}{5} \Omega_0 \gamma \cdot \hat{R}(Q) \right) - \frac{4}{5\alpha} \mathcal{D}_a \mathcal{D}_a \alpha - \frac{2}{5\alpha^2} \mathcal{D}_a \mathcal{D}_a \alpha$$

$$+ \frac{2}{5\alpha^2} \hat{F}_{ab}(A) \hat{F}^{ab} \alpha - \frac{4}{\alpha^2} i_{ij} v_{ij} + \mathcal{O}((\Omega_0^i)^2),$$  \hspace{1cm} (B.12)

where $\mathcal{O}(\Omega_0^i)$ represents terms of higher order in $\Omega_0$.

The relation between the $Q$ and $S$ transformations in 5D and in 6D is finally given by

$$\delta_Q(\varepsilon) = \delta_{PD}^Q(\varepsilon) + \delta_M(\theta_M(\varepsilon)) + \delta_U(\theta_U(\varepsilon)) + \delta_S(\theta_S(\varepsilon)) + \delta_K(\theta_K(\varepsilon)),$$

$$\delta_S(\eta) = \delta_{PS}^S(\eta) + \delta_K(\zeta_K(\eta)), \hspace{1cm} (B.13)$$

where

$$\theta_M^5(\varepsilon) = \frac{2i}{\alpha} \gamma^a \Omega_0 \varepsilon^i,$$

$$\theta_U^i(\varepsilon) = -\frac{2i}{\alpha} \varepsilon^i \Omega_0^i,$$

$$\theta_S^i(\varepsilon) = \frac{1}{4} \gamma \cdot \left( -v + \frac{1}{4\alpha} \hat{F}(A) \right) \varepsilon^i - \frac{i}{2\alpha^2} (\Omega_0^i \varepsilon_j + \frac{i}{\alpha^2} (\Omega_0 \gamma_a \Omega_0^j) \gamma^a \varepsilon_j,$$

$$\theta_K^i(\varepsilon) = -\frac{i}{24} \gamma_a \chi - i\varepsilon \left( \frac{\phi_a}{\gamma_a - \theta_S(\Omega_0^i)} - \phi_a(Q) \right) - \frac{i}{\alpha} \varepsilon \Omega_0^i.$$

(B.14)

Here, the dependent gauge fields $\overline{\phi}^i_a$ and $\overline{\phi}^i_a$ are those determined by the curvature constraint (B.7).

The 6D vector multiplet consists of a real vector field $W_M$, an $SU(2)$-Majorana spinor $Q^i$, and a triplet of the auxiliary scalar field $Y^{ij}$, whereas the 5D vector multiplet consists of a real vector $M_\mu$, a scalar $M$, an $SU(2)$ Majorana spinor $\Omega^i$, and a triplet of the auxiliary scalar $Y^{ij}$. The proper identification of the vector multiplet components is

$$M = -W_5, \quad W_\mu = W_\mu, \quad \Omega^i = \Omega^i + \frac{M}{\alpha} \Omega_0^i,$$

$$Y^{ij} = Y^{ij} + \frac{M}{\alpha} Y^{ij} - \frac{2i}{\alpha} \Omega_0^i \left( \Omega^{ij} - \frac{M}{\alpha} \Omega_0^{ij} \right).$$  \hspace{1cm} (B.15)

The 6D linear multiplet consists of a triplet $L^{ij}$, an $SU(2)$-Majorana spinor $\varphi^i$, and a constrained vector field $E_A$. The components of the 5D linear multiplet are identified as

$$L^{ij} = \frac{1}{\alpha} L^{ij}, \quad \varphi^i = \frac{1}{\alpha} (\varphi^i - 2\Omega_0\varphi^i),$$
\[ E^a = \frac{1}{\alpha} \left( E^a + \frac{4i}{\alpha} \overline{\Omega}_0^a \gamma^a \Omega_0^b L_{ij} - 2i \overline{\Omega}_0^a \varphi \right), \]
\[ N = -\frac{1}{\alpha} (E_5 + 2 L^{ij} t_{ij} + 4i \overline{\Omega}_0 \varphi). \]  
(B.16)

