A COMMENT ON THE ODD FLOWS FOR
THE SUPERSYMMETRIC KdV EQUATION.

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ABSTRACT

In a recent paper Dargis and Mathieu introduced integrodifferential odd flows for the supersymmetric KdV equation. These flows are obtained from the nonlocal conservation laws associated with the fourth root of its Lax operator. In this note I show that only half of these flows are of the standard Lax form, while the remaining half provide us with hamiltonians for an SKdV-type reduction of a new supersymmetric hierarchy. This new hierarchy is shown to be closely related to the Jacobian supersymmetric KP-hierarchy of Mulase and Rabin. A detailed study of the algebra of additional symmetries of this new hierarchy reveals that it is isomorphic to the super-$\mathcal{W}_{1+\infty}$ algebra, thus making it a candidate for a possible interrelationship between superintegrability and two-dimensional supergravity.

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The history of the supersymmetric KdV equation, though short, has already been full of surprises. To the best of my knowledge, the first attempt to supersymmetrize the KdV equation was carried out by Kuperschmidt in [1]. He introduced an extra fermionic variable and was able to define an extended system which was formally integrable and bihamiltonian, but failed to be invariant under supersymmetry. The first fully supersymmetric KdV system—hereafter denoted by SKdV—was introduced by Manin and Radul in [2]. The SKdV equation is most naturally written in (1|1) superspace formalism\(^1\) as

$$\frac{\partial U}{\partial t} = \frac{1}{4} U^{[6]} + \frac{3}{4} (U U^4)''.$$ \hfill (1)

In full analogy with the bosonic case, the SKdV equation was naturally obtained as a reduction of a supersymmetrization of the KP hierarchy (SKP). But unfortunately the analogies stopped there. The Adler-Gel’fand-Dickey scheme which had proved so fruitful in providing us with hamiltonian structures for KP and its reductions of the KdV-type seemed, not to have an analogue for SKP. Nevertheless, some time later, Mathieu [3] showed that the SKdV is hamiltonian with respect to the supervirasoro algebra, \(i.e.\)

$$\{U(X), U(Y)\} = \left( \frac{1}{2} D^5 + \frac{3}{2} U(X) D^2 + \frac{1}{2} U'(X) D + U^4(X) \right) \cdot \delta(X - Y).$$ \hfill (2)

This made evident that the relationship between superintegrability and supersymmetric Gel’fand-Dickey algebras was anything but lost. This result notwithstanding, a bihamiltonian structure—a hallmark of integrability—was still lacking. Mathieu demonstrated that a naive supersymmetrization of the first hamiltonian structure of KdV failed to work and thus concluded that the SKdV equation was not bihamiltonian.

Such was the state of affairs until Oevel and Popowicz [4], and independently J.M. Figueroa-O’Farrill, J. Mas and the author [5], observed that the SKdV equation could be understood as a reduction of an even-order SKP-like operator, SKP\(_2\). This formalism had the important advantage that a direct application of the supersymmetric AGD scheme of [6] equipped the SKdV equation with a bihamiltonian structure—the new hamiltonian structure being a nonlocal deformation of the naive supersymmetrization of the first hamiltonian structure of the KdV equation. Nevertheless, this approach still suffered from two important drawbacks. First of all, the reduction procedure was realized by an explicit computation using Dirac brackets, which did not seem to be easily generalizable to higher order SKdV-type reductions. In fact, the emergence of the supervirasoro algebra appeared unexpectedly due to “miraculous cancellations” of explicit nonlocalities in the formalism. In addition to this, the odd flows of the Manin-Radul SKP hierarchy seemed to be irredeemably lost for the SKP\(_2\) hierarchy.

The way out of this impasse was hinted at by Dargis and Mathieu in [7]. They realised that the first few nonlocal charges for the SKdV equation obtained by Kensten [8] were given by the fourth-order root of the SKdV Lax operator \(D^4 + U D\). In their paper, Dargis and

\(^1\) I will assume in what follows that the reader is familiar with the standard superspace notation. A concise reference, otherwise, is supplied by [2].

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Mathieu assumed without proof that the Adler supertrace of $L^{\frac{2k-1}{4}}$ provided hamiltonians for Lax flows of the type

$$D_{2k-1}L = \left[ L, (L^{\frac{2k-1}{4}})_- \right].$$

(3)

But as I will explicitly show, only half of these flows, the ones corresponding to $k$ an even integer, are consistent. Moreover, the proof that these Lax flows are indeed hamiltonian with respect the supervirasoro algebra, with hamiltonians

$$H_{\frac{4k-1}{4}} = -\frac{4}{4k-1} \text{str} \ L^{\frac{2k-1}{4}},$$

(4)

will be a simple exercise when using the machinery developed in [9].

