The neutrino is related to a symmetry operation in the other numbers of quarks and leptons as follows:

\[
\begin{pmatrix}
-a & b \\
-b & a
\end{pmatrix}
\]

where \(a\) and \(b\) are quark-lepton-charged quark numbers, and \(F\) is the multiplet of all charged particles with at least one charged fermion, together with a third fermion. Quark-lepton-charged numbers are multiplied by a large factor and are neglected. In addition, a fermionic particle belongs to a large multiplet of fermions and weak interactions, and weak interactions, a grand unification theory. Thus, the structure is that in essentially any grand unified theory, that only the strange, charm, and bottom quarks have nonzero masses. The reason for this structure was that, for a long time, the neutrino masses were not known. The reason was that, for a long time, the neutrino masses were not considered. Thus, it would be peculiar if the neutrino did not have nonzero masses. This is not coincidental. There was no hard evidence that they do.

We humans, and the everyday objects around us, are made of neutrinos and electrons. So these particles are the ones most familiar to us. However, for electrons, so these particles are the ones most familiar to us. However, for neutrinos, and the everyday objects around us, are made of neutrinos and electrons.
have a nonzero mass as well.

If neutrinos do have nonzero masses, we must understand why they are nevertheless so light. Perhaps the most appealing explanation of their lightness is the “see-saw mechanism.” To understand how this mechanism works, let us note that, unlike charged particles, neutrinos may be their own antiparticles. If a neutrino is identical to its antiparticle, then it consists of just two mass-degenerate states: one with spin up, and one with spin down. Such a neutrino is referred to as a Majorana neutrino. In contrast, if a neutrino is distinct from its antiparticle, then it plus its antiparticle form a complex consisting of four mass-degenerate states: the spin up and spin down neutrino, plus the spin up and spin down antineutrino. This collection of four states is called a Dirac neutrino. In the see-saw mechanism, a four-state Dirac neutrino $N^D$ of mass $M_D$ gets split by “Majorana mass terms” into a pair of two-state Majorana neutrinos. One of these Majorana neutrinos, $\nu^M$, has a small mass $M_\nu$ and is identified as one of the observed light neutrinos. The other, $N^M$, has a large mass $M_N$ characteristic of some high mass scale where new physics beyond the range of current particle accelerators, and responsible for neutrino mass, resides. Thus, $N^M$ has not been observed. The character of the breakup of $N^D$ into $\nu^M$ and $N^M$ is such that $M_\nu M_N \approx M_D^2$. It is reasonable to expect that $M_D$, the mass of the Dirac particle $N^D$, is of the order of $M_{\text{GUT}}$, the mass of a typical charged lepton $\ell$ or quark $q$, since the latter are Dirac particles too. Then $M_\nu M_N \approx M_{\text{GUT}}^2$. With $M_{\text{GUT}}$ a typical charged lepton or quark mass, and $M_N$ very big, this “see-saw relation” explains why $M_\nu$ is very tiny.

Note that the see-saw mechanism predicts that each light neutrino $\nu^M$ is a two-state Majorana neutrino, identical to its antineutrino.

2 Neutrino Oscillation

To find out whether neutrinos really do have nonzero masses, we need an experimental approach which can detect these masses even if they are very small. The most sensitive approach is the search for neutrino oscillation. Neutrino oscillation is a quantum interference phenomenon in which small splittings between the masses of different neutrinos can lead to large, measurable phase differences between interfering quantum-mechanical amplitudes.

To explain the physics of neutrino oscillation, we must first discuss leptonic “flavor”. Suppose a neutrino $\nu$ is born in the $W$-boson decay

$$W^+ \rightarrow \ell_\alpha^+ + \nu.$$  \hspace{1cm} (2)

Here, $\alpha = e, \mu$ or $\tau$, and $\ell_\alpha^+$ is one of the positively charged leptons: $\ell_e^+ \equiv e^+$, $\ell_\mu^+ \equiv \mu^+$, and $\ell_\tau^+ \equiv \tau^+$. Suppose that, without having time to change its
character, the neutrino $\nu$ interacts in a detector immediately after its birth in the decay (2), and produces a new charged lepton $\ell_\beta^-$ via the reaction $\nu + \text{target} \to \ell_\beta^- + \text{recoils}$. It is found that the “flavor” $\beta$ of this new charged lepton is always the same as the flavor $\alpha$ of the charged lepton with which $\nu$ was born. It follows that the neutrinos produced by the $W$ decays (2) to charged leptons of different flavors must be different objects. We take this fact into account by writing these decays more accurately as

$$W^+ \to \ell_a^+ + \nu_a; \quad a = e, \mu, \tau.$$  \hspace{1cm} (3)

The neutrino $\nu_\alpha$, called the neutrino of flavor $\alpha$, is by definition the neutrino produced in leptonic $W$ decay in association with the charged lepton of flavor $\alpha$. As we have said, when $\nu_\alpha$ interacts to create a charged lepton, the latter lepton is always $\ell_\alpha$. In neutrino oscillation, a neutrino born in association with a charged lepton $\ell_\alpha$ of flavor $\alpha$ then travels for some time during which it can alter its character. Finally, it interacts to produce a second charged lepton $\ell_\beta$ with a flavor $\beta$ different from the flavor $\alpha$ of the charged lepton with which the neutrino was born. For example, suppose a neutrino is born with a muon in the pion decay $\pi^+ \to \text{Virtual } W^+ \to \mu^+ + \nu_\mu$. Suppose further that after traveling down a neutrino beamline, this same neutrino interacts in a detector and produces, not another muon, but a $\tau^-$. At birth, the neutrino was a $\nu_\mu$. But by the time it interacted in the detector, it had turned into a $\nu_\tau$. One describes this metamorphosis by saying the neutrino oscillated from a $\nu_\mu$ into a $\nu_\tau$. As we will see, the probability for it to change its flavor does indeed oscillate with the distance it travels before interacting. As we will also see, the oscillation in vacuum of a neutrino between different flavors requires neutrino mass.

To see how neutrino mass can lead to neutrino oscillation, let us briefly recall the weak interactions of quarks. As we all know, there are three quarks—the $u$ (up), $c$ (charm), and $t$ (top) quarks—which carry a positive electric charge $Q = +2/3$. In addition, there are three quarks—the $d$ (down), $s$ (strange), and $b$ (bottom) quarks—which carry a negative electric charge $Q = -1/3$. Each of these six quarks is a particle of definite mass. As we know, the quarks are arranged into three families or generations, each of which contains one positive quark and one negative quark:

$$\begin{align*}
\text{Family}: \quad & 1 \quad 2 \quad 3 \\
\text{Quarks}: \quad & \begin{pmatrix}
u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}
\end{align*}$$

However, we know experimentally that under the weak interaction, any of the negative quarks, $d$, $s$, or $b$, can absorb a positively-charged $W$ boson and turn
Figure 1: Absorption of a W boson by a quark.

into any of the positive quarks, u, c, or t. This is illustrated in Fig. (1). There, 
$\alpha \equiv d$ is one of the down-type (negative) quarks. That is, $d \equiv d$ is the 
down quark, $d \equiv s$ is the strange quark, and so on. Similarly, $u \equiv u, c, t$ 
is one of the up-type (positive) quarks, with $u \equiv c$ being the charm quark, 
and so on.

In the S(standard) M(oodel) of the electroweak interactions the $W$-quark 
couplings depicted in Fig. (1) are described by the Lagrangian density

$$L_{\text{udW}} = -\frac{g}{\sqrt{2}} \sum_{\alpha = u, c, t} \bar{u}_{\alpha} d \gamma_i W_i^+ + \text{h.c.} \quad (4)$$

Here, the subscript $L$ denotes left-handed chiral projection. For instance, 
$d_{L, (i = d, s, b)}$ is the left-handed strange-quark field. The constant $g$ 
is the semiweak coupling constant, and $V$ is a $3 \times 3$ matrix known as the quark 
mixing matrix. In the SM, $V$ is unitary, because it is basically the matrix for 
the transformation from one basis of quantum states to another.