The 6D nonlinear multiplet consists of a scalar \( \Phi^i_\alpha \), an \( SU(2) \)-Majorana spinor \( \lambda^i_\alpha \), and a vector field \( V_A \). Identification of the 5D nonlinear multiplet is given by
\[ \Phi^i_\alpha = \Phi^i_\alpha, \quad \lambda^i_\alpha = \lambda^i_\alpha - \frac{1}{\alpha} \Omega^i_0, \]
\[ V_a = V_a + \frac{1}{\alpha} D_a \alpha, \quad V^5 = - V_5 - \frac{2i}{\alpha} \overline{\Omega}_0 \lambda. \]  
(B.17)

The 6D hypermultiplet consists of a scalar \( A^i_\alpha \) and a \( SU(2) \)-Majorana spinor \( \zeta^\alpha \). The 5D hyper multiplet is identified as
\[ A^i_\alpha = \frac{1}{\sqrt{\alpha}} A^i_\alpha, \quad \zeta^\alpha = \frac{1}{\sqrt{\alpha}} \zeta^\alpha + \frac{1}{\alpha} \Omega^i_0 A^j_\alpha. \]  
(B.18)

Appendix C

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**Weyl Multiplet in 4D with no Constraints**

In the text we have explained that the constraint for \( Q \) and \( M \) curvatures is not necessarily needed. This fact is not familiar, so we illustrate our formulation by taking an example of the well-known \( N = 1, d = 4 \) superconformal Weyl multiplet.\(^{14}\)

The independent gauge fields of the \( N = 1, d = 4 \) Weyl multiplet are the vierbein \( e^a_\mu \), the gravitino \( \psi_\mu \), the \( D \) gauge field \( b_\mu \), and the \( U(1) \) gauge field \( A_\mu \). In this section, \( \mu, \nu, \ldots \) and \( a, b, \ldots \) are four-dimensional indices. The spinors are Majorana. In the usual formulation, in which the \( Q \) curvature constraint \( \gamma^\nu \hat{R}_{\mu\nu}(Q) = 0 \) is imposed, the \( Q, S \) and \( K \) transformation laws of the Weyl multiplet are given by
\[
\delta e^a_\mu = -2i \bar{\varepsilon} \gamma^a \psi_\mu, \\
\delta \psi_\mu = D_\mu \bar{\varepsilon} + i \gamma_\mu \eta, \\
\delta b_\mu = -2 \bar{\varepsilon} \phi^\text{sol}_\mu + 2i \bar{\eta} \psi_\mu - 2 \xi_{k\mu}, \\
\delta A_\mu = -4i \bar{\varepsilon} \gamma_5 \phi^\text{sol}_\mu + 4i \bar{\eta} \gamma_5 \psi_\mu, \\
(C.1)
\]
where \( \phi^\text{sol}_\mu \) is the solution of \( \gamma^\nu \hat{R}_{\mu\nu}(Q) = 0 \). We note that \( \hat{R}(Q) \) contains the \( S \) gauge field \( \phi_\mu \) in the form \( \hat{R}_{\mu\nu}(Q) = \hat{\Omega}_{\mu\nu}(Q) - 2i \gamma_\mu \phi_\nu, \hat{\Omega}_{\mu\nu}(Q) \equiv \hat{R}_{\mu\nu}(Q)|_{\phi=0} \). Solving this with respect to \( \phi_\mu \), we have
\[ \phi_\mu = \phi_\mu(Q) + \phi^\text{sol}_\mu, \]
\[ \]
\[ \phi_\mu(Q) \equiv \frac{i}{3} \gamma^\nu \hat{R}_{\mu\nu}(Q) - \frac{i}{12} \gamma_\mu^{\nu\rho} \hat{R}_{\nu\rho}(Q), \]
\[ \phi^\text{sol}_\mu \equiv -\frac{i}{3} \gamma^\nu \hat{\Omega}_{\mu\nu}(Q) + \frac{i}{12} \gamma_\mu^{\nu\rho} \hat{\Omega}_{\nu\rho}(Q). \] (C.2)