This result opens the interesting question: which odd flows are obtained from the supertraces of $L^{\frac{2k+1}{4}}$ when we use the supervirasoro algebra as our Poisson structure? The fact stated in [7], that these hamiltonians were not invariant under supersymmetry, seemed to suggest a possible connection with the Mulase-Rabin Jacobian SKP hierarchy (JSKP). The main point of this note is to show that in fact this connection exists, although the precise statement is that the resulting odd flows are a linear combination of the odd flows of the SKP and JSKP hierarchy. More interestingly, I will show that the algebra of additional symmetries of this new hierarchy is nothing else but the super-$\mathcal{W}_{1+\infty}$ algebra. Taking into account the relationship in the bosonic case between additional symmetries of the KP equation and two-dimensional gravity, it does not seem outlandish to expect a relationship between this new superintegrable hierarchy and two-dimensional supergravity.

Understandably, lack of space will prevent me from giving a detailed description of the formalism. Nonetheless, the reader can find the required machinery in [9] and references therein.

The Dargis-Mathieu odd flows

The SKdV Lax operator $L = D^4 + UD$ can be nicely characterized as the unique differential operator of order four obeying the constraint

$$L^* = DLD^{-1}. \quad (5)$$

Compatibility of the odd flows (3) with this constraint requires

$$D_{2k-1}L^* = \left[ L, (L^{\frac{2k-1}{4}})_- \right]^*, \quad (6)$$

or equivalently

$$\left[ L, (L^{\frac{2k-1}{4}})_- \right]_* = -D \left[ L, (L^{\frac{2k-1}{4}})_- \right] D^{-1}, \quad (7)$$

which is only satisfied provided

$$(DL^{\frac{2k-1}{4}}D^{-1})_- = (-)^k D(L^{\frac{2k-1}{4}})_- D^{-1}. \quad (8)$$

On the one hand, the leading term of the above equation clearly implies that $k$ should be an even integer. On the other hand, it is a simple computational matter to check that for
an arbitrary SΨDO $A = \sum_j A_j D^j$ the relation $(DAD^{-1})_+ = DA_+ D^{-1}$ holds if and only if $A_0 = 0$, in other words if $\text{sres } L^{2k-1} D^{-1} = 0$. But this is automatically fulfilled for $k$ an even integer. Indeed

$$\text{sres } L^{2k-1} D^{-1} = \text{sres } (L^{2k-1} D^{-1})^*$$

$$= -\text{sres } D^{-1}(L^*)^{2k-1}$$

$$= (-)^{k+1} \text{sres } L^{2k-1} D^{-1}.$$

(9)

In fact, a direct computation of the first flow with $k = 1$ yields

$$(D_1 U) D = U' D + U U^{[1-1]} + U'',$

(10)

(where we use the notation $U^{[1-1]} = (D^{-1} U)$), which is clearly inconsistent because the left hand side is an operator with no free term.

It was shown in [9] how Poisson brackets for the SKnV variable could be induced from the natural ones associated to the Lax operator $D^3 + U$. All the information about these Poisson brackets, that in this particular case are nothing but the supervirasoro algebra, can be neatly encoded in the Adler-type map

$$J(X) = (LX)_+ L - LD^{-1}(DXLD^{-1})_+ D,$$

(11)

where $X$ is a 1-form in the space of operators $L$, or equivalently

$$X^* = (-)^{|X|+1} DXD^{-1}.$$

(12)

The gradients of the hamiltonians $H_{\frac{2k-1}{4}}$ can be directly computed to yield

$$dH_{\frac{2k-1}{4}} = -L^{\frac{2k-1}{4}}.$$

(13)

It is now a straightforward computation to check that

$$D_{\frac{2k-1}{4}} L = J(dH_{\frac{2k-1}{4}}),$$

(14)

as conjectured by Dargis and Mathieu [7].

REMARK. I have been rather cavalier with the use of the formalism due to the fact that the hamiltonians $H_{\frac{2k-1}{4}}$ are not differential polynomials in $U$, but rather integro-differential polynomials. Nevertheless the reader can check that all the necessary manipulations to arrive to (14) can be extended to the integrodifferential case.