The SM interaction (4) is very well confirmed experimentally. With this 
established behavior of quarks in mind, let us now return to the leptons. Like the 
quarks, the charged leptons $e, \mu,$ and $\tau$ are particles of definite mass. However, 
if leptons behave as quarks do, then the neutrinos $\nu_e, \nu_\mu,$ and $\nu_\tau$ of definite 
flavor are not particles of definite mass. Let us call the neutrinos which do 
have definite masses $\nu_i, i = 1, 2, \ldots, N$. As far as we know, the number of 
$\nu_i, N$, may exceed the number of charged leptons, three. Now, as we recall, 
the neutrino $\nu_\alpha$ of definite flavor $\alpha$ is the neutrino state that accompanies the 
definite-mass charged lepton $\ell_\alpha$ in the decay $W \rightarrow \ell_\alpha + \nu_\alpha$. If leptons behave 
as quarks do, this neutrino state must be a superposition of the neutrinos $\nu_i$ of 
definite mass. To see this, we first note that just as any negative quark $d_i$ of 
definite mass can absorb a $W_i^+$ and turn into any positive quark $u_\alpha$ of definite 
mass, so it must be possible for any neutrino $\nu_i$ of definite mass to absorb a
\( W^- \) and turn into any charged lepton \( \ell^+_{\alpha} \) of definite mass. This absorption is illustrated in Fig. (2). We expect that in analogy with Eq. (4) for the W-quark couplings, the SM interaction that describes the W-lepton couplings is

\[
\mathcal{L}_{\ell W} = -\frac{g}{\sqrt{2}} \sum_{i=1}^{3} \bar{\nu}_{\alpha} \gamma^\lambda U_{\alpha i} \ell L_i W^\lambda_{\alpha} - \frac{g}{\sqrt{2}} \sum_{i} \bar{\nu}_{\alpha} \gamma^\lambda U^\dagger_{i\alpha} \ell L_i W^\pm \lambda_{\alpha} .
\]

Here, \( U \) is an \( N \times N \) unitary matrix which is the leptonic analogue of the quark mixing matrix \( V \). The matrix \( U \) is referred to as the leptonic mixing matrix. If \( N > 3 \), then only the top 3 rows of \( U \) enter in the W-lepton interaction, Eq. (5).

The leptonic decays of the \( W^\pm \) are governed by the second term of \( \mathcal{L}_{\ell W} \), Eq. (5). From this term, we see that when \( W^\pm \rightarrow \ell^+_\alpha + \nu_{\alpha} \), the neutrino state \( |\nu_{\alpha}\rangle \) produced in association with the specific definite-mass charged lepton \( \ell^+_\alpha \) is

\[
|\nu_{\alpha}\rangle = \sum_i U^\dagger_{i\alpha} |\nu_i\rangle .
\]

That is, the “flavor-\( \alpha \)” neutrino \( |\nu_{\alpha}\rangle \) produced together with \( \ell^+_\alpha \) is a coherent superposition of the mass-eigenstate neutrinos \( |\nu_i\rangle \), with coefficients which are elements of the leptonic mixing matrix.

What if \( N \) is bigger than three? Suppose, for example, that \( N = 4 \). Then, with the elements of the bottom row of \( U \), \( U_{\text{lastrow},i} \), we can construct a neutrino state

\[
|\nu_4\rangle = \sum_i U^\dagger_{\text{lastrow},i} |\nu_i\rangle
\]

which does not couple to any of the 3 charged leptons. This state is called a “sterile” neutrino, which just means that it does not participate in the SM weak interactions. It may, however, participate in other interactions beyond the SM whose effects at present-day energies are too feeble to have been observed.
Figure 3: Creation of a neutrino with a charged lepton of flavor $\alpha$ followed by the interaction of this neutrino to produce a charged lepton of flavor $\beta$. The “Source” is the particle whose decay creates the neutrino, $\ell_\alpha$, and other, unlabelled, particles. The “Target” is the particle struck by the neutrino to produce $\ell_\beta$ and other, unlabelled, particles. “$A$” denotes an amplitude.

Owing to the leptonic mixing described by Eq. (25), when the charged lepton of flavor $\alpha$ is created, the accompanying neutrino can be any of the $\nu_i$. Furthermore, if this $\nu_i$ later interacts with some target, it can produce a charged lepton $\ell_\beta$ of any flavor $\beta$. In such a sequence of events, the neutrino itself is an unseen intermediate state. Thus, as shown in Fig. (28), the amplitude for a neutrino to be born with charged lepton $\ell_\alpha$, and then to interact and produce charged lepton $\ell_\beta$, is a coherent sum over the contributions of all the unseen mass eigenstates $\nu_i$.

The birth of a neutrino with charged lepton $\ell_\alpha^{(+)}$ and its subsequent interaction to produce charged lepton $\ell_\beta^{(-)}$ is usually described as the oscillation $\nu_\alpha \rightarrow \nu_\beta$ of a neutrino of flavor $\alpha$ into one of flavor $\beta$ (see earlier discussion).
Using "A" to denote an amplitude, we see from Fig. (2) that
\[ A(\nu \rightarrow \nu) = \sum_i A(\text{neutrino born with } \ell_\alpha^+ \text{ is } \nu_i) \times \]
\[ A(\nu_i \text{ propagates } \ell_\beta^- \text{ when } \nu_i \text{ interacts it makes } \ell_\beta^-) \] .
\[ (8) \]
From $L_{\alpha\beta\nu}$, Eq. (2), we find that apart from irrelevant factors,
\[ A(\text{neutrino born with } \ell_\alpha^+ \text{ is } \nu_i) = U_{\alpha i} \] .
\[ (9) \]
Similarly,
\[ A(\text{when } \nu_i \text{ interacts it makes } \ell_\beta^-) = U_{\beta i} \] .
\[ (10) \]
To find the amplitude $A(\nu_i \text{ propagates})$, we note that in the rest frame of $\nu_i$, where the proper time is $\tau_i$, Schrödinger’s equation states that
\[ i \frac{\partial}{\partial \tau_i} |\nu_i(\tau_i)\rangle = M_i |\nu_i(\tau_i)\rangle \] .
\[ (11) \]
Here, $M_i$ is the mass of $\nu_i$. From Eq. (11),
\[ |\nu_i(\tau_i)\rangle = e^{-iM_i \tau_i} |\nu_i(0)\rangle \] .
\[ (12) \]
Now, for propagation over a proper time interval $\tau_i$, $A(\nu_i \text{ propagates})$ is just the amplitude for finding the original state $|\nu_i(0)\rangle$ in the time-evolved $|\nu_i(\tau_i)\rangle$. That is,
\[ A(\nu_i \text{ propagates}) = \langle \nu_i(0)|\nu_i(\tau_i)\rangle = e^{-iM_i \tau_i} \] .
\[ (13) \]
In terms of the time $t$ and position $L$ in the laboratory frame, the Lorentz-invariant phase factor $\exp(-iM_i \tau_i)$ is
\[ e^{-i(E_it - p_i L)} \] .
\[ (14) \]
Here, $E_i$ and $p_i$ are, respectively, the energy and momentum of $\nu_i$ in the laboratory frame. In practice, our neutrino will be highly relativistic, so if it was born at $(t, L) = (0, 0)$, we will be interested in evaluating the phase factor $e^{-iM_i \tau_i}$ where $t \approx L$, where it becomes
\[ e^{-i(E_i - p_i) L} \] .
\[ (15) \]
Suppose that the neutrino created with $\ell_\alpha$ is produced with a definite momentum $p$, regardless of which $\nu_i$ it happens to be. Then, if it is the particular mass eigenstate $\nu_i$, it has total energy
\[ E_i = \sqrt{p^2 + M_i^2} \approx p + \frac{M_i^2}{2p} \] .
\[ (16) \]
assuming that all the masses $M_i$ are much smaller than $p$. From (15), we then find that

$$A(\nu_i \text{ propagates}) \approx e^{-\frac{M_i^2}{2E}}.$$  \hfill (17)

Alternatively, suppose that our neutrino is produced with a definite energy $E$, regardless of which $\nu_i$ it happens to be. Then, if it is the particular mass eigenstate $\nu_i$, it has momentum

$$p_i = \sqrt{E^2 - M_i^2} \approx E - \frac{M_i^2}{2E}.$$  \hfill (18)

From (15), we then find that

$$A(\nu_i \text{ propagates}) \approx e^{-\frac{M_i^2}{2E}}.$$  \hfill (19)

Since highly relativistic neutrinos have $E \approx p$, the propagation amplitudes given by Eqs. (17) and (19) are approximately equal. Thus, it doesn’t matter whether our neutrino is created with definite momentum or definite energy.