Under the \( \mathcal{Q} \) curvature constraint \( \gamma^\nu \hat{R}_{\mu\nu}(Q) = 0 \), \( \phi_\mu \) equals \( \phi^\text{sol}_\mu \). Then we can replace \( \phi^\text{sol}_\mu \) in (C.1) formally by \( \phi_\mu - \phi_\mu(Q) \), because \( \phi_\mu(Q) = 0 \), and obtain
\[
\begin{align*}
\delta b_\mu &= -2\pi \phi_\mu + 2\pi \phi_\mu(Q) + 2\pi \psi_\mu - 2\xi_\mu, \\
\delta A_\mu &= -4i \pi \gamma_5 \phi_\mu + 4i \pi \gamma_5 \phi_\mu(Q) + 4i \pi \gamma_5 \psi_\mu.
\end{align*}
\] (C.3)

In these expressions, \( \phi_\mu \) is decoupled from the transformation laws, since \( \phi_\mu \) is cancelled by that in \( \phi_\mu(Q) \). This means that the \( \mathcal{Q} \) curvature constraint is not needed. We arrive at the following conclusion. In order to move to the formulation where the \( \mathcal{Q} \) and \( \mathcal{M} \) curvature constraint is not imposed, we only have to replace \( \phi^\text{sol}_\mu \) by \( \phi_\mu - \phi_\mu(Q) \).

**Appendix D**

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**Equivalency to Poincaré Supergravities**

In a previous paper,\(^{10}\) which we refer to as II henceforth, supergravity coupled to a matter-Yang-Millas system in 5D is derived on the basis of the supergravity tensor calculus presented in Ref. 9) which we refer to as I henceforth. However, there, a quite laborious computation was required to obtain the canonical form of the Einstein and Rarita-Schwinger terms. This is due to the redefinitions of fields, in particular that of the Rarita-Schwinger field \( \psi_{i\mu}(5\cdot3) \) in II. These redefinitions are also accompanied by modification of the transformation laws, (6\cdot8)-(6\cdot10) in II. Since we have a full superconformal tensor calculus, it is now easy to reproduce the Poincaré tensor calculus constructed in I and II. The point is that we can obtain this calculus simply by fixing the extraneous gauge freedoms, without making laborious redefinitions of the gauge field.

**D.1. Paper I**

First, we identify one of the Abelian vector multiplets with the central charge vector multiplet, which is a sub-multiplet of the Weyl multiplet in the Poincaré supergravity formulation:
\[
(W_\mu, M, \Omega^i, Y^{ij}) = (A_\mu, \alpha, \Omega^i = 0, -t^{ij} \alpha),
\] (D.1)
where the scalar \( \alpha \) is covariantly constant. That is, we choose the following \( S \) and \( K \) gauges:
\[
S : \Omega^i = 0, \quad K : \alpha^{-1} \hat{\mathcal{D}}_a \alpha = 0.
\] (D.2)
These gauge fixings are achieved by redefinitions of the $Q$ transformation:

$$\begin{align*}
\tilde{\delta}Q(\varepsilon) &= \delta Q(\varepsilon) + \delta S(\eta^i(\varepsilon)) + \delta K(\xi^K_\alpha(\varepsilon)), \\
\eta(\varepsilon)^i &= -\frac{1}{4\alpha} \gamma^i \hat{F}(A) \varepsilon^i - t^i_j \varepsilon^j, \\
\xi^K_\alpha(\varepsilon) &= i \varepsilon (\phi^a - \phi^a(Q)) + \bar{\eta}(\varepsilon) \psi^a \\
&= i \varepsilon (\phi^a - \eta(\psi^a) - \phi^a(Q)).
\end{align*}$$

