Non-Commutative Open \((p, q)\)-String Theories

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Abstract

In this paper we make an SL(2,\(\mathbb{Z}\))-covariant generalisation of the noncommutative theories, NCYM and NCOS on the D3-brane, and NCOS and OD1 on the 5-branes in type IIB. These are studied using the supergravity duals of the D3-brane and the D5-brane, corresponding to a probe brane in the relevant background. We then get different theories on the brane described by different \((p, q)\)-strings. Depending on the choice of charges, some of these will be S-dual to each other. In this way, the S-duality on the D3-brane between NCYM and NCOS is reproduced. We also find new cases where NCOS is S-dual to another NCOS. The analysis for the 5-brane, yields new dualities in the case of higher rank 2-forms. For instance, in the rank 6 case, we find a new open \((p, q)\)-string theory, which is S-dual to NCOS. When one instead transforms both the background and the strings on the probe, an SL(2,\(\mathbb{Z}\)) orbit of equivalent theories, described by \((p, q)\)-strings are obtained.

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1 Introduction

Noncommutative theories on branes have been studied a lot in the recent couple of years. In particular, the theories on the D3-brane have received much attention. Noncommutative Yang-Mills,NCYM, was seen to arise in the zero slope limit of the gauge theory on a $Dp$-brane in a magnetic $B$-field [1]. The effect of the $B$-field is to change the boundary conditions for open strings ending on the $Dp$-brane. This, in turn, changes the propagator, from which one can read off the effective open string metric $G_{\mu\nu}$ and the noncommutativity parameter $\Theta^{\mu\nu}$, which lies in the magnetic directions. We also get an effective open string coupling $G_0$. The $\alpha' \to 0$ limit decouples the physics on the brane from the bulk. If we want to make sense of NCYM in this limit, it is the $G_{\mu\nu}, \Theta^{\mu\nu}$ and $G_0$ which should be kept fixed, and this yields the scaling of the closed string fields $g_{\mu\nu}, B_{\mu\nu}$ and $e^{\phi}$ in the limit. The $\alpha' \to 0$ limit decouples all excited string states, leading to a noncommutative field theory.

When one instead considers D-branes in a critical electric background, the situation is different, since the critical electric field leads to an effective finite open string tension $\left(\alpha'_{\text{crit}}\right)^{-1}$, and the entire open string spectrum is kept. The closed string spectrum is still governed by $\alpha'$, so the result is again a decoupled theory on the brane, but this time a noncommutative theory of open strings, NCOS [2, 3]. Furthermore, we now get space-time noncommutativity in the electric directions. It was also seen that NCOS arises in the strong coupling limit of NCYM and therefore the theories are S-dual.

The theories on the brane can also be studied via the dual description with one probe brane in a background of a stack with a large number of source branes. The supergravity duals of the noncommutative theories have been described in [4, 5] for NCYM and in [2, 6] for NCOS. In [4, 5], general electric and magnetic deformations of D-branes are described in a compact way, using the T-duality group. The open string data are calculated using the supergravity duals [1]. The decoupling limit can be seen as a UV limit for the supergravity dual, i.e., the distance between the probe and source branes becomes large.

In this paper we propose an SL(2,Z)-covariant generalisation of the NCYM and NCOS theories. The idea is as follows. Since type IIB string theory is supposed to have an exact nonperturbative SL(2,Z) symmetry, this should manifest itself in the dynamics of the objects of this theory. One is therefore naturally led to consider an SL(2,Z) family of equivalent open $(p, q)$-string theories which are described by SL(2,Z)-invariant open string data. From the view of the supergravity duals, the equivalent theories are obtained by doing an SL(2,Z)-transformation on both the strings on the probe and the background. The equivalence is therefore quite trivial, the family of theories just corresponds to an SL(2,Z) orbit. When discussing a particular theory, we of course just pick a representative of this orbit. In our analysis, we will focus on the inequivalent theories obtained by just transforming the strings on the probe. This is equivalent to just transforming the background. The duality properties of these theories are then examined. On the D3-brane we reproduce the S-duality between NCOS and NCYM, both for vanishing axion [2] where NCOS and NCYM in our picture are described by $(1,0)$ and $(0,1)$-strings, respectively, and for

\[ \text{This has also been done in [3]. In [4] this was done for the open string metric.} \]
rational axion \[[1]\] where in our picture NCOS, described by \((1,0)\)-strings, is S-dual to NCYM, described by \((p,q)\)-strings, with \(p - q \chi = 0\). For arbitrary axion, we find a new duality between ordinary NCOS, described by \((1,0)\)-strings and another NCOS, described by \((p,q)\)-strings. The analysis is also performed for the D5-brane in all 2-form configurations, ranging from rank 2 to rank 6. In the rank 6 case, NCOS turns out to be S-dual to another NCOS, described by \((p,q)\)-strings, with \(p - q \chi = 0\). For the lower rank cases, we also get several examples of NCOS being S-dual to \((p,q)\) NCOS for arbitrary values of the axion.

In section 2 we present the relevant supergravity dual and in section 3 we present the noncommutative open \((p,q)\)-string theories. We end by discussing the results.

2 The supergravity duals

In this section we briefly discuss the 3-brane and 5-brane solutions, which in the near horizon limit become the supergravity duals of the noncommutative theories. As discussed in the next section, we only need the \((F,D3)\) and the D5-solutions.

As mentioned in the introduction, the noncommutative theories on the brane can be described by the open string data

\[
G_{\mu \nu} = \left((g + B)^{-1}\right)^{\mu \nu} = \left((g + B)^{-1}g(g - B)^{-1}\right)^{\mu \nu} \\
\tilde{G}_{\mu \nu} = \left((g + B)^{-1}\right)^{\mu \nu} = -\left((g + B)^{-1}B(g - B)^{-1}\right)^{\mu \nu} \\
G_{\mu \nu} = g_{\mu \nu} - B_{\mu \lambda} B^\lambda_{\kappa \nu} \\
G_\phi = e^\phi \sqrt{\frac{\det(g + B)}{\det g}} = e^\phi \left(\frac{\det G}{\det g}\right)^{\frac{1}{4}}
\]

where \(\Lambda, S\) refer to the antisymmetric and symmetric parts. Usually, the data are calculated in the decoupling limit of the brane in a fixed flat background. Alternatively, one can use the dual holographic picture and calculate the data using the fields of the supergravity solution. This is the approach we will take.

Our starting point are the type IIB supergravity solutions of Cederwall et al. \[2, 3\]. Our conventions for type IIB supergravity can be found appendix A.