Collecting the various factors that appear in Eq. (8), we conclude that the amplitude $A(\nu_\alpha \to \nu_\beta)$ for a neutrino of energy $E$ to oscillate from a $\nu_\alpha$ to a $\nu_\beta$ while traveling a distance $L$ is given by

$$A(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i}^* e^{-i\frac{M^2_{ij}L}{4E}} U_{\beta i}.$$  \hfill (20)

The probability $P(\nu_\alpha \to \nu_\beta)$ for this oscillation is then given by

$$P(\nu_\alpha \to \nu_\beta) = |A(\nu_\alpha \to \nu_\beta)|^2 = \delta_{\alpha \beta} - 4 \sum_{i>j} \text{Re}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin^2\left(\frac{\delta M^2_{ij}L}{4E}\right)\right) + 2 \sum_{i>j} \text{Im}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin\left(\frac{\delta M^2_{ij}L}{2E}\right)\right).$$  \hfill (21)

Here, $\delta M^2_{ij} \equiv M_i^2 - M_j^2$, and in calculating $|A(\nu_\alpha \to \nu_\beta)|^2$, we have used the unitarity constraint

$$\sum_i U_{\alpha i}^* U_{\beta i} = \delta_{\alpha \beta}.$$  \hfill (22)

Some general comments are in order:

1. From Eq. (22) for $P(\nu_\alpha \to \nu_\beta)$, we see that if all neutrino masses vanish, then $P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha \beta}$, and there is no oscillation from one flavor to another. Neutrino flavor oscillation requires neutrino mass.
2. The probability \( P(\nu_a \rightarrow \nu_b) \) oscillates as a function of \( L/E \). This is why the phenomenon we are discussing is called “neutrino oscillation”.

3. From Eq. (2.8) for \( A(\nu_a \rightarrow \nu_b) \), we see that the \( L/E \) dependence of neutrino oscillation arises from interferences between the contributions of the different mass eigenstates \( \nu_i \). We also see that the phase of the \( \nu_i \) contribution is proportional to \( M_i^2 \). Thus, the interferences can give us information on neutrino masses. However, since these interferences can only reveal the relative phases of the interfering amplitudes, experiments on neutrino oscillation can only determine the splittings \( \delta M_{ij}^2 \equiv M_i^2 - M_j^2 \), and not the underlying individual neutrino masses. This fact is made perfectly clear by Eq. (2.9) for \( P(\nu_a \rightarrow \nu_b) \).

4. With the so-far omitted factors of \( \hbar \) and \( c \) inserted,

\[
\delta M_{ij}^2 \frac{L}{4E} = 1.27 \delta M_{ij}^2 (eV^2) \frac{L(\text{km})}{E(\text{GeV})}.
\]

Thus, from Eq. (2.10) for the oscillation probability \( P(\nu_a \rightarrow \nu_b) \), we see that an oscillation experiment characterized by a given value of \( L(\text{km}) / E(\text{GeV}) \) is sensitive to mass splittings obeying

\[
\delta M_{ij}^2 (eV^2) \gtrsim \left[ \frac{L(\text{km})}{E(\text{GeV})} \right]^{-1}.
\]

To be sensitive to tiny \( \delta M_{ij}^2 \), an experiment must have large \( L/E \). In Table 1, we indicate the \( \delta M_{ij}^2 \) reach implied by Eq. (2.11) for experiments working with neutrinos produced in various ways.

5. There are basically two kinds of oscillation experiments: \textit{appearance} experiments, and \textit{disappearance} experiments.

In an appearance experiment, one looks for the \textit{appearance} in the neutrino beam of neutrinos bearing a flavor not present in the beam initially. For example, imagine that a beam of neutrinos is produced by the decays of charged pions. Such a beam consists almost entirely of muon neutrinos and contains no tau neutrinos. One can then look for the appearance of tau neutrinos, made by oscillation of the muon neutrinos, in this beam.

In a disappearance experiment, one looks for the \textit{disappearance} of some fraction of the neutrinos bearing a flavor which is present in the beam initially. For example, imagine again that a beam of neutrinos is produced by the decays of charged pions, so that almost all the neutrinos
Table 1: The approximate reach in $\delta M^2$ of experiments studying various types of neutrinos. Often, an experiment covers a range in $L$ and a range in $E$. To construct the table, we have used typical values of these quantities.

<table>
<thead>
<tr>
<th>Neutrinos (Baseline)</th>
<th>$L$(km)</th>
<th>$E$(GeV)</th>
<th>$\frac{L}{E}$</th>
<th>$\delta M^2$ (eV$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accelerator (Short Baseline)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Reactor (Medium Baseline)</td>
<td>1</td>
<td>$10^{-3}$</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Accelerator (Long Baseline)</td>
<td>$10^3$</td>
<td>10</td>
<td>$10^2$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Atmospheric</td>
<td>$10^4$</td>
<td>1</td>
<td>$10^4$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Solar</td>
<td>$10^8$</td>
<td>$10^{-3}$</td>
<td>$10^{11}$</td>
<td>$10^{-11}$</td>
</tr>
</tbody>
</table>

in the beam are muon neutrinos, $\nu_{\mu}$. If one knows the $\nu_{\mu}$ flux that is produced initially, one can look to see whether some of this initial $\nu_{\mu}$ flux disappears after the beam has traveled some distance, and the muon neutrinos have had a chance to oscillate into other flavors.

6. Even though neutrinos can change flavor through oscillation, the total flux of neutrinos in a beam will be conserved so long as $U$ is unitary. To see this, note that

$$ \sum_\beta P(\nu_\alpha \rightarrow \nu_\beta) = \sum_\beta |A(\nu_\alpha \rightarrow \nu_\beta)|^2 $$

$$ = \sum_\beta \left( \sum_i U_{\alpha i}^* U_{\beta i} e^{-iM_i^2 L / 2E} \right) \left( \sum_j U_{\alpha j} U_{\beta j}^* e^{+iM_j^2 L / 2E} \right) $$

$$ = \sum_{i,j} U_{\alpha i}^* U_{\alpha j} e^{-iM_i^2 L} \sum_\beta U_{\beta i} U_{\beta j}^* $$

$$ = \sum_i |U_{\alpha i}|^2 = 1. \quad (25) $$

Here, we have used Eq. (24) for the amplitude $A(\nu_\alpha \rightarrow \nu_\beta)$ and the unitarity relations $\sum_\beta U_{\beta i} U_{\beta j}^* = \delta_{ij}$ and $\sum_i |U_{\alpha i}|^2 = 1$. The result $\sum_\beta P(\nu_\alpha \rightarrow \nu_\beta) = 1$ means that if one starts with a certain number $n$ of neutrinos of flavor $\alpha$, then after oscillation the number of neutrinos...
that have oscillated away into new flavors $\beta \neq \alpha$, plus the number that have retained the original flavor $\alpha$, is still $n$. Note, however, that some of the new flavors that get populated by the oscillation might be sterile. If they are indeed sterile, then the number of “active” neutrinos (i.e., neutrinos that participate in the SM weak interactions) remaining after oscillation will be less than $n$.

2.1 Special Cases

Let us now apply the general formalism for neutrino oscillation in vacuum to several special cases of practical interest.

