Localized gravity and mass hierarchy in $D = 6$ with the Gauss-Bonnet term

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We obtain the localized gravity on the intersection of two orthogonal non-solitonic or solitonic 4-branes in $D = 6$ in the presence of the Gauss-Bonnet term. The tension of the intersection is allowed to exist unlike the case without the Gauss-Bonnet term. We show that gravity could be confined to the solitonic 4-branes for a particular choice of the Gauss-Bonnet coupling. If the extra dimensions are compactified with the $T^2/(Z_2 \times Z_2)$ orbifold symmetry, the mass hierarchy between the Planck scale and the weak scale can be explained by putting our universe at the TeV intersection of positive tension located at the orbifold fixed point.

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I. INTRODUCTION

The recent proposals for the fundamental TeV scale physics [1,2] have been a great surprise in high energy physics, which has not been noted for a long period of superstring research. Of particular interest is the first Randall-Sundrum(RSI) model [2] in which only one brane with nonvanishing tension exists, through the compactification of the extra dimension(s). In the RSI model with the Gauss-Bonnet term, enabling one to introduce a TeV scale from the Planck scale with an O(10) ratio of the input parameters, through the compactification of the extra dimension $y$ on $S_1/Z_2$. There are two branes in the RSI model: the brane 1 (B1) located at $y = 0$ and brane 2 (B2) located at $y = y_2$.

Probably a more interesting proposal is the second Randall-Sundrum(RSII) model [3] in which only one brane (B1) located at $y = 0$ is introduced. Thus, the fifth dimension is not compactified, but still this model can describe a meaningful effective 4 dimensional(4D) physics since the gravity is localized around B1. It is an alternative to compactification idea of the extra dimension(s). Both Randall-Sundrum models need AdS spacetime in the bulk.

Subsequently, extensions of the RS type models were proposed toward the hierarchy solution [4–7], for the study of localization of gravity [8,9,10–13,7], and for other aspects [14–16]. In particular, the RSII model have been studied toward finding a self-tuning solution of the cosmological constant problem. It is because, from the beginning of these proposals, the solution of the cosmological constant was sought for in the RS models since the Einstein equations can choose a flat space even with a negative nonvanishing bulk cosmological constant and nonvanishing brane tension(s). But in the first proposals, the nonvanishing parameters should be fine tuned for the universe to be flat in the model [2,3]. It has been suggested that introduction of a bulk real scalar field with a coupling to the brane may give a self-tuning of the cosmological constant but it retains the serious fine tuning problem due to a naked singularity [17]. There exists an example for the selftuning solution with an unconventional interaction of a bulk antisymmetric tensor field [18] which may shed more light toward a final solution of the cosmological constant problem.

In the intersecting brane world scenarios in higher than five dimensions [8,9], our universe is regarded as a 3-brane with higher codimensions given by the common intersection of higher dimensional objects with lower codimensions. However, when we consider discrete sources of higher dimensional objects in the bulk space, no additional contribution is allowed from the brane-brane interaction to the tension of their intersection corresponding to the 3-brane tension, since the Einstein tensor just gives rise to one-dimensional delta function from the intersecting branes. This behavior is well understood in the smooth limit of intersecting branes. For instance, for $n$ orthogonal $(n + 2)$-branes in $D = 4 + n$, each $(n + 2)$-brane has the tension $T_{n+2} = M^{n+4} L$ while the tension of their intersection is $T_3 = M^{n+4} L^n$ by dimensional analysis. ($M$ is the $(4+n)$-dimensional fundamental scale and $L$ is the brane thickness.) Therefore, the 3-brane tension shows up with higher power of $L$, so it gets suppressed in the thin brane limit, which means that higher curvature terms should be taken into account for better resolution to see such a thin 3-brane. Without nonzero 3-brane tension, it is difficult to discuss on the generation of vacuum energy after phase transition on the intersection as our world. Because the corresponding nonzero tension of the intersection is not allowed, the vacuum energy induced by phase transition has no way but at most to leak away along the intersecting branes, whose tensions are allowed to be nonzero. In this context, it is necessary that the nonzero brane-brane interaction or the nonzero tension...
of the intersection should appear in a natural way.

In this paper we consider the RS type solution for the case of two orthogonally intersecting non-solitonic 4-branes* and one 3-brane (or string) on their intersection in D=6 when the Gauss-Bonnet term is added in the bulk action. In that case, we can regard our world as a common intersection of two 4-branes where the localization of gravity arises. In the existence of the Gauss-Bonnet term, in particular, a string tension should be introduced at the beginning to match an additional boundary condition on the intersection. So, our solution with two 4-branes and one string is based on two fine-tuning conditions between input parameters but there is a possibility for naturally regarding the vacuum energy in our world as the string tension in the intersecting brane world scenario, which has not been possible to get without the Gauss-Bonnet term. Thus, it seems that the higher curvature terms know about the inner structure of the intersecting branes whereas the Einstein-Hilbert term has the lower resolution.

For a special relation between the bulk cosmological constant and the Gauss-Bonnet coupling in our model, it is shown that there exists a string solution with codimension-2 by considering the Z_2 × Z_2 symmetry of the extra dimensions as usually imposed in the case of the two orthogonal 4-branes. In that case, the bulk space is found to be a discrete patch of the pure AdS_6 space to make the bulk symmetry manifest and the resultant discontinuities of the derivative of the metric across the symmetry axes are shown to be automatically cancelled between those derived from the Einstein-Hilbert term and the Gauss-Bonnet term in the equations of motion without the need of introducing 4-branes along the symmetry axes. In other words, it is shown that the Einstein-Gauss-Bonnet gravity itself is able to support singularities produced on orbifolding without the need of introducing additional non-solitonic singular sources. From the point of view of the Einstein’s gravity, however, the singularities are interpreted as the so-called solitonic 4-branes [16], of which tensions are determined by the Gauss-Bonnet coupling and the 3-brane tension. Nonetheless, since the solitonic 4-branes are supported by gravity only without sources in D = 6, they don’t give any fine-tuning conditions. Therefore, on patching the AdS_6 bulk in the Z_2 × Z_2 invariant way, there exists a solution of a string residing on the intersection of two solitonic 4-branes, which is based on one fine-tuning condition between bulk parameters but for which the 3-brane cosmological constant Λ_1 can take any positive value without being involved in any fine-tuning. In particular, it is interesting to see that there arises the confinement of gravity to the solitonic 4-branes, which results in exactly two copies of the 5D RSII model in D = 6.

For the string solution with codimension 2 in D = 6, it has been shown that the singular global string solution is possible with a massless scalar field in the flat bulk by the unitarity boundary condition at the singularity [4]. Later, it was pointed out that there exist regular global string solutions by introducing a bulk cosmological constant [11,12,7]. One more interesting observation is that the local string defects were shown to have the localized gravity with no fine-tuning of the bulk cosmological constant, but here the components of the string tension are required to satisfy a certain relation [13], which is a fine tuning. Pertinent to our study of this paper, we note the work of Corradini and Kakushadze in which it has been argued that it is possible to have the localized gravity on a solitonic 3-brane with the Gauss-Bonnet term while freely choosing the 3-brane cosmological constant equivalent to a deficit angle in the extra polar coordinate in the 5th and the 6th space [20]. (Note that a similar result was known in the case with 3-brane sources in the 6D Einstein gravity without a bulk cosmological constant [21] and with a positive bulk cosmological constant [22].) This solution has one fine-tuning condition between bulk parameters and there exists a conical singularity corresponding to the brane tension [20].