2.1 The D3-brane

We start with the D3-brane solution \[12\]. This solution is the most general solution for D3-branes in \(B\)-fields, or equivalently, a bound state of \((F,D1)\)-strings and D3-branes\[4\] and it is parametrised by a complex anti-selfdual 2-form \(F_{(2)}\). The radial dependence of the undeformed D3-brane solution is described by the harmonic function

\[
\Delta = 1 + \frac{R^4}{r^4}
\]

We can then define \(\Delta_\pm = \Delta \pm \nu\), where \(\nu\) describes the deformation. More precisely, \(\nu = 2|\mu|\), where \(\mu = \frac{1}{2}\text{tr}(F_{(2)})^2\). The original deformed solution is an \(SL(2,\mathbb{Z})\)-covariant description of all bound states of D3-branes and \((F,D1)\)-strings. Picking a

\[\text{Later, other solutions have appeared, e.g.} \[4\], \text{but these can be obtained by a rescaling of the coordinates} \[5\].\]
certain string amounts to choosing the scalar doublet $U'$, see appendix A for details. The solution for the (F,D3) bound state in the Einstein frame is [15]

$$
\begin{align*}
 ds^2 &= \Delta_+^\pm \Delta_-^\pm \left( - (dx^0)^2 + (dx^1)^2 \right) + \Delta_+^\pm \Delta_-^\pm \left( (dx^2)^2 + (dx^3)^2 \right) + \Delta_+^\pm \Delta_-^\pm dy^2 \\
 U^1 &= -\frac{i}{\eta} \Delta_+^\pm \Delta_-^\pm \\
 U^2 &= \frac{i}{2} \eta \Delta_+^\pm \Delta_-^\pm \\
 C_1 &= c \sqrt{2\nu} \Delta_-^1 dx^0 \wedge dx^1 \\
 C_2 &= c^{-1} \sqrt{2\nu} \Delta_-^1 dx^2 \wedge dx^3 \\
 e^\phi &= c^2 \sqrt{\frac{\Delta_+}{\Delta_-}} \\
 \chi &= 0
\end{align*}
$$

(3)

where $c$ is an arbitrary real constant (actually $c^2$ is the undeformed asymptotic dilaton) and $\eta = \mu/|\mu|$. As explained in the appendix, the 2-forms, usually written as $B$ and $C$, are now collected in a doublet of 2-forms as $C_1$ and $C_2$. Non-vanishing axion can be obtained by doing an $\text{SL}(2,\mathbb{R})$-transformation which in general changes the string charges $(p^1, p^2) = (p, 0)$ to an arbitrary doublet $(p, q)$

$$
\begin{pmatrix}
 p \\
 q
\end{pmatrix} = \begin{pmatrix}
 1 & p \bar{p} \\
 q/p & \bar{p} \bar{q}
\end{pmatrix} \begin{pmatrix}
 p \\
 0
\end{pmatrix}
$$

where $\bar{p}, \bar{q}$ are real numbers fulfilling $p\bar{q} - q\bar{p} = 1$. So the case $q=0$ yields $\bar{q}=1/p$, and keeping $\bar{p} \neq 0$ gives the solution with general axion. The scalar doublet transforms in the same way as the charge doublet

$$
\begin{pmatrix}
 \tilde{U}^1 \\
 \tilde{U}^2
\end{pmatrix} = \begin{pmatrix}
 U^1 + \bar{p}\bar{q}U^2 \\
 U^2
\end{pmatrix}
$$

and we immediately see that the axion is given by

$$
\chi = p\bar{p}
$$

The doublet of 2-forms becomes

$$
\begin{align*}
 C_1 &= c \sqrt{2\nu} \Delta_-^1 dx^0 \wedge dx^1 \\
 C_2 &= -\chi c \sqrt{2\nu} \Delta_-^1 dx^0 \wedge dx^1 + c^{-1} \sqrt{2\nu} \Delta_-^1 dx^2 \wedge dx^3
\end{align*}
$$

(5)

2.2 The D5-brane

Now turn to the D5-brane solution with arbitrary rank of the $B$-field. In its original form [13], the solution looks quite complicated, but it can be simplified by using a particular basis, see [13] for details. The radial dependence is now given by the harmonic function

$$
\Delta = 1 + \frac{R^2}{r^2}
$$

(6)

Then we can define the deformed harmonic functions

$$
\Delta_{\pm \pm} = \Delta \pm \nu_1 \pm \nu_2 \pm \nu_3
$$

(7)
where $\nu_i$ corresponds to $\frac{1}{2} \nu_i^2$ in \[3\]. Then the solution in the Einstein frame takes the following form

$$
 ds^2 = \Delta_{-+}^{3/4} \left( \Delta_{+-} \Delta_{++} \right)^{1/4} \left( -dx_0^2 + dx_1^2 \right) + \Delta_{--}^{3/4} \left( \Delta_{-+} \Delta_{++} \right)^{1/4} \left( dx_0^2 + dx_1^2 \right) + \Delta_{+-}^{3/4} \left( \Delta_{-+} \Delta_{++} \right)^{1/4} \left( dx_0^2 + dx_1^2 \right) + \left( \Delta_{+-} \Delta_{--} \Delta_{++} \right)^{1/4} dy^2
$$

\begin{align}
(C_1)_{01} &= k^{-1} \sqrt{2 \nu_1 \Delta_1}, & (C_2)_{01} &= -2 k \sqrt{\nu_2 \nu_3} \Delta_1 - k^{-1} \tilde{q} \sqrt{2 \nu_1} \Delta_1 \\
(C_1)_{23} &= k^{-1} \sqrt{2 \nu_2 \Delta_2}, & (C_2)_{23} &= 2 k \sqrt{\nu_1 \nu_3} \Delta_2 - k^{-1} \tilde{q} \sqrt{2 \nu_2} \Delta_2 \\
(C_1)_{45} &= k^{-1} \sqrt{2 \nu_3 \Delta_3}, & (C_2)_{45} &= 2 k \sqrt{\nu_2 \nu_3} \Delta_3 - k^{-1} \tilde{q} \sqrt{2 \nu_3} \Delta_3
\end{align}

\begin{equation}
 e^\phi = k^{-2} \left( \Delta_{+-} \Delta_{++} \Delta_{-+} \right)^{-1/2} \Delta_{++}, \quad \chi = \tilde{q} - k \sqrt{8 \nu_1 \nu_2 \nu_3} \Delta_1
\end{equation}

where $k$ is a real constant related to the asymptotic scalars and $\tilde{q}$ is a real parameter. For general 5-brane charges $(p_1, p_2) = (p, q)$, we instead have a real doublet $(\tilde{p}_1, \tilde{p}_2) = (\tilde{p}, \tilde{q})$, fulfilling $e^\phi \tilde{p} \tilde{p} = 1$. This solution is equivalent to the one obtained without the RR-fields in \[3\]. The rank 2 case was first found in \[7\].

