CAYLEY-KLEIN LIE ALGEBRAS AND THEIR QUANTUM UNIVERSAL ENVELOPING ALGEBRAS

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ABSTRACT. The N-dimensional Cayley-Klein scheme allows the simultaneous description of \(3^N\) geometries (symmetric orthogonal homogeneous spaces) by means of a set of Lie algebras depending on \(N\) real parameters. We present here a quantum deformation of the Lie algebras generating the groups of motion of the two and three dimensional Cayley-Klein geometries. This deformation (Hopf algebra structure) is presented in a compact form by using a formalism developed for the case of (quasi)free Lie algebras. Their quasi-triangularity (i.e., the most usual way to study the associativity of their dual objects, the quantum groups) is also discussed.

1. Introduction

We study here a certain type of Lie algebra deformations (so called “quantum” ones), that have recently appeared in the context of the Quantum Inverse Scattering Method. They are properly defined as deformations of the corresponding universal enveloping algebra \(U_g\) and their dual objects (in certain restricted sense) generate the “quantum groups”–deformations of the algebra of functions on the group, in the spirit of non-commutative geometry \([1,2]\). Their underlying algebraic structure (mainly Hopf algebra properties \([3]\)) is rather rich and was soon described for the classical simple Lie algebras \([4,5,6]\).

However, many physically interesting groups are not simple groups: for instance, the groups of inertial transformations of space–time such as Galilei or Poincaré ones. Some quantum deformations have been built for their associated Lie algebras \([7,8]\) which also arise as symmetries of certain physical problems \([9]\). We present here an attempt towards their characterization based on a Cayley-Klein (CK) geometrical scheme that includes all these groups as well as their transformed by \(\text{Inönü–Wigner}\) contractions \([10]\). We also discuss the problems arising in the definition of the quantum groups as dual objects of these quantum algebras, mainly in connection with the way in which the associativity of the deformed algebra of functions on the group is guaranteed (the \(R\)-matrix problem).

2. The Cayley-Klein Lie algebras

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From a physical point of view, some interesting homogeneous symmetric spaces can be simultaneously described in the framework of CK geometries. To do this, we consider an N-dimensional (N-d) symmetric orthogonal geometry as a group $G$ of dimension $\frac{1}{2}N(N + 1)$ and a set of N commuting involutions $S^{(i)}$ in $g$, the Lie algebra of $G$. If we denote by $h^{(i)}$ the Lie subalgebras of elements invariant under $S^{(i)}$, their corresponding groups $H^{(i)}$ have to be taken as the isotropy groups of a point, a line, ... a $(N - 1)$-flat, so the homogeneous spaces $X^{(i)} \equiv G/H^{(i)}$ turn out to be the spaces of points, lines, ... of the geometry. The involutions must satisfy certain requirements (specially on the dimensions of the subgroups $H^{(i)}$) and the group $G$ is also required to act effectively on all the $X^{(i)}$. Without entering into details, the main result is that the CK Lie algebra depends on $N$ real parameters $(\kappa_1, \ldots, \kappa_N)$ that can take any real value (however, these parameters can be scaled to $-1$, 0 or $+1$, and we have [11].

Let $G$ be the CK Lie group corresponding to an N-d CK geometry.

a) The Lie algebra of $G$ has dimension $\frac{1}{2}N(N + 1)$ with generators $J_{ij}$ ($i < j$; $i, j = 0, 1, \ldots, N$) and is characterized by $N$ real parameters $(\kappa_1, \ldots, \kappa_N)$.

b) The isotopy subgroups of i-flats $H^{(i)}$ ($i = 0, 1, \ldots, N - 1$), correspond to the subalgebras $h^{(i)} = \{J_{ab}, J_{cd} \}$ ($a < b, a, b = 0, 1, \ldots, i; c < d; c, d = i + 1, \ldots, N$).

c) The Lie brackets of the basis elements $J_{ij}$ can be written in terms of the parameters $\kappa_{ij}$ ($i < j$; $i, j = 0, 1, \ldots, N$) defined by

$$[J_{ij}, J_{lm}] = \delta_{im}J_{lj} - \delta_{jl}J_{im} + \delta_{jm}\kappa_{lm}J_{il} + \delta_{il}\kappa_{jm}J_{jm}, \quad (i \leq l, \ j \leq m).$$  \hspace{1cm} (2.1)

d) The CK algebras $g(\kappa_1, \ldots, \kappa_N)$ can be realized in terms of $(N + 1) \times (N + 1)$ real matrices:

$$D(J_{ij}) = -\kappa_{ij}e_{ij} + e_{ji},$$  \hspace{1cm} (2.2)

where $e_{ij}$ are the standard $(N + 1) \times (N + 1)$ matrices with elements $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and commutation relations

$$[e_{ij}, e_{lm}] = \delta_{jl}e_{im} - \delta_{im}e_{lj}.$$  \hspace{1cm} (2.3)