Based on our string solution in the intersecting brane scenario, we can compactify the extra dimensions with the T^2/(Z_2 × Z_2) orbifold symmetry. Then, we can show that the hierarchy problem can be solved if we put the branes at the four fixed points of the orbifold T^2/(Z_2 × Z_2) and the neighboring two 3-branes are connected to each other by one 4-brane. In this case, the positive tension brane diagonally far away from the origin of the extra dimension is regarded as our universe and some other three 3-branes as the hidden branes.

In Sec. II, we obtain a 3-brane (or string) solution in the EGB theory. It is the most relevant generalization of the RSII model. There appears solitonic 4-brane solutions. In Sec. III, we consider the metric perturbation near the background geometry and ensure that there is no tachyonic mode of graviton. Then, in Sec. IV, we make discussions on the gravity confinement to the solitonic 4-branes. In Sec. V, we compactify 6D with the T^2/(Z_2 × Z_2) orbifold symmetry and obtain four fixed points where 3-brane sources can be placed. It is the most relevant generalization of the RSII model in which a TeV 3-brane can occur naturally. Sec. VI is a conclusion.

*Here, solitonic means being supported by gravity only while non-solitonic does by sources.
If we impose the $Z_2$ symmetry on each extra dimension in $D = n + 4$ dimensional generalization of the RS model, we should have $(n + 2)$-branes orthogonally intersecting to each other to match the boundary conditions of the metric [8,9]. Therefore, the 3-brane as our universe only appears as the common intersection of all the $(n + 2)$-branes [8], but without its tension. However, in the presence of the Gauss-Bonnet term, from which no higher than second derivatives are derived in the equations of motion, the intersection of two orthogonal 4-branes in $D = 6$ is required to have a nonzero tension, which will be shown below.

When the Gauss-Bonnet term is added as the next leading-order ghost-free interaction to the Einstein-Hilbert term in 6D with two spacelike extra dimensions, we start with the Einstein-Gauss-Bonnet (EGB) 6D action with singular brane sources,

$$S_6 = \int d^4x dz_2 d2\sqrt{-g}\left[\frac{M^4}{2}R - \Lambda_b\right] + \frac{1}{2}aM^2 (R^2 - 4R_{MN} R^{MN} + R_{MPN} R^{MPN}) + \int d^4x d2\sqrt{-g}(z_{1} = 0)(-\Lambda_{z_1}) + \int d^4x d2\sqrt{-g}(z_{2} = 0)(-\Lambda_{z_2}) + \int d^4x \sqrt{-g}(z_{1} = 0, z_{2} = 0)(-\Lambda_{1}) \tag{1}$$

where $g, g(z_{1} = 0), g(z_{2} = 0)$ and $g(z_{1} = 0, z_{2} = 0)$ are the determinants of the metrics in the bulk, orthogonally intersecting 4-branes and a 3-brane, $M$ is the six dimensional gravitational constant, $\Lambda_b, \Lambda_{z_1}, \Lambda_{z_2}$, and $\Lambda_{1}$ are the bulk and the brane cosmological constants, $\alpha$ is the effective coupling. We considered the 4-branes to write down general equations of motion, but we will see later that there is a possibility of getting the string solution without these 4-brane sources by imposing the $Z_2 \times Z_2$ symmetry in the bulk.

Equations of motion in this EGB theory are,

$$G_{MN} + H_{MN} = M^{-4}T_{MN}. \tag{2}$$

The tensors in the above equation are

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R, \tag{3}$$

$$H_{MN} = \frac{\alpha}{M^2}\left[ -\frac{1}{2}g_{MN}(R^2 - 4R^2_{PQ} + R_{PQRST}R^{PQRST}) + 2RR_{MN} - 4R_{MPN}R^P - 4R_{MP}R^{KP} + 2R_{MQSP}R_{N}^{QSP} \right], \tag{4}$$

$$T_{MN} \equiv -\Lambda_b g_{MN} - \frac{\sqrt{-g(z_{1} = 0)}}{\sqrt{-g}}\Lambda_{z_1}\delta(z_{1})\delta^\mu_M\delta^\nu_Ng_{\mu\nu}^{(z_{1} = 0)} - \frac{\sqrt{-g(z_{2} = 0)}}{\sqrt{-g}}\Lambda_{z_2}\delta(z_{2})\delta^\mu_M\delta^\nu_Ng_{\mu\nu}^{(z_{2} = 0)} - \frac{\sqrt{-g(z_{1} = 0, z_{2} = 0)}}{\sqrt{-g}}\Lambda_1\delta(z_{1})\delta(z_{2})\delta^\mu_M\delta^\nu_Ng_{\mu\nu}^{(z_{1} = 0, z_{2} = 0)}, \tag{5}$$

where the indices $M, N = (0, 1, 2, 3, 5, 6), p, q = (0, 1, 2, 3, 6), a, b = (0, 1, 2, 3, 5)$ and $\mu, \nu = (0, 1, 2, 3)$.

Taking the metric ansatz as a conformally flat one in 6D, which is manifestly 4D Poincaré invariant,

$$ds^2_6 = A^2(z_1, z_2)(\eta_{\mu\nu}dx^\mu dx^\nu + dz_1^2 + dz_2^2), \tag{6}$$

where $(\eta_{\mu\nu}) = \text{diag.}(-1, +1, +1, +1)$, we obtain the tensor components $G_{MN}$ and $H_{MN}$ as follows,

$$G_{\mu\nu} = \frac{2}{A^2}\left[\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2 + 2\frac{A'}{A} + 2\frac{A''}{A^2}\right]\delta_{\mu\nu}, \tag{7}$$

$$G_5^5 = \frac{2}{A^2}\left[\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2 + 2\frac{A'}{A} + 2\frac{A''}{A^2}\right], \tag{8}$$

$$G_5^6 = \frac{4}{A^2}\left[-\frac{A'}{A} + 2\frac{A''}{A^2}\right], \tag{9}$$

$$G_6^6 = \frac{2}{A^2}\left[\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2 + 2\frac{A'}{A} + 2\frac{A''}{A^2}\right], \tag{10}$$

and

$$H_{\mu\nu} = -\frac{12\alpha}{M^2 A^4}\left[ -3\left(\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2\right)^2 + 4\left(\frac{A'}{A}\right)^2\frac{A''}{A^2} + 4\left(\frac{A''}{A}\right)^2\frac{A'}{A} + 2\frac{A'}{A} + 2\frac{A''}{A^2}\right]\delta_{\mu\nu}, \tag{11}$$

$$H_5^5 = \frac{12\alpha}{M^2 A^4}\left[ -2\left(\frac{A'}{A}\right)^2\left(\frac{A''}{A}\right)^2 - 5\left(\frac{A''}{A}\right)^4 + 3\left(\frac{A'}{A}\right)^4 - 4\left(\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2\right)^2\right], \tag{12}$$