Actually, the gauge choices (D.2) are invariant under $\tilde{\delta}Q(\varepsilon)$. Next, we replace the auxiliary fields of the Weyl multiplet, the vector multiplet and the linear multiplet as follows:

$$\begin{align*}
v_{ab} &\rightarrow v_{ab} - \frac{1}{2\alpha} \hat{F}_{ab}(A), \\
\chi^i &\rightarrow 16 \check{\chi}^i - \gamma^i \hat{R}^i(Q), \\
D &\rightarrow 8C + \frac{1}{2} \hat{R}(M) - 6\nu^2 + \frac{2}{\alpha} \nu \cdot \hat{F}(A) + \frac{3}{2\alpha^2} \hat{F}(A)^2 + 20 t^j_i t^i_j, \\
Y^{ij} &\rightarrow Y^{ij} - M_t^{ij}, \\
N &\rightarrow N + 2t^{ij} L_{ij}.
\end{align*}$$

Then, we can show that the Poincaré supergravity tensor calculus in the paper I is exactly reproduced. These gauge choices and the redefinition of the $Q$ transformation must be accompanied by redefinitions of the full covariant curvature $\hat{R}_{i\mu}^{} A$ and the covariant derivative $\hat{D}_\mu$. However, such redefinitions are carried out automatically, and there is no need to do so by hand.

D.2. Paper II

The conditions (5·1) and (5·6) in II require the gauge fixings

$$D : \mathcal{N} = 1, \quad S : \Omega^{ij} \mathcal{N}_I = 0, \quad K : \mathcal{N}^{-1} \hat{D}_a \mathcal{N} = 0.$$  

These gauge fixings are achieved by

$$\begin{align*}
\tilde{\delta}Q(\varepsilon) &= \delta Q(\varepsilon) + \delta S(\eta^i(\varepsilon)) + \delta K(\xi^K_\alpha(\varepsilon)), \\
\eta(\varepsilon)^i &= -\frac{N_I}{12N} \gamma^i \hat{F}(W) \varepsilon^i + \frac{N_I}{3N} Y^{ij} \varepsilon^j + \frac{N_I}{3N} \Omega^{ij} 2i \varepsilon^i \\
&= -\frac{1}{3} \left( \Gamma^i - \gamma \cdot v \right) \varepsilon^i, \\
\xi^K_\alpha(\varepsilon) &= i \varepsilon (\phi^a - \eta(\psi^a) - \phi^a(Q)),
\end{align*}$$

and the following replacements are needed:

$$\begin{align*}
\frac{N_I}{3N} Y^{ij} &\rightarrow -\tilde{\Omega}^{ij}, \\
V^{ij}_\mu &\rightarrow \tilde{V}^{ij}_\mu, \\
v_{ab} &\rightarrow \tilde{v}_{ab}, \\
\chi^i &\rightarrow 16 \check{\chi}^i + 3 \gamma^i \hat{R}^i(Q), \\
D &\rightarrow 8\tilde{C} - \frac{3}{2} \hat{R}(M) + 2\nu^2, \\
\Omega &\rightarrow \lambda, \\
\zeta_\alpha &\rightarrow \xi_\alpha, \\
Y^{ij} &\rightarrow \tilde{Y}^{ij} - M_t^{ij}.
\end{align*}$$

The resultant $Q$ transformation laws of the Weyl multiplet, the vector multiplet and the hypermultiplet are equivalent to (6·8), (6·9) and (6·10) in II, respectively.
References


Note added: After finishing the original form of this paper, the authors became aware of the preprint by Bergshoeff et al.\textsuperscript{22}) treating conformal supergravity in five dimensions. They discussed the two versions of the superconformal Weyl multiplets, which they call the Standard one and the Dilaton one. These correspond to the conventional one and N-R one in the present paper. They did not present the full superconformal tensor calculus which we have given here.