The simplest special case of all is two-neutrino oscillation. This occurs when the SM weak interaction, Eq. (5), couples two charged leptons (say, $e$ and $\mu$) to just two neutrinos of definite mass, $\nu_1$ and $\nu_2$, and only negligibly to any other neutrinos of definite mass. It is then easily shown that the $2 \times 2$ submatrix

\[ \hat{U} = \begin{bmatrix} U_{\nu_1 e} & U_{\nu_2 e} \\ U_{\mu_1 e} & U_{\mu_2 e} \end{bmatrix} \]

of the mixing matrix $U$ must be unitary all by itself. This means that the definite-flavor neutrinos $\nu_e$ and $\nu_\mu$ are composed exclusively of the mass eigenstates $\nu_1$ and $\nu_2$, and do not mix with neutrinos of any other flavor.

From Eq. (26) for the oscillation amplitude, we have for this two-neutrino case

\[ e^{+iM^2_{\nu_12} \frac{\Delta}{\hbar}} A(\nu_e \rightarrow \nu_\mu) = U_{\nu_1 e}^* U_{\mu_1 e} + U_{\nu_2 e}^* U_{\mu_2 e} e^{-i\delta M_{\nu_12} \frac{\Delta}{\hbar}}. \]

Since $\hat{U}$ is unitary, $U_{\nu_1 e}^* U_{\mu_1 e} + U_{\nu_2 e}^* U_{\mu_2 e} = 0$, so Eq. (27) may be rewritten as

\[ e^{+iM^2_{\nu_12} \frac{\Delta}{\hbar}} A(\nu_e \rightarrow \nu_\mu) = -U_{\nu_1 e}^* U_{\mu_2 e} (1 - e^{-i\delta M_{\nu_12} \frac{\Delta}{\hbar}}) \]

\[ = -2i e^{-i\delta M_{\nu_12} \frac{\Delta}{\hbar}} U_{\nu_1 e}^* U_{\mu_2 e} \sin(\delta M_{\nu_12} \frac{\Delta}{\hbar}). \]

Squaring this result, using Eq. (29) to take account of the requisite factors of $\hbar$ and $c$, we find that the probability $P(\nu_e \rightarrow \nu_\mu)$ for a $\nu_e$ to oscillate into a $\nu_\mu$ is given by

\[ P(\nu_e \rightarrow \nu_\mu) = 4|U_{\nu_1 e}|^2 |U_{\mu_2 e}|^2 \sin^2(1.27 \delta M^2 (\text{eV}^2 \frac{L}{E \text{(GeV)}})). \]

Here, we have introduced the abbreviation $\delta M^2_{\nu_12} \equiv \delta M^2$. From the $e \leftrightarrow \mu$ symmetry of the right-hand side of Eq. (26), it is obvious that

\[ P(\nu_\mu \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_\mu). \]
The probability $P(\nu_\alpha \to \nu_\alpha)$ that a neutrino $\nu_\alpha$ of flavor $\alpha = e \text{ or } \mu$ retains its original flavor is given by:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - P(\nu_\alpha \to \nu_{\beta \neq \alpha})$$

$$= 1 - 4|U_{\alpha 2}|^2(1 - |U_{\alpha 3}|^2) \sin^2(1.27 \delta M^2(eV^2) \frac{L(km)}{E(GeV)}) \quad (31)$$

To obtain this expression, we have used the conservation of probability, Eq. (25), the “off-diagonal” oscillation probability, Eqs. (26) and (27), and the unitarity relation $|U_{\alpha 2}|^2 + |U_{\alpha 3}|^2 = 1$.

The unitarity of $U$, Eq. (29), implies that it can be written in the form

$$U = \begin{bmatrix} e^{i\varphi_1} \cos \theta & e^{i\varphi_2} \sin \theta \\ -e^{i(\varphi_1 + \varphi_3)} \sin \theta & e^{i(\varphi_2 + \varphi_3)} \cos \theta \end{bmatrix} \quad (32)$$

Here, $\theta$ is an angle referred to as the leptonic mixing angle and $\varphi_1, \varphi_2, \varphi_3$ are phases. From Eq. (20), $4|U_{\alpha 2}|^2|U_{\alpha 3}|^2 = \sin^2 2\theta$, so that $P(\nu_\alpha \to \nu_\beta)$, Eq. (20), takes the form

$$P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) = \sin^2 2\theta \sin^2(1.27 \delta M^2(eV^2) \frac{L(km)}{E(GeV)}) \quad (33)$$

This is the most commonly quoted form of the two-neutrino oscillation probability.

A second special case which may prove to be very relevant to the real world is a three-neutrino scenario in which two of the neutrino mass eigenstates are nearly degenerate. That is, the neutrino mass^2 spectrum is as in Fig. (3), where

$$|\delta M^2_{12}| \equiv |\delta M^2_{21}| \ll |\delta M^2_{31}| \equiv |\delta M^2_{23}| \equiv \delta M^2_{\text{Big}} \quad (34)$$

All three of the charged leptons, $e, \mu, \tau$, are coupled by the SM weak interaction, Eq. (6), to the neutrinos $\nu_{1,2,3}$.

Suppose that an oscillation experiment has $L/E$ such that $\delta M^2_{\text{Big}} L/E$ is of order unity, which implies that $\delta M^2_{31} L/E \ll 1$. For this experiment, and $\beta \neq \alpha$, the oscillation amplitude of Eq. (24) is given approximately by

$$e^{i\delta M^2_{31} \frac{L}{E}} A(\nu_\alpha \to \nu_{\beta \neq \alpha}) \cong (U_{\alpha 1} U_{\beta 1} + U_{\alpha 2} U_{\beta 2}) e^{i\delta M^2_{31} \frac{L}{E}} + U_{\alpha 3} U_{\beta 3} \quad (35)$$

Using the unitarity constraint of Eq. (24), this becomes

$$e^{i\delta M^2_{31} \frac{L}{E}} A(\nu_\alpha \to \nu_{\beta \neq \alpha}) \cong U_{\alpha 3}^* U_{\beta 3} \left(1 - e^{i\delta M^2_{31} \frac{L}{4E}} \right)$$

$$= -2e^{i\delta M^2_{31} \frac{L}{4E}} U_{\alpha 3}^* U_{\beta 3} \sin(\delta M^2_{31} \frac{L}{4E}) \quad (36)$$

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Taking the absolute square of this relation, using $|\delta M^2_{\text{Big}}|$ $\equiv$ $\delta M^2_{\text{Big}}$, and inserting the omitted factors of $\hbar$ and $c$, we find that the $\nu_{\alpha} \rightarrow \nu_{\beta \neq \alpha}$ oscillation probability is given by

$$P(\nu_{\alpha} \rightarrow \nu_{\beta \neq \alpha}) = 4 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 (1.27 \delta M^2_{\text{Big}} (\text{eV})^2 \frac{L(\text{km})}{E(\text{GeV})}) .$$  \hfill (37)

To find the corresponding probability $P(\nu_{\alpha} \rightarrow \nu_{\alpha})$ that a neutrino of flavor $\alpha$ retains its original flavor, we simply use the conservation of probability, Eq. (25):

$$P(\nu_{\alpha} \rightarrow \nu_{\alpha}) = 1 - \sum_{\beta \neq \alpha} P(\nu_{\alpha} \rightarrow \nu_{\beta}) .$$  \hfill (38)

From Eq. (37) and the unitarity relation $\sum_{\beta \neq \alpha} |U_{\beta 3}|^2 = 1 - |U_{\alpha 3}|^2$, we then find that

$$P(\nu_{\alpha} \rightarrow \nu_{\alpha}) = 1 - 4 |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \sin^2 (1.27 \delta M^2_{\text{Big}} (\text{eV})^2 \frac{L(\text{km})}{E(\text{GeV})}) .$$  \hfill (39)

Comparing Eqs. (37) and (38), and Eqs. (35) and (36), we see that in the three-neutrino scenario with $\delta M^2_{\text{Small}} L/E \ll 1$, the oscillation probabilities are the same as in the two-neutrino case, except that with three neutrinos, the isolated one, $\nu_3$, plays the role played by $\nu_1$ when there are only two neutrinos. This strong similarity between the oscillation probabilities in the two cases is easy to understand. When the three-neutrino case is studied by an experiment for which $\delta M^2_{\text{Small}} L/E \ll 1$, the experiment cannot see the splitting between $\nu_1$ and $\nu_2$. In such an experiment, there appear to be only two neutrinos with distinct masses: the $\nu_1$ and $\nu_2$ pair, which looks like a single neutrino, and $\nu_3$. As a result, the two neutrino oscillation probabilities hold.
3 Neutrino Oscillation in Matter

So far, we have been talking about neutrino oscillation in vacuum. However, some very important oscillation experiments are concerned with neutrinos that travel through a lot of matter before reaching the detector. These neutrinos include those made by nuclear reactions in the core of the sun, which traverse a lot of solar material on their way out of the sun towards solar neutrino detectors here on earth. They also include the neutrinos made in the earth’s atmosphere by cosmic rays. These atmospheric neutrinos can be produced in the atmosphere on one side of the earth, and then travel through the whole earth before being detected in a detector on the other side. To deal with the solar and atmospheric neutrinos, we need to understand how passage through matter affects neutrino oscillation.