$$H_5^6 = -\frac{48\alpha}{M^2 A^4}\left(\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2\right)\left(-\frac{A'}{A} + 2\frac{A''}{A^2}\right), \tag{13}$$

$$H_6^6 = \frac{12\alpha}{M^2 A^4}\left[ -2\left(\frac{A'}{A}\right)^2\left(\frac{A''}{A}\right)^2 - 5\left(\frac{A''}{A}\right)^4 + 3\left(\frac{A'}{A}\right)^4 - 4\left(\left(\frac{A'}{A}\right)^2 + \left(\frac{A''}{A}\right)^2\right)^2\right], \tag{14}$$

where the prime and the dot denote the derivatives with respect to $z_1$ and $z_2$, respectively. The energy momentum tensor $T_{MN}$ is given by
\[ T_M^N = -\Lambda_b \delta^N_M - \frac{1}{A} \Lambda_z \delta(z_1) \delta_p^N \delta_p \]
\[ - \frac{1}{A^2} \Lambda_z \delta(z_2) \delta_M^N \delta_n \]
\[ - \frac{1}{A^2} \Lambda_1 \delta(z_1) \delta(z_2) \delta_M^N \delta_n, \]
\[ (15) \]

Then, the (56) component of the modified Einstein's equations is
\[
\frac{4}{A^2} \left[ 1 - \frac{12a}{M^2} \left( \left( \frac{A'}{A} \right)^2 + \left( \frac{A}{A} \right)^2 \right) \right] \cdot \left( - \frac{A'}{A} + 2 \frac{\dot{A}A'}{A^2} \right) = 0.
\[ (16) \]

Therefore, to assure that the above equation is satisfied, we require that the second factor vanishes,
\[
- \frac{A'}{A} + 2 \frac{\dot{A}A'}{A^2} = 0,
\[ (17) \]

i.e., the general solution of the metric is given by
\[
A(z_1, z_2) \propto \frac{1}{(F(z_1) + G(z_2))}
\[ (18) \]

where \( F \) and \( G \) are undetermined functions of \( z_1 \) and \( z_2 \), respectively. Note that in case of the vanishing first factor in Eq. (16), Eq. (17) is automatically satisfied. To determine the exact solution of the above type, we can rewrite the \((00)\) or \((ii)\), \((55)\) and \((66)\) components under the condition Eq. (17), respectively:

\[
E + e_1 + e_2 + e_3 = M^{-4} \left[ - \Lambda_b - \frac{1}{A} \Lambda_z \delta(z_1) \right.
\]
\[ - \frac{1}{A} \Lambda_z \delta(z_2) - \frac{1}{A^2} \Lambda_1 \delta(z_1) \delta(z_2) \left. \right], \]
\[ (19) \]

\[
E + e_2 = M^{-4} \left[ - \Lambda_b - \frac{1}{A} \Lambda_z \delta(z_2) \right], \]
\[ (20) \]

\[
E + e_1 = M^{-4} \left[ - \Lambda_b - \frac{1}{A} \Lambda_z \delta(z_1) \right], \]
\[ (21) \]

where
\[
E = 10 \left[ 1 - \frac{6a}{M^2} \left( \left( \frac{A'}{A} \right)^2 + \left( \frac{A}{A} \right)^2 \right) \right]
\]
\[ \cdot \frac{1}{A^2} \left[ \left( \frac{A'}{A} \right)^2 + \left( \frac{A}{A} \right)^2 \right], \]
\[ (22) \]
\[ e_1 = \frac{4}{A} \left( \frac{A'}{A^2} \right) \left[ 1 - \frac{12a}{M^2} \left( \left( \frac{A'}{A} \right)^2 + \left( \frac{A}{A} \right)^2 \right) \right], \]
\[ (23) \]

\[
e_2 = \frac{4}{A} \left( \frac{A'}{A^2} \right) \left[ 1 - \frac{12a}{M^2} \left( \left( \frac{A'}{A} \right)^2 + \left( \frac{A}{A} \right)^2 \right) \right], \]
\[ (24) \]
\[ e_3 = -\frac{2\alpha}{M^2} \left( \frac{A'}{A} \right)^{3/2} \left( \frac{A}{A} \right)^{3/2}, \]
\[ (25) \]

Thus, the bulk equation in all the above components, \( E = -\Lambda_b/M^4 \), can be solved only if \( F(z_1) = k_1z_1 + c_1 \) and \( G(z_2) = k_2z_2 + c_2 \) (\( c_1, c_2 \) are integration constants), i.e.
\[
A(z_1, z_2) = \frac{1}{(k_1|z_1| + k_2|z_2| + 1)}
\[ (26) \]

where the \( Z_2 \) symmetry is used along each extra dimension and the integration constants are arbitrarily chosen for \( A \) to be 1 at \((z_1, z_2) = (0, 0)\). \( k_1, k_2 \) are determined by the following relations,
\[
k_1^2 + k_2^2 = \frac{M^2}{12\alpha} \left[ 1 \pm \sqrt{1 + \frac{12\alpha\Lambda_b}{5M^6}} \right] = \kappa^2,
\[ (27) \]
\[ k_1 \left( 1 - \frac{12\alpha k_1^2}{M^2} \right) = \frac{\Lambda_{z_1}}{8M^4}, \]
\[ (28) \]
\[ k_2 \left( 1 - \frac{12\alpha k_2^2}{M^2} \right) = \frac{\Lambda_{z_2}}{8M^4}, \]
\[ (29) \]
\[ \alpha k_1 k_2 = \frac{\Lambda_1}{96M^4}, \]
\[ (30) \]

where the last three equations are derived from the boundary conditions on the branes in Eqs. (19-21). The first and fourth equations determine \( k_1 \) and \( k_2 \) in terms of \( \alpha, \Lambda_b, \Lambda_1 \), and it should be such that \(|\Lambda_1| \leq 48|\alpha|k_2^2M^2\), where the equality implies the existence of exchange symmetry between two extra dimensions, and \( \text{sign}(\Lambda_1) = \text{sign}(\alpha) \) to give real solutions for \( k_1 \) and \( k_2 \). Then, the second and third equations give rise to two fine-tuning conditions between input parameters. Note that the Gauss-Bonnet term requires an additional condition, Eq. (30), on the 3-brane other than those the Einstein-Hilbert action imposes on the 4-branes, Eqs. (28) and (29).

However, if we chose a relation between bulk parameters from the beginning,
\[
\frac{12\alpha\Lambda_b}{5M^6} = -1
\[ (31) \]

such that \( k_1^2 = \frac{M^2}{12\alpha} \) for \( \alpha > 0 \), non-solitonic 4-brane tensions would not be allowed to exist, viz. Eqs. (28) and (29).
where \( k \) words, on patching the bulk space in the four-dimensional space-time with the addition of the Gauss-Bonnet term, but is not possible with the Einstein-Hilbort term alone.\(^1\) In other words, on patching the bulk space in the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetric way as shown in the chosen metric, we naturally obtain a string solution via the cancellation between those derived from the Einstein-Hilbort term and the Gauss-Bonnet term in the equations of motion. However, from the point of view of the Einstein’s gravity, singularities on orbifolding should be seen to stem from solitonic 4-brane tensions, just as in Iglesias and Kakushadze’s [16].