In order to understand what is going on with the flows induced by the other half of the hamiltonians it will be necessary to make a small digression about SKnV-type reductions of the JSKP hierarchy.
The Jacobian SKP hierarchy

It is well known that the SKP hierarchy of Manin and Radul and SKP$_2$ are not the only possible extension to superspace of the standard KP hierarchy. In references [10] and [11] Mulase and Rabin proposed a different supersymmetric extension of KP. Their approach was inspired by the geometrical interpretation of the standard KP equations as flows in a Jacobian variety. From the purely algebraic point of view the Jacobian SKP (JSKP) flows are given as flows in the superVolterra group. An element $\phi$ of the superVolterra group is given by S\(\Psi\)DO's of the form

$$\phi = 1 + \sum_{i=1}^{\infty} s_i D^{-i}. \quad (15)$$

The explicit expression of the JSKP flows is then given by

$$D_{2k} \phi = - \left( \phi \partial_{\phi}^k \phi^{-1} \right)_- \phi \quad (16)$$

$$D_{2k-1} \phi = - \left( \phi \partial_{\phi}^{k-1} \partial_{\theta} \phi^{-1} \right)_- \phi. \quad (17)$$

It is now simple to check that the even flows of JSKP are equivalent to the ones of SKP and SKP$_2$ under the assumption that the Lax operators defining this two hierarchies are dressable. In general, for a dressable S\(\Psi\)DO $\Lambda$ of order $j$, take $\phi$ such that

$$\Lambda = \phi D_j \phi^{-1}. \quad (18)$$

Then

$$D_{2k} \Lambda = \left[ \Lambda, \Lambda^k_- \right] \Leftrightarrow \quad D_{2k} \phi = - \left( \phi \partial_{\phi}^k \phi^{-1} \right)_- \phi. \quad (19)$$

But in contrast the odd flows induced on $\Lambda$ through (17) are given by

$$D_{2k-1} \Lambda = \left[ \Lambda, (\Lambda^{k-1} M)_- \right], \quad (20)$$

with $M = \phi \partial_{\theta} \phi^{-1}$. Since $M$ cannot be simply expressed in terms of $\Lambda$, it is customary to say that the odd flows of JSKP are not of the Lax form, although in fact they are.

The obvious question to ask oneself now is: what happens with the JSKP flows when a reduction of the SKdV-type is imposed upon them? Of course there is nothing new to say about the even flows because they are the “good old ones”, but what about the odd ones?

Let us first study what is the effect of the SKdV constraints at the level of the dressing field $\phi$. Because of the results of [9] we can restrict ourselves without lost of generality to the SKP$_2$ case, i.e. $\Lambda = D^2 + \cdots$. As before we should first consider the SBKP constraint $\Lambda^* = -D \Lambda D^{-1}$. At the level of the $\phi$ this implies that

$$(\phi^{-1})^* \partial \phi^* = D \phi \partial \phi^{-1} D^{-1}. \quad (21)$$

If we now use that $(\phi^{-1})^* = (\phi^*)^{-1}$, (21) can be written as $\partial = P \partial P^{-1}$, with $P = \phi^* D \phi D^{-1}$,
thus
\[ \phi^* = D\phi^{-1}D^{-1}. \]

Thereby,
\[
M^* = - (\phi^{-1})^* \partial_\theta \phi^*
= - D\phi D^{-1} \partial_\theta D\phi^{-1} D^{-1}
= D\phi \partial_\theta \phi^{-1} D^{-1} - D\phi D^{-1} [\partial_\theta, D] \phi^{-1} D^{-1}
= D (\phi \partial_\theta \phi^{-1} - \phi D\phi^{-1}) D^{-1}
= DMD^{-1} - D\Lambda^\frac{1}{2}D^{-1}.
\]

And consequently the odd flows do not reduce nicely. But fortunately not everything is lost. Notice that the combination \( S = 2M - \Lambda^\frac{1}{2} \) has the “correct” symmetry properties, \( i.e. \)
\[ S^* = DS D^{-1}, \quad (22) \]
and moreover \( S^2 = \Lambda \). Nevertheless, I will keep the notation \( \Lambda^\frac{1}{2} \) for the manifestly supersymmetric square root of \( \Lambda \), \( i.e. \), the dressed version of the superderivative. Curiously enough, \( S \) is nothing but \( \phi Q \phi^{-1} \), where \( Q \) is the generator of supersymmetry transformations.

As a result of all of this, it seems natural to introduce the following JSKP-like hierarchy with oddflows given by
\[
D_{2k-1} \Lambda = \left[ \Lambda, (\Lambda^{k-1} S)_- \right], \quad (23)
\]
which can be understood as a linear combination of the original SKP and JSKP odd flows. It is now simple to show that these flows are indeed consistent with the SBKP constraint as long as \( k = 2j + 1 \). As before, the consistency of the oddflows requires
\[ \text{sres} \Lambda^{2j} SD^{-1} = 0, \quad (24) \]
and this follows from
\[
\text{sres} \Lambda^{2j} SD^{-1} = \text{sres} (\Lambda^{2j} SD^{-1})^*
= - \text{sres} D^{-1} S^* (\Lambda^{2j})^*
= - \text{sres} \Lambda^{2j} SD^{-1}.
\]

**Remark.** When there is an explicit dependence on \( \partial_\theta \), in order to compute the super-residue, or equivalently the + and - projections, we should use that
\[ \partial_\theta = D - \theta \partial, \]
and then apply the standard definitions [2].