To be sure, the interaction between neutrinos and matter is extremely feeble. Nevertheless, the coherent forward scattering of neutrinos from many particles in a material medium can build up a big effect on the oscillation amplitude.

For both the solar and atmospheric neutrinos, it is a good approximation to take just two neutrinos into account. Furthermore, it is convenient to treat the propagation of neutrinos in matter in terms of an effective Hamiltonian. To set the stage for this treatment, let us first derive the Hamiltonian for travel through vacuum. For the sake of illustration, let us suppose that the two neutrino flavors that need to be considered are $\nu_e$ and $\nu_\mu$. The most general time-dependent neutrino state vector $|\nu(t)\rangle$ can then be written as

$$|\nu(t)\rangle = \sum_{\alpha=e,\mu} f_{\alpha}(t) |\nu_\alpha\rangle,$$

where $f_{\alpha}(t)$ is the time-dependent amplitude for the neutrino to have flavor $\alpha$. If $\mathcal{H}_V$ (short for $\mathcal{H}_{\text{Vacuum}}$) is the Hamiltonian for this two-neutrino system in vacuum, then Schrödinger’s equation for $|\psi(t)\rangle$ reads

$$i \frac{\partial}{\partial t} |\nu(t)\rangle = \sum_{\alpha} i f_{\alpha}(t) |\nu_\alpha\rangle$$

$$= \mathcal{H}_V |\nu(t)\rangle = \sum_{\beta} f_{\beta}(t) \mathcal{H}_V |\nu_\beta\rangle$$

$$= \sum_{\beta} f_{\beta}(t) \sum_{\alpha} |\nu_\alpha\rangle \langle \nu_\alpha | \mathcal{H}_V |\nu_\beta\rangle$$

$$= \sum_{\alpha} \left[ \sum_{\beta} (\mathcal{H}_V)_{\alpha\beta} f_{\beta}(t) \right] |\nu_\alpha\rangle .$$

(41)
Here, $\langle H_V \rangle_{\alpha\beta} \equiv \langle \nu_\alpha | H_V | \nu_\beta \rangle$. Comparing the coefficients of $|\nu_\alpha\rangle$ at the beginning and end of Eq. (41), we clearly have

$$\dot{f}_\alpha(t) = H_V \left[ f_\alpha(t) \right] ,$$

where $H_V$ is now the $2 \times 2$ matrix with elements $\langle H_V \rangle_{\alpha\beta}$. The Schrödinger equation (41) is completely analogous to the familiar one for a spin-$1/2$ particle. The roles of the two spin states are now being played by the two flavor states.

Let us call the two neutrino mass eigenstates out of which $\nu_e$ and $\nu_\mu$ are made $\nu_1$ and $\nu_2$. To find the matrix $H_V$, let us assume that our neutrino has a definite momentum $p$, so that its mass-eigenstate component $\nu_i$ has definite energy $E_i$ given by Eq. (48). That is, $H_V |\nu_i\rangle = E_i |\nu_i\rangle$, and the different mass eigenstates $|\nu_i\rangle$, like the eigenstates of any Hermitean Hamiltonian, are orthogonal to each other. Then, in view of Eq. (60), the elements $\langle H_V \rangle_{\alpha\beta}$ of the vacuum Hamiltonian are given by

$$\langle H_V \rangle_{\alpha\beta} = \langle \nu_\alpha | H_V | \nu_\beta \rangle = \sum_i U_{\alpha i}^* U_{\beta i} E_i .$$

In this expression, the two-neutrino mixing matrix $U$ may be taken from Eq. (32). However, we have seen that the complex phase factors in Eq. (32) have no effect on the two-neutrino oscillation probabilities, Eq. (33). Indeed, it is not hard to show that when there are only two neutrinos, complex phases in $U$ have no effect whatsoever on neutrino oscillation. Thus, since oscillation is our only concern here, we may remove the complex phase factors from the $U$ of Eq. (32). If, in addition, we relabel the mixing angle $\theta_V$ (short for $\theta_{\text{Vacuum}}$), $U$ becomes

$$U = \begin{bmatrix} \cos \theta_V & \sin \theta_V \\ -\sin \theta_V & \cos \theta_V \end{bmatrix} .$$

Inserting in Eq. (43) the elements $U_{\alpha i}$ of this matrix and the energies $E_i$ given by Eq. (18), we can obtain all the $\langle H_V \rangle_{\alpha\beta}$.

The matrix $H_V$ can be put into a more symmetric and convenient form if we add to it a suitably chosen multiple $(\Delta E) I$ of the identity matrix $I$. Such an addition will not change the predictions of $H_V$ for neutrino oscillation. To see why, we note first that the identity matrix is invariant under the unitary transformation that diagonalizes $H_V$. Thus, adding $(\Delta E) I$ to $H_V$ in the flavor basis, where its elements are $\langle H_V \rangle_{\alpha\beta}$, is equivalent to adding $(\Delta E) I$ to $H_V$ in the mass eigenstate basis, where it is diagonal. Hence, if the eigenvalues of $H_V$ are $E_i, i = 1, 2$, those of $H_V + (\Delta E) I$ are $E_i + \Delta E, i = 1, 2$. That is, both eigenvalues are displaced by the same amount, $\Delta E$. To see that such a
common shift of all eigenvalues does not affect neutrino oscillation, suppose a neutrino is born at time $t=0$ with flavor $\alpha$. That is, $|\nu(0)\rangle = |\nu_\alpha\rangle$. After a time $t$, this neutrino will have evolved into the state $|\nu(t)\rangle$ given, according to Schrödinger’s equation, by

$$|\nu(t)\rangle = e^{-iH_V t} |\nu(0)\rangle = e^{-iH_V t} \sum_{i} U_{\alpha i}^* |\nu_i\rangle = \sum_{i} U_{\alpha i}^* e^{-iE_i t} |\nu_i\rangle .$$

(45)

The amplitude $A(\nu_\alpha \rightarrow \nu_\beta)$ for this neutrino to have oscillated into a $\nu_\beta$ in the time $t$ is then given by

$$A(\nu_\alpha \rightarrow \nu_\beta) = \langle \nu_\beta | \nu(t) \rangle = \sum_{i} U_{\alpha i}^* e^{-iE_i t} U_{\beta i} .$$

(46)

Clearly, if we add $(\Delta E) I$ to $H_V$ so that its eigenvalues $E_i$ are replaced by $E_i + \Delta E$, then $A(\nu_\alpha \rightarrow \nu_\beta)$ is just multiplied by the overall phase factor $\exp[-i(\Delta E)t]$. Obviously, this phase factor has no effect on the oscillation probability $P(\nu_\alpha \rightarrow \nu_\beta) = |A(\nu_\alpha \rightarrow \nu_\beta)|^2$. Thus, the addition of $(\Delta E) I$ to $H_V$ does not affect neutrino oscillation.