In our case, the solitonic 4-brane tensions \( f_1(f_2) \) located at \( z_1 = 0(z_2 = 0) \) are determined to be positive as

\[
f_1 = 8k_1 M^4, \quad f_2 = 8k_2 M^4
\]

where \( k_1 \) and \( k_2 \) are given by solving Eqs. (27) and (30) under the condition Eq. (31).

Then, after integrating the extra dimensions with the 4D part of the metric as \( g_{\mu\nu}(x) = \eta_{\mu\nu} \) in Eq. (6), we obtain the 4D effective action as follows,

\[
S_{\text{eff}} = \frac{M^2_{\text{P,eff}}}{2} \int d^4x \sqrt{-\hat{g}^{(4)}} \left[ \tilde{R} + \frac{\alpha_{\text{eff}}}{M^2_{\text{P,eff}}} (\tilde{R}^2) - 4\tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu}\right]
\]

where the 4D Planck mass and the 4D Gauss-Bonnet coupling are given by

\[
M^2_{\text{P,eff}} = M^4 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^\infty dz_2 \left[ A^4 \left( 1 + \frac{12\alpha}{M^2} \right) \left( \frac{A'}{A} \right)^2 + \left( \frac{A'}{A} \right)^2 - \frac{12\alpha}{M^2} (AA')' + (\bar{A}A)' \right]
\]

\[
= \frac{2M^4}{3k_1 k_2} \left( 1 + \frac{12\alpha k_1^2}{M^2} \right) = \frac{64\alpha M^6}{\Lambda_1} \left( 1 + \frac{12\alpha k_1^2}{M^2} \right)
\]

\[
\geq \frac{4M^4}{3k_1 k_2} \left( 1 + \frac{12\alpha k_1^2}{M^2} \right),
\]

\begin{equation}
\alpha_{\text{eff}} = \alpha M^2 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 A^2
\end{equation}

where \((AA')'\) and \((\bar{A}A)')\) terms in the first line vanish after integration. For a negative Gauss-Bonnet coupling \( \alpha \), the 4D Planck mass would not be positive definite due to the contribution from the Gauss-Bonnet term. Therefore, the positivity condition gives \( |\alpha| < \frac{M^2}{12k_1^2} \) for \( \alpha < 0 \) and any value for \( \alpha > 0 \). On the other hand, the 4D Gauss-Bonnet coupling is shown to become logarithmically divergent after integration. This seems to be a generic feature of higher curvature terms, which is rephrased as the delocalization of gravity in warped geometry [15]. Nonetheless, there does not arise a problem in our case since the Gauss-Bonnet term is a total derivative in \( D = 4 \) and thus it does not modify the equation of motion for graviton in the 4D spacetime. Therefore, we can drop the 4D Gauss-Bonnet term in Eq. (33) to get the 4D effective Einstein gravity [15].

III. METRIC PERTURBATION NEAR THE BACKGROUND GEOMETRY

Now that we have obtained the background solution, it is of interest to examine the perturbation effects of gravity near the background solution. Since the effects inform us how the gravitational interaction between matter is described at low energy scales under a background geometry, it is indispensable to study the perturbative expansion and compare it with the well-known gravitational interaction. The perturbation in higher dimensional space-time is usually interpreted as the graviton in the corresponding space-time dimension, and is, in 6 dimension case, decomposed into a 4 dimensional graviton, two kinds of vectors and three kinds of scalars. In this section, however, we assume that the vector and scalar modes are decoupled by some physics due to their absence at the low energy scale, and we focus on the gravitational interaction mediated by the 4 dimensional graviton.

Thus, for the study, let us assume the metric as the following,

\[
ds^2 = \left[ A^2(z_1, z_2) n_{\mu\nu} + h_{\mu\nu}(x, z_1, z_2) \right] dx^\mu dx^\nu + A^2(z_1, z_2) (dz_1^2 + dz_2^2)
\]

\[
= A^2(z_1, z_2) \left[ n_{\mu\nu} + h_{\mu\nu}(x, z_1, z_2) \right] dx^\mu dx^\nu + dz_1^2 + dz_2^2
\]

where \( x \) denotes the 4 dimensional coordinate, and we would keep the linear parts in \( h_{\mu\nu} \) in the full expression.
of the Einstein equation. Here, $A(z_1, z_2)$ is the background solution given by Eq. (26) and $h_{\mu\nu}$ represents a small perturbation near it. With Eq. (36), the linearized variations for $G_{\mu\nu}$, $H_{\mu\nu}$ and $T_{\mu\nu}$ are given by

$$\delta G_{\mu\nu} = -\frac{1}{2A^2} \Box_4 + \frac{2}{A^2} \left( \partial^2_{z_1} + \partial^2_{z_2} \right) - 26 \left( k^2_1 + k^2_2 \right)$$

$$\delta H_{\mu\nu} = \frac{\alpha}{M^2} \left[ \frac{1}{A^2} \left( 6(k^2_1 + k^2_2) - 8k_1 \delta(z_1) - 8k_2 \delta(z_2) \right) \Box_4 \right.$$  

$$+ \frac{1}{A^2} \left( 6(k^2_1 + k^2_2) - 8k_2 \delta(z_1) \right) \partial^2_{z_1} \right.$$  

$$+ \frac{1}{A^2} \left( 6(k^2_1 + k^2_2) - 8k_1 \delta(z_1) \right) \partial^2_{z_2} \right.$$  

$$+ \frac{8k_1}{A} \left( 3 \delta(z_1) \right) sgn(z_1) \partial_{z_1}$$  

$$+ \frac{8k_2}{A} \left( 3 \delta(z_2) \right) sgn(z_2) \partial_{z_2}$$  

$$- 96 \left( k^2_1 + k^2_2 \right)^2$$  

$$+ \frac{k_1}{A} \delta(z_1) \left( 168k^2_1 + 152k^2_2 \right)$$  

$$+ \frac{k_2}{A} \delta(z_2) \left( 168k^2_2 + 152k^2_1 \right)$$  

$$- 160 \left( k_1 k_2 \delta(z_1) \delta(z_2) \right) \left( \Box_4 \right.$$  

$$+ \frac{k_2}{A} \left( 10 - \frac{\alpha}{M^2}(168k^2_2 + 152k^2_1) \right) - \frac{\Lambda_z}{A^2} \frac{\Lambda_1}{A^2} \right] h_{\mu\nu} \right.$$  

$$\delta T_{\mu\nu} = -\Lambda_0 h_{\mu\nu} - \frac{1}{A} \Lambda_2 \delta(z_1) h_{\mu\nu} - \frac{1}{A} \Lambda_2 \delta(z_2) h_{\mu\nu}$$

$$- \frac{1}{A^2} \Lambda_1 \delta(z_1) \delta(z_2) h_{\mu\nu} \right.$$  

where $\Box_4 \equiv \eta_{\mu\nu} \partial_\mu \partial_\nu$, and we choose the traceless transverse gauge conditions, $\partial^\mu h_{\mu\nu} = h_{\nu\mu} = 0$.