A representation of the one-parameter subgroups associated to the generators $J_{ij}$ is obtained by exponentiation of the matrices $D(J_{ij})$. The matrix entries of these subgroup elements are easily written by using the “generalized” trigonometric functions $\sin_{\kappa}(x) \equiv S_{\kappa}(x)$ and

$$\cos_{\kappa}(x) \equiv C_{\kappa}(x),$$

defined in terms of power series as follows:

$$S_{\kappa}(x) = \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l+1}}{(2l+1)!}, \quad C_{\kappa}(x) = \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l}}{(2l)!}.$$  \hspace{1cm} (2.4)

These functions constitute an intrinsic tool throughout any explicit computation within the CK scheme (either classical or quantum), and can be considered as
deformations of the “Galilean” or “parabolic” functions 1 and x, to which they tend in the limit $\kappa \to 0$:

\[
C_\kappa(x) = \begin{cases} 
\cos \sqrt{\kappa}x & \text{if } \kappa > 0 \\
1 & \text{if } \kappa = 0 \\
\cosh \sqrt{-\kappa}x & \text{if } \kappa < 0
\end{cases}
\]

\[
S_\kappa(x) = \begin{cases} 
\frac{1}{\sqrt{-\kappa}} \sin \sqrt{-\kappa}x & \text{if } \kappa > 0 \\
x & \text{if } \kappa = 0 \\
\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa}x & \text{if } \kappa < 0
\end{cases}
\]

(2.5)
3. Quantum Universal Enveloping algebras

Following Drinfel’d [4], a “quantized universal enveloping algebra” (QUE algebra) of a Lie algebra \( \mathfrak{g} \) is a Hopf algebra \( \mathcal{A} \) over the formal power series \( \mathbb{C}[z] \) on a deformation indeterminate \( z \) such that \( \mathcal{A} \) is topologically free \( \mathbb{C}[z] \)-module and \( \mathcal{A}/z\mathcal{A} \) is isomorphic (as Hopf algebra) to \( U\mathfrak{g} \).

Recall that a \( \mathbb{C} \)-algebra is a Hopf algebra if there exist two homomorphisms called coproduct \( (\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}) \) and counit \( (\epsilon: \mathcal{A} \to \mathbb{C}) \), as well as an antihomomorphism (the antipode \( \gamma: \mathcal{A} \to \mathcal{A} \)) such that, \( \forall a \in \mathcal{A} \):

\[
(id \otimes \Delta)(a) = (\Delta \otimes id)(a), \tag{3.1}
\]

\[
(id \circ \epsilon)(a) = (\epsilon \circ id)(a) = a, \tag{3.2}
\]

\[
m((id \otimes \gamma)(\Delta(a))) = m((\gamma \otimes id)(\Delta(a))) = \epsilon(a)1, \tag{3.3}
\]

where \( m \) is the usual multiplication \( m(a \otimes b) = ab \). Counit and antipode are derived from the coproduct in a unique way.

Roughly speaking, we can think of the elements in \( \mathcal{A} \) as formal power series in \( z \) with coefficients in \( U\mathfrak{g} \). Provided we are not going to take into account topological properties, the topological freeness is translated into algebraic terms by imposing that the QUE algebra must be free, or at least torsion-free as a \( \mathbb{C}[z] \)-module [12]. Both the Hopf homomorphisms and the torsion condition restrict the number of formal power series suitable for quantization [13].

The “classical limit” property \( \mathcal{A}/z\mathcal{A} \simeq U\mathfrak{g} \) is also a very strong constrain on the quantization. Our aim is to obtain quantum CK algebras, so the \( z \to 0 \) limit must lead to the Lie brackets described in (2.1). The standard quantization of the classical Cartan series of simple Lie algebras uses the \( A_1 \) quantum structure as building block [4,5]. For the quantum CK algebras already obtained [14,15], the analogous generalization seems not to be immediate. However, CK geometries of a given dimension do contain lower dimension subgeometries. Any compact way of writing of the Hopf homomorphisms embodying somehow this embedding would help to obtain the general structure. In this sense, the following proposition [16] ensures the consistency of the deformation for a “quasi”-free algebra (the only condition is the commutativity of the primitive generators) and will be used to write the quantum CK algebras in the next section.