For our purposes, the most convenient choice of $\Delta E$ is $-\left[p + (M_1^2 + M_2^2)/2\right]$. Then, from Eq. (43), the new effective Hamiltonian $H'_V \equiv H_V + (\Delta E) I$ has the matrix elements

$$(H'_V)_{\alpha \beta} = \sum_{i} U_{\alpha i} U_{\beta i}^* E_i - \left[p + \frac{M_1^2 + M_2^2}{2p}\right] \delta_{\alpha \beta} .$$

(47)

From Eqs. (45) and (47), this gives

$$H'_V = \frac{\delta M_2^2}{4E} \begin{bmatrix} -\cos 2\theta_V & \sin 2\theta_V \\ \sin 2\theta_V & \cos 2\theta_V \end{bmatrix} .$$

(48)

Here, we have used the fact that $p \equiv E$, the energy of the neutrino averaged over its two mass-eigenstate components. We leave to the reader the instructive exercise of verifying that, inserted into the Schrödinger Eq. (42), the $H'_V$ of Eq. (48) does indeed lead to the usual two-neutrino oscillation probability, Eq. (33).

With the Hamiltonian that governs neutrino propagation through the vacuum in hand, let us now ask how neutrino propagation is modified by the presence of matter. Matter, of course, consists of electrons and nucleons. When passing through a sea of electrons and nucleons, a (non-sterile) neutrino can undergo the forward elastic scatterings depicted in Fig. (5). Coherent fo-
Forward elastic scattering of a neutrino from a particle of matter. (a) W-exchange-induced scattering from an electron, which is possible only for a $\nu_e$. (b) Z-exchange-induced scattering from an electron, proton, or neutron. This is possible for $\nu_\alpha = \nu_e, \nu_\mu, \nu_\tau$. According to the Standard Model, the amplitude for this Z exchange is the same, for any given target particle, for all three active neutrino flavors.

ward scatterings, via the pictured processes, from many particles in a material medium will give rise to an interaction potential energy of the neutrino in the medium. Since one of the reactions in Fig. (5) can occur only for electron neutrinos, this interaction potential energy will depend on whether the neutrino is a $\nu_e$ or not. The interaction potential energy for a neutrino of flavor $\alpha$ must be added to the matrix element $\langle H' \rangle_{\alpha\alpha}$ to obtain the Hamiltonian for propagation of a neutrino in matter.

An important application of this physics is to the motion of solar neutrinos through solar material. The solar neutrinos are produced in the center of the sun by nuclear reactions such as $p + p \rightarrow ^2H + e^+ + \nu_e$. The neutrinos produced by these reactions are all electron neutrinos. Let us suppose that the only neutrinos with which electron neutrinos mix appreciably are muon neutrinos, so that we have a two-neutrino system of the kind we have just been discussing. The solar neutrinos stream outward from the center of the sun in all directions, some of them eventually arriving at solar neutrino detectors here on earth. The passage of these solar neutrinos through solar material on their way out of the sun modifies their oscillation. Any neutrino which is still a $\nu_e$, as it was at birth, can interact with solar electrons via the W exchange of Fig. (5a). This interaction leads to an interaction potential energy $V_W(\nu_e)$ of an electron neutrino in the sun. This $V_W(\nu_e)$ is obviously proportional to the Fermi coupling constant $G_F$, which governs the amplitude for the process in Fig. (5a). It is also proportional to the number of electrons per unit volume,
$N_e$, at the location of the neutrino, since $N_e$ measures the number of electrons which can contribute coherently to the forward $\nu_e$ scattering. One can show that in the Standard Model,

$$V_W(\nu_e) = \sqrt{2} G_F N_e.$$  \hspace{1cm} (49)

This energy must be added to $(H'_V)_{ee}$ to obtain the Hamiltonian for propagation of a neutrino in the sun.

In principle, the interaction energy produced by the $Z$ exchanges of Fig. (5b) must also be added to $H'_V$. However, since these $Z$ exchanges are both flavor diagonal and flavor independent, their contribution to the Hamiltonian is a multiple of the identity matrix. As we have already seen, a contribution of this character does not affect neutrino oscillation. Thus, we may safely ignore it.

In incorporating $V_W(\nu_e)$ into the Hamiltonian, it is convenient, in the interest of symmetry, to add as well the multiple $-\frac{1}{2} V_W(\nu_e) I$ of the identity matrix. Thus, with $H'_V$, the Hamiltonian for propagation in vacuum given by Eq. (48), the Hamiltonian $H_\odot$ for propagation in the sun is given by

$$H_\odot = H'_V + \frac{G_F}{\sqrt{2}} N_e \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{\delta M^2_{21} D_\odot}{4E} \begin{bmatrix} -\cos 2\theta_\odot & \sin 2\theta_\odot \\ \sin 2\theta_\odot & \cos 2\theta_\odot \end{bmatrix}.$$  \hspace{1cm} (50)

In this expression,

$$D_\odot = \sqrt{\sin^2 2\theta_\odot + (\cos 2\theta_\odot - x_\odot)^2},$$  \hspace{1cm} (51)

and

$$\sin^2 2\theta_\odot = \frac{\sin^2 2\theta_V}{\sin^2 2\theta_V + (\cos 2\theta_V - x_\odot)^2},$$  \hspace{1cm} (52)

where

$$x_\odot = \frac{2\sqrt{2} G_F N_e E}{\delta M^2_{21}}.$$  \hspace{1cm} (53)

The angle $\theta_\odot$ is the effective neutrino mixing angle in the sun when the electron density is $N_e$.

We note that $H_\odot$, Eq. (50), has precisely the same form as $H'_V$, Eq. (48). The only difference between these two Hamiltonians is that the parameters—the mixing angle and the effective neutrino $(\text{Mass})^2$ splitting out in front of the matrix—have different values. Of course, the electron density $N_e$ is not a constant, but depends on the distance $r$ from the center of the sun. Thus,
the parameters $\theta_\odot$ and $(\Delta M^2_{31})_{\odot}$ are not constant either, unlike their counterparts, $\theta_V$ and $\Delta M^2_{12}$, in $H'_V$. However, let us imagine for a moment that $N_e$ is a constant. Then $H_\odot$, like $H'_V$, is independent of position, and must lead to the same oscillation probability, Eq. (53), as $H'_V$ does, except for the substitutions $\theta \rightarrow \theta_\odot$ and $\Delta M^2_{31} \rightarrow (\Delta M^2_{31})_{\odot}$. That is, in matter of constant electron density $N_e$, $H_\odot$ leads to the oscillation probability

$$P(\nu_e \rightarrow \nu_\mu) = P(\nu_\mu \rightarrow \nu_e) = \sin^2 2\theta_\odot \sin^2 [1.27 \Delta M^2_{31}(eV^2) D/\text{E(km)}] \cdot$$

(54)

Now let $N_e$ vary with $r$ as it does in the real world. However, suppose that it varies slowly enough that the constant-$N_e$ picture we have just painted applies at any given radius $r$, but with $N_e(r)$, hence $x_\odot(r)$, slowly decreasing as $r$ increases. Suppose also that $\Delta M^2_{31}$ and $E$ are such that $x_\odot(r = 0) > \cos 2\theta_V$. Then, assuming that $\cos 2\theta_V > 0$, there must be a radius $r = r_c$ somewhere between $r = 0$ and the outer edge of the sun, where $N_e \rightarrow 0$, such that $x_\odot(r_c) = \cos 2\theta_V$. From Eq. (54), we see that at this special radius $r_c$, there is a kind of “resonance” with $\sin^2 2\theta_\odot = 1$, even if $\theta_V$ is tiny. That is, mixing can be maximal in the sun even if it is very small in vacuo. As a result, the oscillation probability, which is proportional to the mixing factor $\sin^2 2\theta_\odot$ as we see in Eq. (54), can be very large.