The above expressions lead to the linearized Einstein equation,

$$-\frac{1}{2A^2} \left( 1 - \frac{12\alpha}{M^2}(k^2_1 + k^2_2) \right) \left[ \Box_4 + \partial^2_{z_1} + \partial^2_{z_2} - 6A(k^2_1 + k^2_2) \right] h_{\mu\nu}$$

$$- \delta(z_1) \left[ \frac{8\alpha}{M^2} \frac{k_1}{A} \left( \frac{1}{A^2} \left( \Box_4 + \partial^2_{z_2} \right) \right) \right.$$  

$$+ \frac{k_2}{A} sgn(z_2) \partial_{z_2} - 3k \frac{\Lambda_1}{\Lambda_1} \frac{\Lambda_2}{\Lambda_2} \right] \left[ \frac{1}{A^2} \left( \Box_4 + \partial^2_{z_1} \right) \right.$$  

$$+ \frac{k_1}{A} \left( 10 - \frac{\alpha}{M^2}(168k^2_2 + 152k^2_1) \right) - \frac{\Lambda_1}{A^2} \frac{\Lambda_2}{A^2} \right] h_{\mu\nu}$$

$$- \delta(z_2) \left[ \frac{8\alpha}{M^2} \frac{k_2}{A} \left( \frac{1}{A^2} \left( \Box_4 + \partial^2_{z_1} \right) \right) \right.$$  

$$+ \frac{k_1}{A} sgn(z_1) \partial_{z_1} - 3k \frac{\Lambda_1}{\Lambda_1} \frac{\Lambda_2}{\Lambda_2} \right] \left[ \frac{1}{A^2} \left( \Box_4 + \partial^2_{z_2} \right) \right.$$  

$$- \frac{1}{2} \left( 1 - \frac{12\alpha}{M^2} \right) \left[ \Box_4 + \partial^2_{z_1} + \partial^2_{z_2} \right.$$  

$$- 4A\left\{ k_1 sgn(z_1) \partial_{z_1} + k_2 sgn(z_2) \partial_{z_2} \right\} \left[ \frac{8\alpha}{M^2} \frac{k_1}{A} \left( \frac{1}{A^2} \left( \Box_4 + \partial^2_{z_2} \right) \right) \right.$$  

$$- \delta(z_1) \delta(z_2) \left[ \frac{160\alpha k_1 k_2}{M^2} \right.$$  

$$- \frac{1}{A^2} \frac{\Lambda_1}{A^2} \right] h_{\mu\nu} = 0 \right.$$  

(41)

where we use Eq. (27). The above equation for $h_{\mu\nu}$ is more simplified in the conformal coordinate,

$$- \frac{1}{2} \left( 1 - \frac{12\alpha}{M^2} \right) \left[ \Box_4 + \partial^2_{z_1} + \partial^2_{z_2} \right.$$  

$$- \delta(z_1) \delta(z_2) \left[ \frac{160\alpha k_1 k_2}{M^2} \right.$$  

$$- \frac{1}{A^2} \frac{\Lambda_1}{A^2} \right] h_{\mu\nu} = 0 \right.$$  

(42)

The bulk contribution in the above equation comes only from the first term (41). The second and third parts of (41) and (42) describe the behavior of the graviton on the corresponding 4 brane, and the last part of (42) just gives a boundary condition of $h_{\mu\nu}$ at the origin(i.e. at the 3-brane), which is consistent with Eq. (30). In general, the bulk equations, the first part of Eq. (41) (or (42)) cannot be solved easily, but the solution for the massless mode is trivial. If we assume $\partial_{z_1} h_{\mu\nu} = \partial_{z_2} h_{\mu\nu} = 0$ and put the background relations Eq. (28)–(30) into the above equation, we obtain

$$\Box_4 \tilde{h}_{\mu\nu}(x) = 0 \right.$$  

(43)

Hence, the massless graviton has the following profile in the bulk,

$$h_{\mu\nu}^0(x, z_1, z_2) = A^2(z_1, z_2) \tilde{h}_{\mu\nu}^0(x) = A^2(z_1, z_2) \epsilon_{\mu\nu} e^{ipx} \right.$$  

(44)

where $\epsilon$ is the polarization tensor of the 4 dimensional graviton.

As the effective 4 dimensional theory would be described by the massless graviton dominantly, let us calculate the effective 4 dimensional Planck mass $M_{P, eff}$
approximately. After integrating the extra dimensions with the 4D part of the metric as \( g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + h_{\mu\nu} \) in Eq. (6), we obtain the 4D effective action as follows,

\[
S_{\text{eff}} = \frac{M_{\text{eff}}}{2} \int d^4x \sqrt{-g^{(4)}} \left[ \tilde{R} + \cdots \right], \tag{45}
\]

where \( \tilde{R} \) is the 4D Ricci scalar. The 4D Planck mass is calculated by reading off the coefficients of ‘\( \Box \)’ in Eq. (41) or (42) and integrating those with respect to \( z_1 \) and \( z_2 \),

\[
M_{\text{eff}}^2 = M_{-}^4 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 A^4 \left[ 1 - \frac{12\alpha}{M^2} k_{\pm}^2 \right] + \frac{1}{16\alpha} A \left( k_1 \delta(z_1) + k_2 \delta(z_2) \right) = \frac{2M^4}{3k_1k_2} \left( 1 + \frac{12\alpha k_{\pm}^2}{M^2} \right), \tag{46}
\]

which gives a finite value. Therefore, we can explain gravitational interactions consistently even in the non-compact 6 spacetime dimensions. Note that our effective 4D Planck mass obtained above from the Einstein equation is the same as the one obtained from the action itself by integrating out \( z_1 \) and \( z_2 \), as given in Eq. (34).

In case of the absence of the 4-branes, i.e. \( k_1^2 + k_2^2 = M^2/(12\alpha) \), the bulk kinetic term in Eq. (41) or (42) does not contribute to the linearized Einstein equation and thus the graviton is not allowed to propagate in the bulk. But by higher order terms in the \( h_{\mu\nu} \) expansion, a certain “gravity interaction” could exist in the bulk even though the mediating particle cannot be defined as the graviton.