We are going to use the solutions above for the generalised open $(p, q)$-string theories in the next section.

### 3 Open $(p, q)$-string theories

As mentioned, NCYM and NCOS can be described in a setup where fundamental strings are attached to a probe brane, sitting in a background magnetic and electric $B$-field, respectively. In particular, the important quantities, the open string metric, the noncommutativity parameter and the open string coupling are derived from perturbation theory of the fundamental string. The S-duality of NCYM and NCOS has so far been seen as the S-duality of the backgrounds, but alternatively one can get an S-dual description by using the dual strings, i.e., D-strings to describe the theory on the probe. In the same spirit, if we S-dualise both the background and the strings on the probe, we get an equivalent theory. NCYM can thus either be described by F-strings in a background of a magnetic $B$-field or by D-strings in a background of a magnetic $C$-field. Similarly, NCOS can be described by an F-string in a background of an electric $B$-field or by a D-string in a background of an electric $C$-field.

On the type IIB 5-branes, we get a similar picture. NCOS is described by an F-string in a background of an electric $B$-field. By S-dualising both the strings on the probe and the background we get an equivalent description of a D-string in a background of an electric $C$-field, i.e., the OD1 theory.\(^6\)

Since type IIB string theory has an SL(2,$\mathbb{Z}$)-invariance, we can generalise the above. Transforming both the background and the strings on the probe, we get an SL(2,$\mathbb{Z}$) orbit of equivalent noncommutative theories from $(p, q)$-strings. On the D3-brane, the NCOS orbit corresponds to $(p, q)$-strings ending on an $(F,D1,D3)$ bound state. We can pick a representative of this orbit, e.g., $(1,0)$-strings ending on $(F,D3)$.

\(^5\)We are grateful to M. Cederwall and B.E.W. Nilsson for pointing this out to us.

\(^6\)We do not consider the little strings on the 5-branes. Including these in the discussion might alter the equivalence of NCOS and OD1.
Similarly on the 5-branes, the NCOS orbit corresponds to \((p,q)\)-strings ending on \((p',q')\) 5-branes (with lower dimensional 1-branes and/or 3-branes in them according to what kind of 2-form configuration we have), and here we can pick \((1,0)\)-strings ending on D5-branes as a representative. If we want to check the duality properties of the theories, we can do an SL(2,\(\mathbb{Z}\)) transformation on just the background or the strings on the probe. The first approach was pursued in \([14, 15]\) for the D3-brane (and in \([9]\), but they only consider the usual S-duality, which is just one specific SL(2,\(\mathbb{Z}\))-element, on the general background). Here we will concentrate on the second approach and the general SL(2,\(\mathbb{Z}\))-duality.

We want an SL(2,\(\mathbb{Z}\))-covariant description of the open string data relevant for the noncommutative theories. Thus, our starting point is the Einstein metric, the doublet of 2-forms and the scalar doublet instead of the string metric, the \(B\)-field, the dilaton and the axion. As should be clear from the discussion above, when considering the 2-forms, it is important what kind of strings we are using for the description. To be precise, it is the angle between the open string charges and the 2-forms that matters. In the open string data, we should therefore replace \(B\) with \(\hat{p}'C_r\), where \((\hat{p}', p^2) = (p, q)\) are the charges of the open strings ending on the probe brane. The charges with indices downstairs are \(p_r = e_r p^s\), and therefore \((p_1, p_2) = (-q, p)\). In the Im(\(\mathcal{U}^2\))=0 gauge, the \((p, q)\) string tension in the Einstein frame is, in units of \(\frac{1}{\sqrt{2}}\) \([15, 17]\).

\[
[\mathcal{U}' p_r] = \sqrt{e^\phi (p - q \chi)^2 + q^2 e^{-\phi}}
\]

And we get the SL(2,\(\mathbb{Z}\))-covariance by replacing \(e^{i \phi/2}\) everywhere with \([\mathcal{U}' p_r]\). Then the \((p, q)\) open string data become

\[
G^{\mu \nu} = \frac{1}{\mu \nu} \frac{1}{\pi} \int \left( \frac{g^\mu}{g^a} + \frac{e_c}{g^a} \right)^{-1} \mu^\nu = \frac{1}{\mu \nu} \frac{1}{\pi} \int \left( \frac{g^\mu}{g^a} + \frac{e_c}{g^a} \right)^{-1} \mu^\nu
\]

\[
\frac{G^{\mu \nu}}{\alpha'} = \frac{1}{\mu \nu} \frac{1}{\pi} \int \left( \frac{g^\mu}{g^a} + \frac{e_c}{g^a} \right)^{-1} \mu^\nu = \frac{1}{\mu \nu} \frac{1}{\pi} \int \left( \frac{g^\mu}{g^a} + \frac{e_c}{g^a} \right)^{-1} \mu^\nu
\]

\[
G_{\mu \nu} = [\mathcal{U}' p_r] \left( g_{\mu \nu} - \frac{e_c}{g^a} \right) \frac{\partial g^a}{\partial g^\mu} g^\nu \right) \quad G_o = [\mathcal{U}' p_r] \left( \frac{\det G}{\det g^a} \right)^{\frac{1}{2}}
\]

It is easy to see that the above formulae reduce to the usual when \((p, q) = (1, 0)\).

### 3.1 The D3-brane

The supergravity duals are obtained in certain scaling limits, corresponding to \(\alpha' \to 0\). Here, we will use the limits obtained in \([8]\) and translate them to our coordinates. The result for the D3-brane is that the coordinate scalings are the same in both the electric and the magnetic near horizon limits

\[
\hat{x} = \ell \frac{x}{\sqrt{\alpha'}}, \quad u = \ell \frac{r}{\sqrt{\alpha'}}, \quad c \quad \text{fixed}
\]

where \(\ell\) is a fixed length scale. For general charges we can then calculate the data, using fixed coordinates and starting with the electric background with constant

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We use the convention, \(e^{12} = 1\).
axion from the previous section

\[ \frac{G_{\mu\nu}}{\alpha} = \frac{1}{\ell^2} e^{-1} (q^2 + e^4 (p - q \chi)^2) (q^2 \Delta_+ + e^4 (p - q \chi)^2 \Delta_+) - \frac{1}{\ell^3} \eta_{\mu\nu} \]

\[ \Theta^{01} = -\ell^2 \sqrt{2 \nu} e^3 (p - q \chi) (q^2 + e^4 (p - q \chi)^2)^{-1} \]

\[ \Theta^{23} = -\ell^2 \sqrt{2 \nu} e^3 (q^2 + e^4 (p - q \chi)^2)^{-1} \]

\[ G_0 = e^{-2} (q^2 + e^4 (p - q \chi)^2) \]  

(12)