**Proposition 3.1.** Let \( \{1, H_1, \ldots, H_n, X_1, \ldots, X_m\} \) be the generators of a “free” (up to the conditions \( [H_i, H_j] = 0 \) \( \forall i, j \)) associative algebra \( E \) over \( \mathbb{C} \). Let \( \alpha_i, \beta_j \) \( i,j = 1, \ldots, n \) be a set of \( (m \times m) \) matrices with entries in \( \mathbb{C}[z] \) such that \( [\alpha_i, \beta_j] = [\alpha_i, \alpha_j] = [\beta_i, \beta_j] = 0 \) \( \forall i, j \). Let \( \tilde{X} \) be a “vector” with components \( X_l, l = 1, \ldots, m \). The coproduct

\[
\Delta 1 = 1 \otimes 1, \\
\Delta H_i = 1 \otimes H_i + H_i \otimes 1, \\
\Delta \tilde{X} = \exp(\sum_{i=1}^{n} \alpha_i H_i) \otimes \tilde{X} + \sigma \left( \exp(\sum_{i=1}^{n} \beta_i H_i) \otimes \tilde{X} \right), \tag{3.4}
\]


turns the completion $B$ of formal power series on $z$ with coefficients in $E$ into a Hopf algebra.

We have denoted by $\sigma$ the permutation map $\sigma(a \otimes b) = b \otimes a$ and, if $P \equiv (p_{kl})$ is a $(m \times m)$ matrix with entries in $B$, the $k$-th component of $(P \otimes \tilde{X})$ is defined as

$$ (P \otimes \tilde{X})_k = \sum_{l=1}^{m} p_{kl} \otimes X_l. $$(3.5)

4. QUE Cayley-Klein algebras

We restrict in this section to the study of the algebras generating the two and three dimensional CK systems. The former is a family of 3-d Lie algebra depending on two parameters which contains as particular cases $so(3), so(2,1)$, the 2-d euclidean $e(2)$ and the $(1+1)$ Newton-Hooke, Galilei and Poincaré algebras. A simultaneous quantization of all of these systems is given in the following theorem.

**Theorem 4.1.** Let $\mathfrak{g}_{(\kappa_1, \kappa_2)}$ be the Lie algebra generating the 2-d CK systems and whose infinitesimal generators are $\{J_{12}, P_1, P_2\}$. The coproduct

$$ \Delta P_2 = 1 \otimes P_2 + P_2 \otimes 1, $$

$$ \Delta \left( \begin{array}{c} P_1 \\ J_{12} \end{array} \right) = \exp \left\{ \left( \begin{array}{cc} -\frac{\kappa_2}{2} P_2 & 0 \\ 0 & -\frac{\kappa_1}{2} P_2 \end{array} \right) \right\} \otimes \left( \begin{array}{c} P_1 \\ J_{12} \end{array} \right) + \sigma \left( \exp \left\{ \left( \begin{array}{cc} -\frac{\kappa_2}{2} P_2 & 0 \\ 0 & -\frac{\kappa_1}{2} P_2 \end{array} \right) \right\} \otimes \left( \begin{array}{c} P_1 \\ J_{12} \end{array} \right) \right), $$

and the commutation relations

$$ [J_{12}, P_1] = S_{-z^2} (P_2), \quad [J_{12}, P_2] = -\kappa_2 P_1, \quad [P_1, P_2] = \kappa_1 J_{12}. $$

(4.1)

define the QUE algebra $U_z \mathfrak{g}_{(\kappa_1, \kappa_2)}$.

**Corollary 4.1.** Counit and antipode are deduced from (4.1) and, for $X \in \{P_1, P_2, J_{12}\}$, read

$$ \epsilon(X) = 0, \quad \gamma(X) = -e^{\frac{\kappa_2}{2} P_2} X e^{-\frac{\kappa_2}{2} P_2}. $$

(4.2)

**Corollary 4.2.** The center of $U_z \mathfrak{g}_{(\kappa_1, \kappa_2)}$ is generated by

$$ C_z = 4 \, C_{\kappa_1 \kappa_2} \left( \frac{z}{2} \right) \left[ S_{-z^2} (\frac{z}{2} P_2) \right]^2 + \frac{\kappa_2}{2} S_{\kappa_1 \kappa_2} \left( \frac{z}{2} \right) \{ \kappa_2 P_1^2 + \kappa_1 J_{12}^2 \}. $$