Now let us discuss the Kaluza-Klein(KK) modes of the graviton. We will get a bulk solution first using Eq. (41) or (42), and then apply the boundary conditions with the delta functions in the above equations. Eq. (41) is easier to treat rather than Eq. (42) because the former does not have any first derivative terms in the bulk equation. It is possible to separate the variables, \( h_{\mu\nu}(x, z_1, z_2) = \psi(z_1, z_2) e^{ip\cdot x} \), where \( x^\mu \) and \( p^\mu \) are 4D coordinate and momentum, respectively. Then, the bulk part of Eq. (41), which is a two dimensional differential equation, is

\[
\left[ -\partial_{z_1}^2 - \partial_{z_2}^2 + \frac{6(k_1^2 + k_2^2)}{(k_1|z_1| + k_2|z_2| + 1)^2} \right] \psi(z_1, z_2) = m^2 \psi(z_1, z_2), \tag{47}
\]

where \( p^2 = -m^2 \). To separate the bulk variables, let us introduce a new coordinate \((s, t)\),

\[
s \equiv k_1|z_1| + k_2|z_2| + 1 \quad \text{and} \quad t \equiv k_2|z_1| - k_1|z_2| + 1. \tag{48}
\]

Then Eq. (47) becomes

\[
\left( k_1^2 + k_2^2 \right) \left[ -\partial_s^2 - \partial_t^2 + \frac{6}{k_1|z_1| + k_2|z_2| + 1} \right] \psi(s, t) = m^2 \psi(s, t), \tag{49}
\]

where \( \psi(s, t) \equiv \psi(z_1, z_2) \). It is separable as

\[
\left[ -\partial_s^2 + \frac{6}{k_1} \right] \phi_s(s) = m_s^2 \phi_s(s) \tag{50}
\]

\[-\partial_t^2 \phi_t(t) = m_t^2 \phi_t(t), \tag{51}
\]

where \( \phi_s(s), \phi_t(t), m_s^2 \) and \( m_t^2 \) are defined as

\[
\psi(s, t) = \phi_s(s) \phi_t(t) \Rightarrow \frac{m^2}{(k_1^2 + k_2^2)} = m_s^2 + m_t^2. \tag{52}
\]

From Eqs. (50) and (51), we can see that \( m_s^2, m_t^2 \) and so \( m^2 \) should be positive definite, because they could be regarded as a ‘Hamiltonian’ in quantum mechanics, and have positive and flat ‘potentials’, respectively. Hence, they have positive ‘energies’ or eigenvalues. Thus we conclude that there do not exist any tachyonic KK modes.

Eqs. (50) and (51) are easily solved and have the following solutions,

\[
\phi_s(s) = c_1 \sqrt{3} J_{5/2}(m_s s) + c_2 \sqrt{s} Y_{5/2}(m_s s) = \sqrt{\frac{2}{\pi m_s}} \left[ c_1 \left( \frac{3}{(m_s s)^2} - 1 \right) \sin(m_s s) - \frac{3}{m_s^2 s^2} \cos(m_s s) + \frac{3}{m_s^2 s^2} \sin(m_s s) \right], \tag{53}
\]

\[
\phi_t(t) = d_1 \sin(m_t t) + d_2 \cos(m_t t), \tag{54}
\]

where \( J_{5/2} \) and \( Y_{5/2} \) are Bessel functions. \( c_1, c_2, d_1 \) and \( d_2 \) are arbitrary constants but should be determined by the boundary conditions. Note that for large \( m_s s \), we have

\[
\phi_s(s) \approx -\sqrt{\frac{2}{\pi}} \left[ c_1 \sin(m_s s) + c_2 \cos(m_s s) \right], \tag{55}
\]

i.e. KK modes behave like free particles.

On integrating Eq. (41) near the extra dimension axes and the origin, the boundary conditions for the spin-2 graviton modes are given as follows respectively,

\[
\left[ \left( 1 - \frac{12\alpha k_1^2}{M^2} \right) \xi \right. \left. + \frac{8\alpha k_1}{M^2 A} \left( -\xi' + A(k_2 \eta - k_1 \xi) \right) \right]_{z_1=0^+} = 0, \tag{56}
\]
of the four-dimensional space-time. However, the above boundary conditions are not satisfied by the KK massive modes of a function of $s$ only except for $m_s^2 = 0$, i.e., the zero mode. Moreover, the situation would not be different for the more general KK modes of type $\hat{\psi}^{(2)}$. Therefore, even though the bulk equation for the 4D massive gravitons is exactly solvable, there would not exist bulk solutions satisfying the boundary conditions along the extra dimension axes with the simple ansatz for separation of variables, Eq. (48). It is shown that this situation does not become different even without the Gauss-Bonnet term.

IV. CONFINING GRAVITY TO THE SOLITONIC 4-BRANES

Let us discuss the case with the orthogonal 4-branes being regarded as solitonic by choosing the relation between bulk parameters Eq. (31), for which there is no six-dimensional bulk propagation of graviton but the gravity is confined to the solitonic 4-branes as shown in Eq. (41) or (42). In this case, we can rewrite the linearized equation (42) with $\tilde{h}_{\mu\nu} = A^{-3/2}\tilde{\psi}(z_1, z_2)e^{i\nu_x}g_{\mu\nu}$ as

$$-\delta(z_1)\frac{8\alpha k_1}{M^2}\left[\frac{d}{dz_1} + \frac{15}{4}k_2 A^2 + 3k_2 A\delta(z_2)\right]\tilde{\psi} + 24\frac{k_2^2}{M^2}\text{sgn}(z_1)\tilde{A}\delta(z_1)(\partial_{z_1} + \frac{3}{2}k_2 A\tilde{\psi})$$

$$-\delta(z_2)\frac{8\alpha k_2}{M^2}\left[\frac{d}{dz_2} + \frac{15}{4}k_1 A^2 + 3k_1 A\delta(z_1)\right]\tilde{\psi} + 24\frac{k_1^2}{M^2}\text{sgn}(z_2)\tilde{A}\delta(z_2)(\partial_{z_2} + \frac{3}{2}k_1 A\tilde{\psi}) = 0.$$  

8
Then, the above equation is decomposed into two five-dimensional bulk equations of graviton and three boundary conditions:

\[
\left(-\partial_{z_1}^2 + \frac{15}{4} k_1^2 A^2\right) \tilde{\psi} = m^2 \tilde{\psi}, \quad \text{along } z_1 \text{ axis} \tag{69}
\]

\[
\left(-\partial_{z_2}^2 + \frac{15}{4} k_2^2 A^2\right) \tilde{\psi} = m^2 \tilde{\psi}, \quad \text{along } z_2 \text{ axis} \tag{70}
\]

\[
\left(\partial_{z_1} + \frac{3}{2} k_1 A\right) \tilde{\psi} |_{z_1=0} = 0, \tag{71}
\]

\[
\left(\partial_{z_2} + \frac{3}{2} k_2 A\right) \tilde{\psi} |_{z_2=0} = 0, \tag{72}
\]

\[
\left[\left(\partial_{z_1} + \frac{3}{2} k_1 A\right) \tilde{\psi} + \left(\partial_{z_2} + \frac{3}{2} k_2 A\right) \tilde{\psi}\right] |_{z_1=z_2=0} = 0 \tag{73}
\]

where we note that the last equation is a necessary consequence in the case that the third and fourth ones are satisfied and vice versa for our case as will be shown later. From Eqs. (69) and (70), the zero mode solution for \(m^2 = 0\) becomes the same as in the non-solitonic case,

\[
\tilde{\psi}_0 = (k_1 |z_1| + k_2 |z_2| + 1)^{-3/2} \tag{74}
\]

which automatically satisfies the boundary conditions, Eqs. (71-73). Note that the zero mode wave \(\tilde{\psi}_0\) is chosen to be nonvanishing only along the solitonic 4-branes.