As in [2], \( \Theta \) and \( G_0 \) are \( r \)-independent. In [3] infinite magnetic deformation parameter is obtained in the \( \alpha' \to 0 \) limit and the critical electrical field is obtained in the limit \( \frac{\alpha'}{\ell} \to 1 \). In our case both the limits corresponds to \( \nu \to 1 \). In this sense the electric and magnetic deformations are described in a unified way with our coordinates. In particular, \( \nu \to 1 \) can be interpreted as a critical field limit independently of the field being electric or magnetic. The UV limit corresponds to large \( u \) and therefore we can write the harmonic function as

\[ \Delta = 1 + \frac{\ell^4}{u^4} \sim 1 + \epsilon \]  

(13)

Using the critical field limit, the result for the deformed harmonic functions is

\[ \Delta_- \sim \epsilon, \quad \Delta_+ \sim 2 \]  

(14)

The usual S-duality between electric and magnetic backgrounds now holds for the \((p, q) = (1, 0)\) and the \((0, 1)\) probes but with a background with vanishing axion, since then we get electric and magnetic noncommutativity parameter, respectively. The open string data for the \((1, 0)\)-probe are

\[ \frac{G_{\mu\nu}}{\alpha} = \frac{1}{\sqrt{2 \nu}} \eta_{\mu\nu}, \quad \Theta^{01} = -\ell^2 \sqrt{2} e^{-1} \]

\[ G_0 = e^2, \quad \Theta^{23} = 0 \]  

(15)

and for the \((0, 1)\)-probe we have

\[ \frac{G_{\mu\nu}}{\alpha} = \frac{1}{\ell^2} e^{-1} e^{-\frac{1}{\ell^3}} \eta_{\mu\nu}, \quad \Theta^{23} = -\ell^2 \sqrt{2} e \]

\[ G_0 = e^{-2}, \quad \Theta^{01} = 0 \]  

(16)

and thus the open string coupling for the NCOS is the inverse of that of the NCYM and the result is therefore a strong/weak coupling duality.

If we instead consider probe charges \((1, 0)\) and non-vanishing axion we still get NCOS, \( \Theta^{01} \neq 0 \) and \( \Theta^{23} = 0 \), as we see from [12] with \((p, q) = (1, 0)\). Now do an \( SL(2, \mathbb{Z}) \)-transformation to new charges \((p, q)\), with \( q \neq 0 \). The open string data are still given by [12], but now with the general charges. Then we get a NCYM if \( p - q \chi = 0 \), which can only be achieved if the axion is rational. The open string coupling for the NCOS is \( G_0 = e^2 \), and for the NCYM it is \( G_0 = e^{-2} q^2 \), so if we let \( e \) go to zero or infinity we see that a strongly coupled NCOS corresponds to a weakly coupled NCYM and vice versa, i.e., the theories are S-dual. This was first derived in [11] (and it was also discussed in [22] but only for the theories on the brane and not for the supergravity duals).
For arbitrary axion and general charges for the strings on the probe, we always get NCOS theories if \( p - q \chi \neq 0 \), since then \( \Theta^{01} \neq 0 \). This case certainly includes all irrational values of the axion and arbitrary charges for the strings, and almost all cases of rational values of the axion with arbitrary string charges (the obvious exception being the one set of charges fulfilling \( p - q \chi = 0 \)). When \( q \neq 0 \) we also get \( \Theta^{23} \neq 0 \). In fixed coordinates, the open string data are

\[
\frac{G_{\mu \nu}}{\alpha'} = \frac{c^{-2}(q^2 + c^2(p - q \chi)^2)}{p - q \chi} \eta_{\mu \nu} \\
\Theta^{01} = -\ell^2 \sqrt{2} c^3 (p - q \chi)(q^2 + c^4(p - q \chi)^2)^{-1} \\
\Theta^{23} = -\ell^2 \sqrt{2} c (q^2 + c^4(p - q \chi)^2)^{-1} \\
G_{\alpha} = c^{-2} q^2 + c^2(p - q \chi)^2
\]  

(17)

But two such theories can in general never be S-dual, as noted in [22], since the coupling has the form \( G_{\alpha} = c^{-2} q^2 + c^2(p - q \chi)^2 \), and the second term is always non-vanishing. Consider for instance charges \((1,0)\) and \((p, q)\). Even though the couplings go to zero and infinity, respectively, when \( c \to 0 \), both go to infinity for \( c \to \infty \). There is, however, a special case where an S-duality can be obtained. For arbitrary axion and large \( c \), we can find charges \((p, q)\) such that \( p - q \chi \sim c^{-\beta} \), where \( \beta > 1 \). This statement just corresponds to the fact that for any real number, we can find a rational number as close to the real number as we want. Then the coupling scales with a negative power of \( c \) and the result is a strong/weak coupling case between two NCOS theories. For \( \beta = 2 \) we get by simply replacing \( p - q \chi \) with \( c^{-2} \)

\[
\frac{G_{\mu \nu}}{\alpha'} \simeq \frac{1}{\alpha'} c^{-1}(q^2 + 1) \eta_{\mu \nu} \\
\Theta^{01} \simeq -\ell^2 \sqrt{2} c(q^2 + 1)^{-1} \\
\Theta^{23} \simeq -\ell^2 \sqrt{2} c q(q^2 + 1)^{-1} \\
G_{\alpha} \simeq c^{-2}(q^2 + 1)
\]  

(18)

We would like to stress that this S-duality is somewhat different than the usual, for the following reason. Once we have picked charges \((p, q)\) such that \( p - q \chi \sim c^{-2} \), we want to keep these charges to have a well-defined \((p, q)\) NCOS. If one then increases the coupling of \((1,0)\) NCOS, one must change the axion such that the required scaling is still obeyed. The S-duality thus involves a change in the parameters \( c \) and \( \chi \), i.e., both the scalars of the undeformed solution.

In general it is seen that \( \frac{G_{\mu \nu}}{\alpha'} \) is finite in the UV when \( p - q \chi \neq 0 \). When \( p - q \chi = 0 \), this quantity diverges in the UV. The effective open string tension can be read off from the open string metric

\[
\frac{1}{\alpha'_{\text{eff}}} = \frac{G_{00}}{\alpha'}
\]  

(19)

and hence as expected the effective open string tension is finite for the NCOS-theories but diverges for NCYM.

We have seen that the usual NCOS theory can be described by an orbit of equivalent open \((p, q)\)-string theories, corresponding to an \( \text{SL}(2, \mathbb{Z}) \)-transformation on both the probe strings and the background. We can single out a member of the orbit,
\( e.g. \) the string with charges \((1,0)\), and then study the dual by just transforming the probe charges. For arbitrary axion, we can find charges such that NCOS, described by \((1,0)\)-strings, is S-dual to NCOS, described by \((p,q)\)-strings, and for the particular case \(p - q\chi = 0\), \((1,0)\) NCOS is S-dual toNCYM, described by \((p,q)\)-strings.