(4.3)

**Proposition 4.1.** The fundamental representation $D_q$ of $U_z \mathfrak{g}_{(\kappa_1, \kappa_2)}$ in terms of the “classical” one $D$ is defined as follows

$$ D_q(P_2) = D(P_2), \quad D_q(P_1) = \sqrt{\frac{S_{\kappa_1 \kappa_2} (z)}{z}} D(P_1), \quad D_q(J_{12}) = \sqrt{\frac{S_{\kappa_1 \kappa_2} (z)}{z}} D(J_{12}). $$

(4.4)

Note that the “generalized trigonometric functions” appear as natural deformation functions in this context. Moreover, they are consistent formal power series in
the sense that, for instance, $\Delta(S_\kappa(P_2)) = S_\kappa(P_2) \otimes C_\kappa(P_2) + C_\kappa(P_2) \otimes S_\kappa(P_2)$. The classical limit $z \to 0$ is always well defined and straightforwardly leads to the algebra defined in (2.1). It is also worth remarking that different algebras are obtained by specialization of the $\kappa_i$ parameters. This deformation preserves always non-Lie character in (4.2), since the deformed bracket cannot vanish whatever the $\kappa_i$ are.

For the case of 3-d geometries, the CK Lie algebra is now 6-d and depends on three measure coefficients $(\kappa_1, \kappa_2, \kappa_3)$. The number of different geometries included is now $3^3$. The algebras $so(4), so(3,1), so(2,2)$, the 3-d Euclidean algebra $\epsilon(3)$, the $(2+1)$-d versions of the Newton-Hooke, Galilei and Poincaré algebras can be found among the set $\mathfrak{g}_{(\kappa_1, \kappa_2, \kappa_3)}$.

**Theorem 4.2.** Let $\mathfrak{g}_{(\kappa_1, \kappa_2, \kappa_3)}$ be the Lie algebra generating the 3-d CK systems. We denote the infinitesimal generators as $J_{ij}$, $(i < j; i, j = 0, 1, 2, 3)$. The coproduct

$$
\Delta(J_{03}) = 1 \otimes J_{03} + J_{03} \otimes 1,
\Delta(J_{12}) = 1 \otimes J_{12} + J_{12} \otimes 1,
$$

$$
\Delta \begin{pmatrix} J_{01} \\ J_{02} \\ J_{13} \\ J_{23} \end{pmatrix} = \exp \{\alpha_1 J_{03} + \alpha_2 J_{12}\} \otimes \begin{pmatrix} J_{01} \\ J_{02} \\ J_{13} \\ J_{23} \end{pmatrix} + \sigma \exp \{\beta_1 J_{03} + \beta_2 J_{12}\} \otimes \begin{pmatrix} J_{01} \\ J_{02} \\ J_{13} \\ J_{23} \end{pmatrix},
$$

$$
\alpha_1 = -\beta_1 = \begin{pmatrix} -\frac{\kappa_1}{2} & 0 & 0 & 0 \\ 0 & -\frac{\kappa_2}{2} & 0 & 0 \\ 0 & 0 & -\frac{\kappa_3}{2} & 0 \\ 0 & 0 & 0 & -\frac{\kappa_3}{2} \end{pmatrix}, \quad \alpha_2 = -\beta_2 = \begin{pmatrix} 0 & 0 & 0 & -\frac{\kappa_1}{2} \\ 0 & 0 & \frac{\kappa_2}{2} & 0 \\ 0 & \frac{\kappa_3}{2} & 0 & 0 \\ -\frac{\kappa_3}{2} & 0 & 0 & 0 \end{pmatrix},
$$

and the following non-vanishing commutation relations

$$
[J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -\kappa_2 J_{01}, \quad [J_{01}, J_{02}] = \kappa_1 S_{-z^2, \kappa_1, \kappa_3}(J_{12}) C_{-z^2}(J_{03}),
$$

$$
[J_{13}, J_{01}] = S_{-z^2}(J_{03}) C_{-z^2, \kappa_1, \kappa_3}(J_{12}), \quad [J_{13}, J_{03}] = -\kappa_2 S_{-z^2, \kappa_1, \kappa_3}(J_{12}), \quad [J_{01}, J_{03}] = \kappa_1 J_{13},
$$

$$
[J_{23}, J_{01}] = S_{-z^2}(J_{03}) C_{-z^2, \kappa_1, \kappa_3}(J_{12}), \quad [J_{23}, J_{03}] = -\kappa_3 S_{-z^2, \kappa_1, \kappa_3}(J_{12}), \quad [J_{02}, J_{03}] = \kappa_1 \kappa_2 J_{23},
$$

$$
[J_{23}, J_{12}] = J_{13}, \quad [J_{23}, J_{13}] = -\kappa_3 S_{-z^2, \kappa_1, \kappa_3}(J_{12}) C_{-z^2}(J_{03}), \quad [J_{12}, J_{13}] = \kappa_2 \kappa_3.
$$