A nice picture of this enhanced probability for flavor transitions in matter can be gained by considering the neutrino energy eigenvalues and eigenvectors. If we neglect the inconsequential $Z$ exchange contribution, and take $H_\odot$ from Eq. (55), then the true Hamiltonian for propagation in the sun is

$$H_{\text{true}} = H_\odot + [p + M_1^2 + M_2^2/4p + 1/2V_W(\nu_e)] \cdot$$

(55)

since the second term in this expression was subtracted from the true Hamiltonian to get $H_\odot$. Now, $p + (M_1^2 + M_2^2/4p) \cong E$, the energy our neutrino would have in vacuum, averaged over its two mass-eigenstate components. Thus, from Eqs. (55) and (60), (48), (49), and (53), we have in the $\nu_e - \nu_\mu$ basis

$$H_{\text{true}}(r) = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} + \frac{\Delta M^2_{31}}{4E} \begin{bmatrix} -\cos 2\theta_V + 2x_\odot(r)/\sin 2\theta_V & \sin 2\theta_V \\ \sin 2\theta_V & \cos 2\theta_V \end{bmatrix}. \quad (56)$$

If we continue to assume that $N_e(r)$, and consequently $x_\odot(r)$, varies slowly, we may diagonalize this Hamiltonian for one $r$ at a time to see how a solar neutrino will behave. We find from Eq. (56) that for a given $r$, the energy
eigenvalues \( E_\pm(r) \) are given by

\[
E_\pm(r) = E + \frac{\Delta M_{21}^2}{4E} \left[ x_\odot(r) \pm \sqrt{(x_\odot(r) - \cos 2\theta_V)^2 + \sin^2 2\theta_V} \right].
\]  

(57)

To explore the implications of these energy levels, let us suppose that \( \Delta M_{21}^2 \) is such that at \( r = 0 \), where \( N_e(r) \) and hence \( x_\odot(r) \) has its maximum value, \( x_\odot \gtrsim 1 \). From Eq. (55), the value of \( G_F \), the value (\( \sim 10^{26}/\text{cc} \)) of \( N_e(r = 0) \), and the typical energy \( E (\sim 1 \text{ MeV}) \) of a solar neutrino, we find that the required \( \Delta M_{21}^2 \) is of order \( 10^{-5} \text{ eV}^2 \). The dominant term in the energies \( E_\pm(r) \) of Eq. (55) will be the first one, \( E (\sim 1 \text{ MeV}) \). However, very interesting physics will result from the second term, despite the fact that this term is only of order \( \Delta M_{21}^2/4E \sim (10^{-5}\text{eV}^2)/1\text{MeV} \sim 10^{-17} \text{ MeV}! \)

The neutrino states which propagate in the sun without mixing significantly with each other are the eigenvectors of \( \mathcal{H}_{\text{Dirac}}(r) \). To study these eigenvectors, let us assume for simplicity that the vacuum mixing angle \( \theta_V \) is small. Then it quickly follows that, except in the vicinity of the special radius \( r_c \) where \( x_\odot(r_c) = \cos 2\theta_V \), one of the eigenvectors of \( \mathcal{H}_{\text{Dirac}}(r) \), Eq. (56), is essentially pure \( \nu_e \), while the other is essentially pure \( \nu_\mu \). The evolution of a neutrino traveling outward through the sun is then as depicted in Fig. (56). The neutrino follows the trajectory indicated by the arrows. Produced by some nuclear process, the neutrino is born at small \( r \) as a \( \nu_e \). From Eq. (56), the eigenvector that is essentially \( \nu_e \) at \( r = 0 \) is the one with the higher energy, \( E_+(r) \). Thus, our neutrino begins its outward journey through the sun as the eigenvector belonging to the upper eigenvalue, \( E_+ \). Since the eigenvectors do not cross at any \( r \) and do not mix appreciably, the neutrino will remain this eigenvector. However, the flavor content of this eigenvector changes dramatically as the neutrino passes through the region near the radius \( r = r_c \) where \( x_\odot(r) = \cos 2\theta_V = 0 \). For \( r < r_c \), the eigenvector belonging to \( E_+(r) \) is essentially a \( \nu_e \), as one may see from Eq. (56) if one neglects the small off-diagonal terms proportional to \( \sin 2\theta_V \), and imposes the small-\( r \) condition \( x_\odot > \cos 2\theta_V \). But for \( r > r_c \), the eigenvector belonging to the higher energy level \( E_+(r) \) is essentially a \( \nu_\mu \), as one may also see from Eq. (56) if one neglects the off-diagonal terms and imposes the large-\( r \) condition \( x_\odot < \cos 2\theta_V \). Thus, the eigenvector corresponding to the higher energy eigenvalue \( E_+(r) \), along which the solar neutrino travels, starts out as a \( \nu_e \) at the center of the sun, but ends up as a \( \nu_\mu \) at the outer edge of the sun. The solar neutrino, born a \( \nu_e \) in the solar core, emerges from the rim of the sun as a \( \nu_\mu \). Furthermore, it does this with high probability even if the vacuum mixing angle \( \theta_V \) is very small, so that oscillation in vacuum would not have much of an effect. This very efficient conversion of solar electron neutrinos into neutrinos of another flavor as a result of interaction with matter
is known as the Mikheyev-Smirnov-Wolfenstein (MSW) effect.

We turn now from solar neutrinos to atmospheric neutrinos, whose propagation through the earth entails a second important application of the physics of neutrinos traveling through matter. As already mentioned, an atmospheric neutrino can be produced in the atmosphere on one side of the earth, and then journey through the whole earth to be detected in a detector on the other side. While traveling through the earth, this neutrino will undergo interactions that can significantly modify its oscillation pattern.

There is very strong evidence that the atmospheric neutrinos born as muon neutrinos oscillate into neutrinos $\nu_e$ of another flavor. It is known that $\nu_e$ is not a $\nu_\tau$. It could be a $\nu_\mu$, or a sterile neutrino $\nu_s$, or sometimes one of these and sometimes the other. One way to find out what is becoming of the oscillating muon neutrinos is to see whether their oscillation is affected by their passage through earth-matter. The oscillation $\nu_\mu \rightarrow \nu_s$ will be affected, but $\nu_\mu \rightarrow \nu_\tau$ will not be.

To see why this is so, let us first assume that the oscillation is $\nu_\mu \rightarrow \nu_\tau$. 

Figure 6: Propagation of a neutrino in the sun. The horizontal axis is linear in $N_e$ and $x_\odot$. The solid line shows the upper eigenvalue, $E_{+}(r)$, for small $\delta m^2$. The dashed lines show the two eigenvalues, $E_{\pm}(r)$, when there is no vacuum mixing ($\theta_{13} = 0$). A neutrino born as a $\nu_e$ follows the path indicated by the arrows.
Then the Hamiltonian $\mathcal{H}_E$ (short for $\mathcal{H}_{\text{Earth}}$) that describes neutrino propagation in the earth is a $2 \times 2$ matrix in $\nu_\mu - \nu_\tau$ space. Now, either a $\nu_\mu$ or a $\nu_\tau$ will interact with earth-matter via the $Z$ exchange of Fig. (5b). This interaction will give rise to an interaction potential energy of the neutrino in the earth. However, according to the Standard Model, the $Z$-exchange amplitude is the same for a $\nu_\tau$ as it is for a $\nu_\mu$. Thus, in the case of $\nu_\mu \to \nu_\tau$ oscillation, the contribution of neutrino-matter interaction to $\mathcal{H}_E$ is a multiple of the identity matrix. As we have already seen, such a contribution has no effect on oscillation.

Now, suppose the oscillation is not $\nu_\mu \to \nu_\tau$ but $\nu_\mu \to \nu_s$. Then $\mathcal{H}_E$ is a $2 \times 2$ matrix in $\nu_\mu - \nu_s$ space. A $\nu_\mu$ will interact with earth-matter, as we have discussed, but a $\nu_s$, of course, will not. Thus, the contribution of neutrino-matter interaction to $\mathcal{H}_E$ is now of the form

$$
\begin{pmatrix}
\nu_\mu & \nu_s \\
\nu_s & V_{Z}(\nu_\mu) \quad 0 \\
0 & 0
\end{pmatrix}
$$

where $V_{Z}(\nu_\mu)$ is the interaction potential energy of muon neutrinos produced by the $Z$ exchange of Fig. (5b). Since the matrix (58) is not a multiple of the identity, neutrino-matter interaction does affect $\nu_\mu \to \nu_s$ oscillation.