### 3.2 \((p,q)\) 5-branes

We can now do the same analysis for the \((p,q)\) 5-branes, using the general rank solution from the previous section. Actually, we only need to consider the D5 background, since the SL(2,Z)-duals are obtained by transforming the charges of the strings on the probe. We consider each rank separately, beginning with the highest rank.

#### 3.2.1 Rank 6

In the near horizon limit, the following quantities are fixed

\[
\hat{x} = \ell \frac{x}{\sqrt{\alpha'}} , \quad u = \ell \frac{r}{\sqrt{\alpha'}} , \quad k \quad \text{fixed} \tag{20}
\]

where \(\ell\) is a fixed length scale. In the rank 6 case the critical field is obtained for \(\nu_1 + \nu_2 + \nu_3 \to 1\). The values of each of the parameters are undetermined, but each must be finite in the rank 6 case. One might worry whether it is allowed to consider SL(2,Z)-dual strings, since D-strings cannot end on D5-branes. However, the background solution correspond to an \((F,D1,D3,D3,D5)\) bound state, which can be seen from the 2- and 4-form potentials. We have electric NS-NS and RR 2-forms, yielding the strings in the bound state. From the 4-forms \(\text{[7]}\) we see that we get a D3-brane along \(x_0, x_1, x_2, x^3\) and another along \(x_0, x_1, x^4, x^5\), and the D-strings can end on these. Define

\[
f = \left( Q \Delta_{++} + 2\sqrt{2} k^2 q \sqrt{\frac{\nu_2 \nu_3}{\nu_1}} \right)^2 + k^4 q^2 \Delta_{--} \Delta_{++} \tag{21}
\]

with

\[
Q = p - q\hat{q} \tag{22}
\]

Then the open string data for arbitrary probe charges become

\[
\begin{align*}
\Theta^{01} &= \ell^2 k \left( 2k^2 q \sqrt{\nu_2 \nu_3} - \sqrt{2\nu_1} Q \right) \Delta_{--} \Delta_{++} \left( f - \Delta_{--} (2k^2 q \sqrt{\nu_2 \nu_3} - \sqrt{2\nu_1} Q)^2 \right) \\
\Theta^{23} &= -\ell^2 k \left( 2k^2 q \sqrt{\nu_1 \nu_3} + \sqrt{2\nu_2} Q \right) \Delta_{++} \Delta_{++} \left( f + \Delta_{--} (2k^2 q \sqrt{\nu_1 \nu_3} + \sqrt{2\nu_2} Q)^2 \right) \\
\Theta^{45} &= -\ell^2 k \left( 2k^2 q \sqrt{\nu_1 \nu_2} + \sqrt{2\nu_3} Q \right) \Delta_{++} \Delta_{++} \left( f + \Delta_{--} (2k^2 q \sqrt{\nu_1 \nu_2} + \sqrt{2\nu_3} Q)^2 \right) \\
\alpha^{1+} &= \frac{1}{\ell^2 k} k^{-1} \Delta_{--} \Delta_{++} f^{-\frac{1}{2}} \left( f - \Delta_{--} (2k^2 q \sqrt{\nu_2 \nu_3} - \sqrt{2\nu_1} Q)^2 \right) \\
\alpha^{2+} &= \frac{1}{\ell^2 k} k^{-1} \Delta_{++} \Delta_{++} f^{-\frac{1}{2}} \left( f + \Delta_{--} (2k^2 q \sqrt{\nu_1 \nu_3} + \sqrt{2\nu_2} Q)^2 \right) \\
\alpha^{3+} &= \frac{1}{\ell^2 k} k^{-1} \Delta_{++} \Delta_{++} f^{-\frac{1}{2}} \left( f + \Delta_{--} (2k^2 q \sqrt{\nu_1 \nu_2} + \sqrt{2\nu_3} Q)^2 \right) \\
G_0 &= k^{-2} (\Delta_{--} \Delta_{++} \Delta_{++})^{-\frac{1}{2}} \Delta_{++} f^{-\frac{1}{2}} \left( f - \Delta_{--} (2k^2 q \sqrt{\nu_2 \nu_3} - \sqrt{2\nu_1} Q)^2 \right) \tag{23}
\end{align*}
\]
\[
\left( f + \Delta_{+-} \left( 2k^2 q \sqrt{\nu_1 \nu_3} + \sqrt{2} \nu_2 Q \right)^{\frac{1}{2}} \right) \left( f + \Delta_{-+} \left( 2k^2 q \sqrt{\nu_1 \nu_3} + \sqrt{2} \nu_2 Q \right)^{\frac{1}{2}} \right)
\]

where the open string metric consists of three two-dimensional blocks. In contrast to the D3-brane case, the \( \Theta \)'s and \( G_\alpha \) are now \( r \)-dependent. We now want to analyse these quantities in the UV limit which we define as follows

\[
\Delta = 1 + \frac{\hat{R}^2}{r^2} \sim 1 + \epsilon \tag{24}
\]

Taking the critical field limit into account, the harmonic functions of the 5-brane become

\[
\Delta_{+-} \sim \epsilon, \quad \Delta_{-+} \sim 1 + \nu_1 + \nu_2 - \nu_3 \equiv a
\]

\[
\Delta_{++} \sim 1 + \nu_1 - \nu_2 + \nu_3 \equiv b, \quad \Delta_{--} \sim 1 + \nu_1 - \nu_2 - \nu_3 \equiv c
\]

and \( a, b \) and \( c \) are finite constants in the genuine rank 6 case. The strategy is now to insert different choices of charges of the strings on the probe and see what kind of theory emerges. We start with the charges \((1, 0)\), corresponding to an \( F \)-string, and then we consider \((p, q)\)-strings to see if we can get an S-dual description. In the \((1, 0)\) case we get

\[
\Theta^{01} = -\ell^2 k \sqrt{2\nu_1}, \quad \Theta^{23} = -\ell^2 k \sqrt{2\nu_2}, \quad \Theta^{45} = -\ell^2 k \sqrt{2\nu_3}
\]

\[
\frac{G_{\mu\nu}}{\alpha'} = \frac{1}{\alpha'} k^{-1} e^{-\frac{1}{2} q \mu \nu}, \quad G_\alpha = k^{-2} e^{-\frac{1}{2}}
\]

As expected we get an NCOS, but with both space-time and space-space noncommutativity. The effective open string tension can be read off from the effective open string metric

\[
\frac{1}{\alpha_\pi} = -\frac{G_{00}}{\alpha'} = \frac{1}{\ell^2} k^{-1} e^{-\frac{1}{2}}
\]

and the result is finite as it should be for an NCOS. The effective open string coupling is also finite and given by the undeformed coupling specified by the asymptotic dilaton.