Because \( D_{4j+1} \) acts as a superderivation it follows that
\[
D_{4j+1} \Lambda^n = \left[ \Lambda^n, (\Lambda^{2j} S)_- \right], \quad (25)
\]
and it is consistent to impose the constraint \( (\Lambda^n)_- = 0 \). In particular for \( n = 2 \) we recover the standard SKdV Lax operator.
I will give strong evidence in the following that for $n = 2$ the flows (25) are hamiltonian with respect the superVirasoro algebra, and hamiltonians

$$H_{j+1} \sim \text{str} L^{j+1}. \quad (26)$$

Moreover, the formalism is such that the generalization of this result to higher order SKdV-type reductions is self-evident.

The key point is supplied by the relationship

$$dH_{j+1} \sim L^{-1}S. \quad (27)$$

Notice first that as given above $dH_{j+1}$ has the correct weight and symmetry properties, i.e.

$$dH_{j+1} = DdH_{j+1}D^{-1},$$

as corresponds to an odd one-form (12). It is now straightforward to show that

$$D_{4j+1}L \sim J(dH_{j+1}). \quad (28)$$

Unfortunately, I do not have at this point a proof of (27), although I have checked its validity for the first few flows.

Notice that the first flow of the hierarchy is given by

$$D_{1}U = [Q, U] = U' - \theta U'',$$

which is nothing but the supersymmetric variation of $U$. This property of $D_{1}$ is not a peculiarity of the SKdV reduction and is also shared by the whole hierarchy. This is in contrast to the SKP and JSKP hierarchies where the first flow contains an explicit dependence on the potentials.

**Additional symmetries**

It is now simple to show, following standard techniques [12] [13][14], that the algebra of additional symmetries of the new supersymmetric hierarchy given by the standard even flows plus (23) is isomorphic to the algebra of superdifferential operators in $(1|1)$ superspace (SDOP), which is itself isomorphic to super-$\mathcal{W}_{1+\infty}$ [15].

The algebra of the flows is given by

$$[D_{2i}, D_{2j}] = 0, \quad [D_{2i}, D_{2j-1}] = 0, \quad \text{and} \quad [D_{2i-1}, D_{2j-1}] = 2D_{2i+2j-2}.$$

A representation of these flows in terms of even and odd time parameters is given by

$$D_{2k} = \frac{\partial}{\partial t_{2k}} \quad (30)$$

$$D_{2k-1} = \frac{\partial}{\partial \tau_{2k-1}} + \sum_{j \geq i} \tau_{2j-1} \frac{\partial}{\partial t_{2k+2j-2}} - 7 -$$

$$+ \frac{\partial}{\partial \tau_{2k-1}} + \sum_{j \geq i} \tau_{2j-1} \frac{\partial}{\partial t_{2k+2j-2}}$$

$$- 7 -$$
In complete analogy with the bosonic case the additional symmetries are going to be generated by additional flows of the form

$$\partial \Gamma \phi = - (\phi \Gamma \phi^{-1})\phi, \tag{32}$$

with

$$[D_{2k} - \partial^k, \Gamma] = 0, \quad \tag{33}$$
$$[D_{2k-1} - \partial^{k-1}Q, \Gamma] = 0. \tag{34}$$

An obvious solution for $\Gamma$ is given by the superderivative $D$. It is also possible to obtain two other solutions by the standard procedure of deforming the operators $x$ and $\theta$, i.e.

$$\Gamma_x = x + \sum_{k \geq 1} kt_{2k}\partial^{k-1} + \sum_{k \geq 1} (k - 1)\tau_{2k-1}\partial^{k-2}Q +$$
$$\theta \sum_{k \geq 1} \tau_{2k-1}\partial^{k-1} - \frac{1}{2} \sum_{k,j \geq 1} (k - j)\tau_{2k-1}\tau_{2j-1}\partial^{i+j-2}, \tag{35}$$

and

$$\Gamma_\theta = \theta + \sum_{k \geq 1} \tau_{2k-1}\partial^{k-1}. \tag{36}$$

Therefore any $\Gamma$ of the form

$$\Gamma = \Gamma_x^{n}\Gamma_\theta^k D^m, \tag{37}$$

with $n \geq 0, k = 0,1,$ and $m \in \mathbb{Z}$ defines a symmetry of the hierarchy. Notice that the commutation relations

$$[D, \Gamma_x] = \Gamma_\theta, \quad \text{and} \quad [D, \Gamma_\theta] = 1, \tag{38}$$

directly imply that the algebra of additional symmetries is isomorphic to SDOP.

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