According to the SM, the forward $Z$-exchange amplitudes for a target $e$ and a target $p$ are equal and opposite. Thus, assuming that the earth is electrically neutral so that it contains an equal number of electrons and protons per unit volume, the $e$ and $p$ contributions to $V_{Z}(\nu_\mu)$ cancel. Then $V_{Z}(\nu_\mu)$ is proportional to the neutron number density, $N_n$. Taking the proportionality constant from the SM, we have

$$V_{Z}(\nu_\mu) = - \frac{G_F}{\sqrt{2}} N_n .$$

To obtain the Hamiltonian $\mathcal{H}_E$ for $\nu_\mu \to \nu_s$ oscillation in the earth, we add to the vacuum Hamiltonian $\mathcal{H}_V'$ of Eq. (48) the contribution (58) from matter interactions, using Eq. (59) for $V_{Z}(\nu_\mu)$. Of course, it must be understood that $\mathcal{H}_V'$ is now to be taken as a matrix in $\nu_\mu - \nu_s$ space, and that the vacuum (Mass)$^2$ splitting $\delta M_{21}^2$ and mixing angle $\theta_V$ in $\mathcal{H}_V'$ are now different parameters than they were when we obtained from $\mathcal{H}_V'$ the Hamiltonian $\mathcal{H}_V$ for neutrino propagation in the sun. The quantities $\delta M_{21}^2$ and $\theta_V$ are now new parameters appropriate to the vacuum oscillation of atmospheric, rather than solar, neutrinos. To obtain a more symmetrical and convenient $\mathcal{H}_E$ from $\mathcal{H}_V'$, we also add $-\frac{1}{2} V_{Z}(\nu_\mu) I$, a multiple of the identity which will not affect the
implications of $H_E$ for oscillation. The result is

$$H_E = \frac{\delta M^2_{21}}{4E} \begin{pmatrix} -\cos 2\theta_E & \sin 2\theta_E \\ \sin 2\theta_E & \cos 2\theta_E \end{pmatrix}.$$  \quad (60)

Here,

$$D_E = \sqrt{\sin^2 2\theta_V + (\cos 2\theta_V - x_E)^2},$$  \quad (61)

and

$$\sin^2 2\theta_E = \frac{\sin^2 2\theta_V}{\sin^2 2\theta_V + (\cos 2\theta_V - x_E)^2},$$  \quad (62)

where

$$x_E = -\frac{\sqrt{2}G_F N_n E}{\delta M^2_{21}}.$$

As a rough approximation, we may take the neutron density $N_n$ to be constant throughout the earth. Then, $x_E$ is also a constant, and the Hamiltonian $H_E$ of Eq. (60) is identical to the vacuum Hamiltonian $H_V$ of Eq. (48), except that the constant $\delta M^2_{21}$ is replaced by the constant $(\delta M^2_{21}) D_E$, and the constant $\theta_V$ by the constant $\theta_E$. Thus, from the fact that $H'_V$ leads to the vacuum oscillation probability of Eq. (59) (with $\nu_e \rightarrow \nu_\mu$ replaced by $\nu_\mu \rightarrow \nu_s$ for the present application), we immediately conclude that the $H_E$ of Eq. (60) leads to the oscillation probability

$$P(\nu_\mu \rightarrow \nu_s) = \sin^2 2\theta_E \sin^2 [1.27 \delta M^2_{21}(\text{eV}^2) D_E \frac{L[\text{km}]}{E[\text{GeV}]^2}].$$  \quad (64)

As we see from Eq. (63), $x_E$, which is a measure of the influence of matter effects on atmospheric neutrinos, grows with energy $E$. In a moment we will see that for $E \sim 1\text{ GeV}$, matter effects are negligible. Fits to data on atmospheric neutrinos with roughly this energy have led to the conclusion that atmospheric neutrino oscillation involves a neutrino (Mass)$^2$ splitting $\delta M^2_{\text{Atmos}}$ given by

$$\delta M^2_{\text{Atmos}} \sim 3 \times 10^{-3}\text{eV}^2,$$

and a neutrino mixing angle $\theta_{\text{Atmos}}$ given by

$$\sin^2 2\theta_{\text{Atmos}} \sim 1.$$  \quad (66)

That is, the mixing when matter effects are negligible is very large, and perhaps maximal. The quantities $\delta M^2_{\text{Atmos}}$ and $\theta_{\text{Atmos}}$ are to be taken, respectively, for $\delta M^2_{21}$ and $\theta_V$ in Eqs. (60) - (64) to find the implications of these equations for $\nu_\mu \rightarrow \nu_s$ within the earth.
Since atmospheric neutrino oscillation involves maximal mixing when matter effects are negligible, the matter effects cannot possibly enhance the oscillation, but can only suppress it. From Eq. (62), we see that if, as observed, \( \sin^2 2\theta_V \approx 1 \), then matter effects will lead to a smaller effective mixing \( \sin^2 2\theta_E \) in the earth, given by

\[
\sin^2 2\theta_E = \frac{1}{1 + x_E^2}.
\]  

As is clear in Eq. (64), this will result in a smaller oscillation probability \( P(\nu_\mu \to \nu_e) \) than one would have in vacuum, where \( \sin^2 2\theta_V \) is replaced by \( \sin^2 2\theta_E \) (\( \sim 1 \)).

Since \( x_E \) grows with energy, the degree to which matter effects suppress \( \nu_\mu \to \nu_e \) grows as well. From Eq. (63) for \( x_E \), the known values of \( G_F \) and \( N_n \), and the value (65) required for \( \delta M^2_{12} \) by the data, we find that \( x_E \ll 1 \) when \( E \sim 1 \) GeV. Thus, matter effects are indeed negligible at this energy, so it is legitimate to determine the vacuum parameters \( \delta M^2_{\text{Atmos}} \) and \( \sin^2 2\theta_{\text{Atmos}} \) by analyzing the \( \sim 1 \) GeV data neglecting matter effects. However, at sufficiently large \( E \), the matter-induced suppression of \( \nu_\mu \to \nu_e \) will obviously be significant. From Eq. (64), we find that \( \sin^2 2\theta_E \) is below 1/2 when \( E \gtrsim 20 \) GeV. The consequent suppression of oscillation at these energies has been looked for, and is not seen.\(^\text{13}\) This absence of suppression is a powerful part of the evidence that the neutrinos into which the atmospheric muon neutrinos oscillate are not sterile neutrinos, or at least not solely sterile neutrinos.\(^\text{12}\)

4 Conclusion

Evidence has been reported that the solar neutrinos, the atmospheric neutrinos, the accelerator-generated neutrinos studied by the Liquid Scintillator Neutrino Detector (LSND) experiment at Los Alamos, and the accelerator-generated neutrinos studied by the K2K experiment in Japan, actually do oscillate. Some of this evidence is very strong. The neutrino oscillation experiments, present and future, are discussed in this Volume by John Wilkerson.\(^\text{12}\)

In these lectures, we have tried to explain the basic physics that underlies neutrino oscillation, and that is invoked to understand the oscillation experiments. As we have seen, neutrino oscillation implies neutrino mass and mixing. Thus, given the compelling evidence that at least some neutrinos do oscillate, we now know that neutrinos almost certainly have nonzero masses and mix. This knowledge raises a number of questions about the neutrinos:

- How many neutrino flavors, including both interacting and possible sterile flavors, are there? Equivalently, how many neutrino mass eigenstates are there?
• What are the masses, $M_i$, of the mass eigenstates $\nu_i$?

• Is the antiparticle $\overline{\nu}_i$ of a given mass eigenstate $\nu_i$ the same particle as $\nu_i$, or a different particle?

• What are the sizes and phases of the elements $U_{\alpha i}$ of the leptonic mixing matrix? Equivalently, what are the mixing angles and complex phase factors in terms of which $U$ may be described? Do complex phase factors in $U$ lead to CP violation in neutrino behavior?

• What are the electromagnetic properties of neutrinos? In particular, what are their dipole moments?

• What are the lifetimes of the neutrinos? Into what do they decay?

• What is the physics that gives rise to the masses, the mixings, and the other properties of the neutrinos?

Seeking the answers to these and other questions about the neutrinos will be an exciting adventure for years to come.

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