Now consider general probe charges, which is obtained by an \( \text{SL}(2, \mathbb{Z}) \)-transformation on the \( F \)-string. Starting with the case \( Q = 0 \), which can be obtained for a rational value of the constant part of the axion, the result is

\[
\Theta^{01} = 2\ell^2 k^{-1} q^{-1} c \sqrt{\nu_2 \nu_3} \left( ab - 4\nu_2 \nu_3 \right)^{-1}, \quad \Theta^{23} = -\frac{1}{2} \ell^2 k^{-1} q^{-1} c \sqrt{\nu_2 \nu_3}
\]

\[
\Theta^{45} = -\frac{1}{2} \ell^2 k^{-1} q^{-1} c \sqrt{\nu_1 \nu_2}, \quad \frac{G_{\mu\nu}}{\alpha'} = \frac{1}{\alpha'} e^{-\frac{1}{2} q k \left( 2\nu_1 \nu_2 \nu_3 \right)^{-1} \nu_1 \nu_2}, \quad \frac{G_{\alpha}}{\alpha'} = \frac{4}{\alpha'} e^{-\frac{1}{2} q k \left( 2\nu_1 \nu_2 \nu_3 \right)^{-1} \nu_1 \nu_2}
\]

\[
G_\alpha = 4k^4 q^3 c^{-1} \nu_1 \nu_2 \nu_3 |ab - 4\nu_2 \nu_3|^2
\]

One might worry about potential divergencies and zeros caused by the factor involving \( a \) and \( b \), but writing this in terms of the \( \nu \)'s we get

\[
ab - 4\nu_2 \nu_3 = (1 + \nu_1)^2 - (\nu_2 + \nu_3)^2
\]

\[
ab - 4\nu_2 \nu_3 = (1 + \nu_1)^2 - (\nu_2 + \nu_3)^2
\]
which is strictly positive in the genuine rank 6 case. The theory with the above open string data is also an NCOS, since \( \Theta^{01} \neq 0 \) and the effective open string tension is finite. The strong and weak coupling limits of ordinary NCOS, obtained in the \((1,0)\) case, corresponds to letting \( k \) go to zero and infinity, respectively. Since the effective coupling for the open \((p,q)\)-string in question scales with a positive power of \( k \), we see that the NCOS theory defined by this string is S-dual to ordinary NCOS.

Turning to the case \( Q \neq 0 \), corresponding to arbitrary values of the constant part of the axion, and charges such that \( p - q \tilde{q} \neq 0 \), \( \Theta^{01} \) scales like \( \epsilon \) and therefore goes to zero in the UV limit. The space-space noncommutativity parameters on the other hand are finite (and are just obtained from \([23]\)). The first block of the open string metric diverges whereas the other two are finite. Hence the effective open string tension diverges. The open string coupling scales like \( \epsilon^{-1/2} \) and therefore diverges in the UV. We are thus led to believe that the \((p,q)\)-strings describe a noncommutative Yang-Mills theory in this case. For a general \( D_p \)-brane in \( B \)-fields, the coupling of NCYM is

\[
g_{YM}^2 = G_{\alpha} (\alpha')^{k=2} \sim G_{\alpha} \epsilon^{-1/2}
\]

where we have used the usual scaling of \( \alpha' \). Thus we see that we get a finite coupling for the NCYM theory above. This theory should only be considered as an effective low energy theory, since it is not UV-complete. We, however, get the peculiarity that the open string metric only blows up in the electric directions without noncommutativity, and in this sense the theory differs from ordinary NCYM.

The open string data of the \((1,0)\) NCOS do not depend on the axion, but as we have seen, this is not true for the \( SL(2,\mathbb{Z}) \)-transformed theory. For rational \( \tilde{q} \) and charges fulfilling \( p - q \tilde{q} = 0 \) we get an S-dual NCOS described by \((p,q)\)-strings. For arbitrary \( \tilde{q} \) we can choose charges such that we get NCYM which is, however, not UV-complete.

### 3.2.2 Electric rank 4

Now consider the electric rank 4 case with \( \nu_3 = 0 \). The background corresponds to an \((F_1,D_1,D_3,D_5)\) bound state, with the D3-brane along \( x^0, x^1, x^4, x^5 \). Define

\[
f_4 = Q^2 \Delta_{++} + k^4 q^2 \Delta_{--} \Delta_{++}
\]

In the fixed coordinates, the open string data for arbitrary charges of the strings on the probe are

\[
\begin{align*}
\Theta^{01} &= -\frac{\epsilon^2 k}{\sqrt{2 \nu_1}} Q (Q^2 + k^4 q^2 \Delta_{++})^{-1} \\
\Theta^{23} &= -\frac{\epsilon^2 k}{\sqrt{2 \nu_2}} Q (Q^2 + k^4 q^2 \Delta_{--})^{-1} \\
\Theta^{45} &= -\frac{\epsilon^2 k^3}{\sqrt{2 \nu_1 \nu_2}} Q (Q^2 + 4 k^4 q^2 \nu_1 \nu_2)^{-1} \\
\phi_{+} &= \frac{1}{\nu_1} k^{-1} f_4^{q-1/2} (Q^2 + k^4 q^2 \Delta_{++}) \\
\phi_{-} &= \frac{1}{\nu_2} k^{-1} f_4^{q-1/2} (Q^2 + k^4 q^2 \Delta_{--}) \\
\phi_{\pm} &= \frac{1}{\nu_1 \nu_2} k^{-1} f_4^{q-1/2} (f_4 + 4 k^4 q^2 \nu_1 \nu_2) \\
G_{\alpha} &= k^{-2} \Delta_{+}^{q-1/2} f_4^{q-1/2} (Q^2 + k^4 q^2 \Delta_{++})^{1/2} (Q^2 + k^4 q^2 \Delta_{--})^{1/2} (f_4 + 4 k^4 q^2 \nu_1 \nu_2)^{1/2}
\end{align*}
\]
Now consider the UV limit
\[ \Delta_{-} \sim \epsilon , \quad \Delta_{+} \sim 1 + \nu_1 + \nu_2 \equiv a , \quad \Delta_{+-} \sim 1 + \nu_1 - \nu_2 \equiv b \]  \hfill (33)

For the charges \((1,0)\) we then get
\[ \Theta^{01} = -\ell^2 k \sqrt{2 \nu_1} , \quad \Theta^{23} = -\ell^2 k \sqrt{2 \nu_2} ; \quad \Theta^{45} = 0 \]
\[ \frac{a_{\alpha \mu}}{\eta_{\nu^\mu}} = \frac{1}{b} k^{-1} b^{-\frac{1}{2}} \eta_{\mu \nu} , \quad G_\alpha = k^{-2} b^{-\frac{1}{2}} \]  \hfill (34)

We see that we get space-time as well as space-space noncommutativity, and that the effective open string tension is finite, and hence we get an NCOS as expected.