On the other hand, solving Eqs. (69) and (70), the KK mode solutions are given as linear combinations of Bessel functions of order two as in the RS case, propagating along solitonic 4-branes located at the \(z_1\) and \(z_2\) axes:

\[
\tilde{\psi}_m = N^{(1)}_m(|z_1| + 1/k_1)^{1/2} Y_2(m(|z_1| + 1/k_1))
+ B_m J_2(m(|z_1| + 1/k_1)), \quad \text{along } z_1 \text{ axis} \tag{75}
\]

\[
\tilde{\psi}_m = N^{(2)}_m(|z_2| + 1/k_2)^{1/2} Y_2(m(|z_2| + 1/k_2))
+ C_m J_2(m(|z_2| + 1/k_2)), \quad \text{along } z_2 \text{ axis} \tag{76}
\]

where \(N^{(1,2)}_m, B_m \) and \(C_m \) are constants to be determined by boundary conditions and normalization. Then, for the KK modes with small masses, i.e., \(m(|z_{1,2}| + 1/k_{1,2}) \ll 1\), the constants \(B_m \) and \(C_m \) are determined approximately from the boundary conditions, Eqs. (71) and (72), as the following,

\[
B_m \simeq \frac{4k_1^2}{\pi m^2}, \quad C_m \simeq \frac{4k_2^2}{\pi m^2}. \tag{77}
\]

Furthermore, from the plane wave normalization such that

\[
1 = \int_0^{z_c} dz_1 |\tilde{\psi}_m|^2 + \int_0^{z_c} dz_2 |\tilde{\psi}_m|^2, \tag{78}
\]

we also obtain the normalization constant \(N^{(1,2)}_m\) as

\[
N^{(1)}_m \sim B_m^{-1} \sqrt{\frac{\pi m}{z_c}} \left(1 + \frac{k_2}{k_1}\right)^{-1/2} = \left(\frac{k_2}{k_1}\right)^{3/2} N^{(2)}_m. \tag{79}
\]

Therefore, the Newtonian potential for two point sources \(m_1, m_2\) separated by \(r\) on the 3-brane is found in a conventional way to be

\[
V(r) \simeq \frac{G \psi^{(1)} \psi^{(2)}}{r} + (16ak_2M^2)^{-1} \int_0^{\infty} \frac{dm}{r} \frac{m_1 m_2 e^{-mr}}{r} |\tilde{\psi}_m(0)|^2
\]

\[
+ (16ak_1M^2)^{-1} \int_0^{\infty} \frac{dm}{r} \frac{m_1 m_2 e^{-mr}}{r} |\tilde{\psi}_m(0)|^2
\]

\[
\simeq \frac{G \psi^{(1)} \psi^{(2)}}{r} \left[1 + \left(\frac{k_2}{k_1}\right)^2 \left(\frac{1}{(k_1 r)^2}\right)\right] \tag{80}
\]

where we used \(G_N = M_p^{-2} = (3k_1k_2)/(4M^4)\) from Eq. (34), \(|\tilde{\psi}_m(0)|^2 \sim m/(k_1 + k_2)\) and the effective 5D gravity couplings for KK modes are read off from coefficients of the 5D kinetic terms in Eq. (68). As a result, corrections due to the KK massive modes are five-dimensional due to the confinement of gravity to the solitonic 4-branes and suppressed in comparison with the Newton force at larger length scales than the curvature scales. Consequently, the confinement of gravity exactly gives rise to two copies of the five-dimensional RSII model. In addition, since gravity does not propagate into the bulk, one fine-tuning condition between bulk parameters, Eq.(31), remains intact at the quantum level of linearized gravity.

**V. THE MASS HIERARCHY WITH THE ORBITFOLD \(T^2/(Z_2 \times Z_2)\)**

We have just shown that there exists two orthogonal 4-brane solution with the nonzero tension of the intersection (or 3-brane) in 6D with the Gauss-Bonnet term. Therefore, it is possible to put another 3-brane in the appropriate position of the bulk as the additional intersection of 4-branes to solve the hierarchy problem as in the RS I case. But, it should be guaranteed that the additional brane should be located at the fixed point of the orbifold to be stable, i.e., the bulk should end at the position of the additional brane. Thus, we assume that there exist the compact extra dimensions with the orbifold \(T^2/(Z_2 \times Z_2)\), where \(Z_2\) acts on each extra dimension once. And let us set the range of the extra coordinates.
as \( z_1 \in (-a, a) \) and \( z_2 \in (-b, b) \). Here we assumed the periodicity of \( 2a(2b) \) along \( z_1(z_2) \) direction. Then, with the \( Z_2 \times Z_2 \) symmetric solution Eq. (26), we need four 3-branes to match the boundary conditions at the four fixed points of the torus, \((z_1, z_2) = (0, 0), (a, 0), (a, b)\) and \((0, b)\). Let us denote the 3-brane tensions as \( \Lambda_1, \Lambda_2, \Lambda_3 \) and \( \Lambda_4 \) in order. And the neighboring two 3-branes are connected to each other by one 4-brane denoted as \( \Lambda_{12}, \Lambda_{23}, \Lambda_{34} \) and \( \Lambda_{41} \) in cyclic order. If the boundary equations in Eqs. (19)-(21) are changed into the following,

\[
e_1 = -M^{-4} \frac{1}{4} (\Lambda_{23} \delta(z_1) + \Lambda_{23} \delta(z_1 - a)), \tag{81}
\]

\[
e_2 = -M^{-4} \frac{1}{4} (\Lambda_{12} \delta(z_2) + \Lambda_{34} \delta(z_2 - b)), \tag{82}
\]

\[
e_3 = -M^{-4} \sum_{i=1}^{4} \frac{1}{A^2} \Lambda_i \delta(z_1 - z_1^{(i)}) \delta(z_2 - z_2^{(i)}), \tag{83}
\]

where \( z_1^{(i)}, z_2^{(i)} \) are positions of the branes, then we obtain the following relations between the 4-brane tensions and similarly for the 3-brane tensions,

\[
\Lambda_{41} = -\Lambda_{23} = k_1 \left( 1 - \frac{12 \alpha k_1^2}{M^2} \right), \tag{84}
\]

\[
\Lambda_{12} = -\Lambda_{34} = k_2 \left( 1 - \frac{12 \alpha k_2^2}{M^2} \right), \tag{85}
\]

\[
\Lambda_1 = \Lambda_3 = -\Lambda_2 = -\Lambda_4 = 96 \alpha k_1 k_2 M^2. \tag{86}
\]

In general, in view of Eqs. (27-30), for fixed bulk parameters, two orthogonal 4-brane tensions should be fine-tuned with the 3-brane tension on their intersection (e.g., between \( \Lambda_{41}(\Lambda_{12}) \) and \( \Lambda_1 \) and etc.). When we adopt the string solution with two solitonic 4-branes, each 3-brane tension can take an arbitrary value of either sign irrespective of the bulk parameters as argued in the previous section, but it should be fine-tuned to one another as shown in Eq. (86). Then, to explain the large mass hierarchy for both the string solution with non-solitonic 4-branes for \( \alpha > 0 \) and the string solution with solitonic 4-branes, we may take the \( \Lambda_3 \) brane with positive tension as the visible brane, whereas the \( \Lambda_1 \) brane can be considered as the hidden brane of the Planck scale. In addition, if \( \Lambda_2 \) brane and \( \Lambda_4 \) branes are considered as the second and the third generation family branes while the \( \Lambda_3 \) brane is interpreted as the first family brane, we may understand the mass hierarchy between families and neutrino oscillation. In this case, the gauge fields are required to live in the bulk. But, we do not digress into this family problem here.