(4.7)

define the QUE algebra $U_3 \mathfrak{g}_{(\kappa_1, \kappa_2, \kappa_3)}$.

**Corollary 4.3.** Count and antipode are

$$
\epsilon(J_{ij}) = 0, \quad \gamma(J_{ij}) = -\epsilon^z J_{03} J_{ij} e^{-z J_{03}}.
$$

(4.8)

**Corollary 4.4.** The following deformed second order elements belong to the center of $U_3 \mathfrak{g}_{(\kappa_1, \kappa_2, \kappa_3)}$

$$
C^q_1 = 4 C_{03}(z) \left[ S_{-z^2}(\frac{1}{2} J_{03}) C_{-z^2, \kappa_1, \kappa_3}(\frac{1}{2} J_{12}) + \kappa_1 \kappa_3 S_{-z^2, \kappa_1, \kappa_3}(\frac{1}{2} J_{12}) C_{-z^2}(\frac{1}{2} J_{03}) \right]
$$

$$
+ \frac{1}{2} S_{03}(z) \left[ \kappa_2 \kappa_3 J_{01} + \kappa_3 J_{02} + \kappa_1 J_{13} + \kappa_1 \kappa_2 J_{23} \right],
$$

$$
C^q_2 = C_{03}(z) S_{-z^2}(J_{03}) S_{-z^2, \kappa_1, \kappa_3}(J_{12}) + \frac{1}{2} S_{03}(z)[\kappa_2 J_{01} J_{23} - J_{02} J_{13}].
$$

(4.9)
\textbf{Proposition 4.2.} The fundamental representation $D_q$ of $U_q(\mathfrak{g}_{(\kappa_1, \kappa_2, \kappa_3)})$ is given by

$$
D_q(J_{ij}) = \sqrt{i} S_{\kappa_3}(z) D(J_{ij}), \quad \text{if } ij = 01, 02, 13, 23, \\
D_q(J_{ij}) = D(J_{ij}), \quad \text{if } ij = 03, 12. 
$$

(4.10)

5. Quasitriangular Hopf algebras

A quasitriangular Hopf algebra [4] is a pair $(A, R)$ where $A$ is a Hopf algebra and $R \in A \otimes A$ is invertible and obeys

$$
\sigma \circ \Delta h = R(\Delta h)R^{-1}, \quad \forall \ h \in A \\
(\Delta \otimes id)R = R_{12} R_{23}, \quad (id \otimes \Delta)R = R_{13} R_{12},
$$

(5.1)

where, if $R = \sum_i a_i \otimes b_i$, we denote $R_{12} = \sum_i a_i \otimes b_i \otimes 1$, $R_{13} = \sum_i a_i \otimes 1 \otimes b_i$ and $R_{23} = \sum_i 1 \otimes a_i \otimes b_i$. If $A$ is a quasitriangular Hopf algebra, then $R$ is called an “universal” $R$–matrix and satisfies the Quantum Yang–Baxter Equation:

$$
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
$$

(5.2)

Given a matrix representation $\rho : A \rightarrow Mat(n, \mathbb{C})$, the matrix elements $t_{ij}$ of its dual $A^*$ (the “quantum group”) satisfy the commutation relations

$$
RT_1 T_2 = T_2 T_1 R,
$$

(5.3)

where $R = (\rho \otimes \rho)(R)$ and $T = (t_{ij})$, $T_1 = T \otimes 1_n$ and $T_2 = 1_n \otimes T$. These matrix solutions $R$ are relevant for physical applications. If $R$ satisfies (5.2), this equation ensures that the non–commutative algebra generated by the $t_{ij}$ is associative (third order commutation relations are derived from second order ones). However, for non–semisimple quantum algebras (in our scheme, if some $\kappa_i = 0$), neither the universal $R$ matrices nor even some of their particular realizations are easy to find [7,17]. In fact, they could not exist.

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References