In the rank 4 case we also have the possibility of having SL(2,\(\mathbb{Z}\))-duals, since \(e.g.\) D-strings can end on the D3-brane in the bound state. For the rank 4 and 2 cases, the axiom in the D5-brane solution is constant, \(\chi = \hat{q}\). In the case of \(p,q\) probe strings with \(Q = 0\), which can be obtained for rational axion, we get
\[ \Theta^{01} = \Theta^{23} = 0 , \quad \Theta^{45} = -\frac{1}{2} \ell^2 k^{-1} q^{-1} b(\nu_1 \nu_2)^{-\frac{1}{2}} \]
\[ \frac{a_{\alpha \mu}}{\eta_{\nu^\mu}} = \frac{1}{b} k^{-1} q a \hat{q}^{-\frac{1}{2}} , \quad \frac{a_{\alpha \mu}}{\eta_{\nu^\mu}} = \frac{1}{b} k^{-1} q a \hat{q}^{-\frac{1}{2}} , \quad \frac{a_{\alpha \mu}}{\eta_{\nu^\mu}} = \frac{1}{b} k^{-1} q a \hat{q}^{-\frac{1}{2}} \]
\[ G_\alpha = 2k^{2} q^{2} b^{-\frac{1}{2}} \sqrt{\nu_1 \nu_2} \]  \hfill (35)

Thus we only get space-space noncommutativity and we also see that the effective open string tension diverges, so the resulting theory appears to be a NCYM. Regarded as a theory on the D5-brane, the Yang-Mills coupling goes to zero in the UV, yielding a free theory. However, it is seen that the open string metric blows up on the D3-brane, in agreement with the fact that the dual strings only can end on the D3-brane. The Yang-Mills coupling on the D3-brane equals the open string coupling, so it seems reasonable that we get a well-defined NCYM on the D3-brane. From the \(k\)-scaling of the couplings we see that we get a strong/weak coupling relation between this NCYM theory and NCOS, but it is not clear how this should be understood, since we are dealing with four- and six-dimensional theories.

For arbitrary axion and charges \((p,q)\) for the strings on the probe, such that \(Q \neq 0\), the open string data become
\[ \Theta^{01} = -\ell^2 k \sqrt{2 \nu_1} Q (Q^2 + k^4 q^2 a)^{-1} , \quad \Theta^{23} = -\ell^2 k \sqrt{2 \nu_2} Q^{-1} \]
\[ \Theta^{45} = -2\ell^2 k^3 q \sqrt{\nu_1 \nu_2} b (Q^2 b + 4k^4 q^2 \nu_1 \nu_2)^{-1} , \quad \frac{a_{\alpha \mu}}{\eta_{\nu^\mu}} = \frac{1}{Q^2} k^{-1} b^{-\frac{1}{2}} (Q^2 + k^4 q^2 a) \]
\[ G_\alpha = k^{-2} b^{-1} (Q^2 + k^4 q^2 a)^{\frac{1}{2}} (Q^2 b + 4k^4 q^2 \nu_1 \nu_2)^{\frac{1}{2}} \]  \hfill (36)

Thus we both get space-time and space-space noncommutativity. Furthermore, the effective open string tension is finite, so we get an NCOS. In general this NCOS is not S-dual to the ordinary NCOS obtained from the \((1,0)\) charges, since both couplings diverge when \(k\) goes to zero. As for the D3-brane, we have a special case, for certain values of the charges. To be specific, the values should be such that \(Q \) scales like \(k^\beta\), with \(\beta > 1\). The coupling of the \((p,q)\) NCOS then scales with a positive power of \(k\), yielding a weakly coupled theory when the \((1,0)\) NCOS is strongly coupled and vice versa. Hence the two NCOS theories are S-dual.
We have seen that for arbitrary axion we can choose charges such that (1,0) NCOS is S-dual to \((p,q)\) NCOS. In the special case \(p - q\chi = 0\), which can be obtained for rational axion, the S-dual seems to be NCYM on the D3-brane sitting in the D5-brane.

### 3.2.3 Magnetic rank 4

This case corresponds to \(\nu_1 = 0\) and the background is then a \((D1,D3,D3,D5)\) bound state, where the D3-branes share the electric directions as in the rank 6 case. Define

\[
h_4 = Q^2 \Delta_\pm + k^4 q^2 \Delta_{+} \Delta_{-}
\]

In the fixed coordinates, the open string data then become

\[
\begin{align*}
\Theta^{01} &= 2\ell^2 k^3 q \sqrt{\nu_2 \nu_3} \Delta_- (h_4 - 4k^4 q^2 \nu_2 \nu_3)^{-1} \\
\Theta^{23} &= -\ell^2 k \sqrt{2 \nu_2} Q (Q^2 + k^4 q^2 \Delta_+)^{-1} \\
\Theta^{45} &= -\ell^2 k \sqrt{2 \nu_3} Q (Q^2 + k^4 q^2 \Delta_+)^{-1} \\
G^{\mu \nu} &= \frac{1}{\ell^2} k^{-1} h_4^{-\frac{1}{2}} (h_4 - 4k^4 q^2 \nu_2 \nu_3)^{-1} \\
\Phi^{\mu \nu} &= \frac{1}{\ell^2} k^{-1} h_4^{-\frac{1}{2}} (Q^2 + k^4 q^2 \Delta_+) \\
G_\alpha &= k^{-2} \Delta_- \frac{1}{2} (h_4 - 4k^4 q^2 \nu_2 \nu_3)^{-\frac{1}{2}} (Q^2 + k^4 q^2 \Delta_+)^{-\frac{1}{2}}
\end{align*}
\]

The open string data are then analysed in the UV limit

\[
\Delta_- \sim \epsilon, \quad \Delta_+ \sim 1 + \nu_2 - \nu_3 \equiv a, \quad \Delta_\pm \sim 1 - \nu_2 + \nu_3 \equiv b
\]

For the \((1,0)\) probe string we get

\[
\begin{align*}
\Theta^{01} &= 0, \quad \Theta^{23} = -\ell^2 k \sqrt{2 \nu_2}, \quad \Theta^{45} = -\ell^2 k \sqrt{2 \nu_3} \\
G^{\mu \nu} &= \frac{1}{\ell^2} k^{-1} h_4^{-\frac{1}{2}} \eta_{\mu \nu}, \quad G_\alpha = k^{-2} \epsilon^{-\frac{1}{2}}
\end{align*}
\]

We get space-space noncommutativity and a diverging effective open string tension, suggesting a NCYM. It is seen that the open string metric blows up on the entire D5-brane. We also see that we get a finite Yang-Mills coupling on the D5-brane and therefore we get an effective NCYM theory described by the \((1,0)\) strings.