Before considering how the mass hierarchy is generated in this model, let us rewrite the metric as

\[
d s_6^2 = A^2(z_1, z_2)(\eta_{\mu\nu} dx^\mu dx^\nu + dz_1^2 + dz_2^2) = A^2(y_1, y_2) \eta_{\mu\nu} dx^\mu dx^\nu + B^2(y_1, y_2) dy_1^2 + C^2(y_1, y_2) dy_2^2 \tag{87}
\]

by the following bulk coordinate transformations:

\[
d z_1 = \frac{B}{A} dy_1, \quad dz_2 = \frac{C}{A} dy_2, \tag{88}
\]

i.e. \( k_1 z_1 = \text{sign}(y_1)(e^{k_1|y_1|}-1), k_2 z_2 = \text{sign}(y_2)(e^{k_2|y_2|}-1) \). Then, we can have the metric functions in the new coordinate: \( A = (e^{k_1|y_1|} + e^{k_2|y_2|}-1), B = e^{k_1|y_1|} A \) and \( C = e^{k_2|y_2|} A \). So, the 4D Planck mass becomes

\[
M^2_{P, eff} = M^4 \left[ \int_a^b d z_1 \int_{-b}^b d z_2 \left[ A^4 \left( 1 + \frac{12 \alpha}{M^2} \left( \frac{A'}{A} \right)^2 \right) \right] \right]
\]

\[
\left( \left( AA' \right)' + (AA') \right) \left( 1 + (k_1 b_1 + k_2 b_2 - 1)^{-2} \right) \right.
\]

\[
- e^{-2k_1 b_1} - e^{-2k_2 b_2} \right] \tag{89}
\]

where \( (AA')' \) and \( (AA') \) terms in the first line vanish after integration due to the periodicity of the extra dimensions and \( b_1, b_2 \) are the range of the extra dimensions in the new coordinate and in the limit of \( b_1 \to \infty \) and \( b_2 \to \infty \), Eq. (34) can be reproduced. Note that the 4D Planck mass has a finite value if \( k_1 k_2 \neq 0 \), i.e., \( \Lambda_i \neq 0 \) for all \( i \) from Eqs. (30) and (83) and its positiveness is assured for \( |\alpha| < \frac{M^2}{2k_i^2} \) for \( \alpha < 0 \) and any value for \( \alpha > 0 \).

In this new coordinate, let us consider the action for the Higgs scalar field at the \( \Lambda_3 \) brane,

\[
S_{vis} \supset \int d x^4 \sqrt{-g}^{(vis)} \left[ g^{\mu\nu} \partial_\mu H \partial_\nu H - (H^2 - m_3^2)^2 \right],
\]

\[
= \int d x^4 \sqrt{-g}^{(4)} A^4 \left[ A^{-2}(\partial H)^2 - (H^2 - m_3^2)^2 \right], \tag{90}
\]

which becomes of a canonical form by redefining the scalar field as \( \tilde{H} = AH \),

\[
\int d x^4 \sqrt{-g}^{(4)} \left[ (\partial \tilde{H})^2 - (\tilde{H}^2 - m_3^2)^2 \right] \tag{91}
\]

where the Higgs mass parameter on the visible brane is given by
Therefore, when we regard the $\Lambda$ branes, $\Lambda$ Similarly, we obtain the effective mass scales on the other scale:

$$m_3 = \Lambda m_0 = (e^{k_1 b_1} + e^{k_2 b_2} - 1)^{-1/2} m_0.$$  

Similarly, we obtain the effective mass scales on the other branes, $\Lambda_2$ and $\Lambda_4$, respectively:

$$m_2 = e^{-k_1 b_1} m_0, \quad m_4 = e^{-k_2 b_2} m_0.$$  

Therefore, when we regard the $\Lambda_3$ brane as our universe, we can obtain the hierarchy between the Planck scale ($m_0$) and the weak scale ($m_4$) by choosing $k_1 b_1$ and/or $k_2 b_2$ as about 37. It is interesting to see that the mass parameters on the branes are related by

$$\frac{1}{m_2} + \frac{1}{m_4} - \frac{1}{m_3} = \frac{1}{m_0}. \quad (94)$$

where $m_0$ is the mass scale of order the Planck mass at the 3-brane located at $(0,0)$. Since the RHS of Eq. (94) is negligible, the magnitudes of at least two of $m_2, m_3$ and $m_4$ are of the same order, which may allow to a deeper understanding of the family structure. Instead of putting different families in the different 3-branes, one can put all the fermions and the Higgs doublet in the $(a,b)$ brane or in the $(a,0)$ and $(0,b)$ branes with $b \gg a$. Then the $(a,0)$ brane can be used for an intermediate scale brane. However, it is not necessarily needed as proposed in [24] toward a solution of the $\mu$ problem with supersymmetry [25], because the visible sector fields here are already put at the TeV brane. On the other hand, if the visible sector fields with supersymmetric extension are put at the two Planck scale branes at $(0,0)$ and $(a,0)$ with $b \gg a$, then it is needed to introduce intermediate scale brane(s) at $(0,b)$ and $(a,b)$ [24]. In this case, there can be two intermediate scales in principle due to the two 3-branes at the intermediate scales.

**VI. CONCLUSION**

In this paper we obtained the localized gravity on the intersection of two orthogonal non-solitonic or solitonic 4-branes in the Einstein-Gauss-Bonnet theory in 6D. The nonzero 3-brane tension is allowed, which has been possible due to the presence of the Gauss-Bonnet term. The Gauss-Bonnet term can contain a product of two terms with two derivatives of the metric on each term. Therefore, in the EGB theory 3-brane solutions are not possible beyond 6D. To have 3-brane solutions beyond 6D, we have to introduce higher derivative gravity than the Gauss-Bonnet term.

The solution has a warp factor which decreases exponentially at large distance from the origin in the extra dimension. If the $Z_2 \times Z_2$ symmetry is assumed on the bulk space even without non-solitonic 4-branes, one can consider a solution of a 3-brane residing on the intersection of two solitonic 4-branes for the localization of gravity and also for a possible solution of the cosmological constant problem as in the RSII model [18]. With this solution, it is interesting to make the confinement of gravity to the solitonic 4-branes possible, which results in nothing but two copies of the 5D RSII model. In addition, the extra dimension can be compactified. The $T^2/(Z_2 \times Z_2)$ orbifold symmetry gives four fixed points where 3-branes resides on intersections of two 4-branes. In this case, the electroweak scale versus the Planck scale hierarchy can be understood. We also pointed out the possibility of understanding the family structure, which will be studied in a future publication.

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