Now we look at the SL(2,\(\mathbb{Z}\))-transformed \((p,q)\)-strings. Starting with rational axion and \(Q = 0\), the open string data are

\[
\begin{align*}
\Theta^{01} &= \ell^2 k^{-1} q^{-1} \sqrt{\nu_2 \nu_3}, \quad \Theta^{23} = \Theta^{45} = 0 \\
G^{\mu \nu} &= \frac{1}{\ell^2} k q (ab)^{-\frac{1}{2}}, \quad G^{\alpha \mu} = \frac{1}{\ell^2} k q b (ab)^{-\frac{1}{2}}, \quad G^{\alpha \nu} = \frac{1}{\ell^2} k q a (ab)^{-\frac{1}{2}} \\
G_\alpha &= \sqrt{2} k q^2
\end{align*}
\]

where we have used the following UV-behaviour for a term entering many of the quantities

\[
\Delta_+ \Delta_- - 4 \nu_2 \nu_3 = 1 - (\nu_2 + \nu_3)^2 + 2 \frac{R^2}{u^2} + \frac{R^4}{u^4} \sim 2 \epsilon
\]

13
We only get space-time noncommutativity. Furthermore, the effective open string tension is finite so we get an NCOS. From the scaling of the coupling with respect to $k$, it is seen that this NCOS is S-dual to the NCYM above.

For the $(p,q)$ probe string with arbitrary axion and $Q \neq 0$, we get

$$
\Theta^{01} = 2\ell^2 k^3 q \sqrt{\nu_2 \nu_3} (Q^2 + 2k^4 q^2 a)^{-1}, \quad \Theta^{23} = -\ell^2 k \sqrt{2\nu_2} Q (Q^2 + k^4 q^2 b)^{-1}
$$

$$
\Theta^{45} = -\ell^2 k \sqrt{2\nu_3} Q (Q^2 b + 2k^4 q^2 a)^{-1}, \quad \frac{G_{ab}}{a} = \frac{1}{\ell^2 k^{-3} q^{-1}} (ab)^{-\frac{1}{2}} (Q^2 + 2k^4 q^2 b), \quad \frac{G_{ab}}{b} = \frac{1}{\ell^2 k^{-3} q^{-1}} (ab)^{-\frac{1}{2}} (Q^2 + k^4 q^2 a)
$$

$$
G_0 = \sqrt{2k^{-2}} (ab)^{-\frac{1}{2}} (Q^2 + k^4 q^2 a)^{\frac{1}{2}} (Q^2 + k^4 q^2 b)^{\frac{1}{2}} (44)
$$

We get space-time as well as space-space noncommutativity and the effective open string tension is finite so we get an NCOS. In general this NCOS is not S-dual to the NCYM obtained with $(1,0)$-strings, since both couplings diverge when $k$ goes to zero. But again, having charges, yielding a $Q$ which scales like $k^{1/2}$, $\beta > 1$, the NCOS coupling will scale with a positive power of $k$ and the two theories are therefore S-dual.

### 3.2.4 Electric rank 2

This case corresponds to $\nu_2 = \nu_3 = 0$. The background is an $(F,D1,D5)$ bound state and this is the only case where there is no D3-brane present in the bound state. We might therefore run into problems when using general strings to describe the theory on the brane. Take a $(D1,NS5)$ bound state as example. The F-string cannot end on the NS5-brane, but it can end on the D-string sitting inside the NS5-brane. A general $(p,q)$-string, with $q \neq 0$, may however end on the NS5-brane. By $SL(2,\mathbb{Z})$-covariance, a D-string may end on the F-string in an $(F,D5)$ bound state. When the string charge in the bound state is the opposite as that of the probe string we therefore get the above kinematical constraint. $SL(2,\mathbb{Z})$-covariantising this statement gives the result that when $\epsilon_{rs} \hat{p}^r \hat{p}^s = \hat{p}^r \hat{p}_r = 0$, where $\hat{p}_r$ are the string charges in the bound state, the $(p,q)$-strings can only end on the strings in the 5-brane. Otherwise, the strings may end on the entire 5-brane. Since the charges of the strings in the bound state can be obtained from the 2-forms of the brane solution, we arrive at the following requirement for when a $(p,q)$-string only can end on the strings in the 5-brane

$$
p^r C_r = 0
$$

As seen from the D5-brane solution in the electric rank 2 case, this corresponds to

$$
p - q\chi = Q = 0.
$$

In fixed coordinates, the open string data are

$$
\Theta^{01} = -\ell^2 k \sqrt{2\nu_2} Q (Q^2 + k^4 q^2 a)^{-1}, \quad \Theta^{23} = \Theta^{45} = 0
$$

$$
\frac{G_{ab}}{a} = \frac{1}{\ell^2 k^{-1}} \Delta_{+}^{-\frac{1}{2}} (Q^2 + k^4 q^2 a)(Q^2 + k^4 q^2 a)^{-\frac{1}{2}}
$$

$$
\frac{G_{ab}}{b} = \frac{G_{ab}}{a} = \frac{1}{\ell^2 k^{-1}} \Delta_{-}^{-\frac{1}{2}} (Q^2 + k^4 q^2 a)^{\frac{1}{2}} (45)
$$

$$
G_0 = k^{-2} \Delta_{+}^{-\frac{1}{2}} (Q^2 + k^4 q^2 a)^{\frac{1}{2}} (Q^2 + k^4 q^2 a)^{\frac{1}{2}}
$$
In this case, the UV limit is
\[ \Delta_- \sim \epsilon, \quad \Delta_+ \sim 2 \] (46)

For (1,0) probe strings the open string data become
\[ \Theta^{01} = -\ell^2 k \sqrt{2}, \quad \Theta^{23} = \Theta^{45} = 0 \]
\[ G_{\alpha\beta} = \frac{1}{\sqrt{2} \epsilon} k \eta_{\alpha\beta}, \quad G_{0} = \frac{1}{\sqrt{2} k} \] (47)

With space-time noncommutativity and finite effective open string tension we get an NCOS as expected.

Consider the case \( Q = 0 \). Then the open strings can only end on the strings sitting in the 5-brane. In this case, the open string data are
\[ \Theta^{01} = \Theta^{23} = \Theta^{45} = 0 \]
\[ G_{\alpha\beta} = \frac{1}{\sqrt{2} \epsilon} k \eta_{\alpha\beta}, \quad G_{0} = \frac{1}{\sqrt{2} k} \] (48)

The open string metric blows up in the electric directions and shrinks in the magnetic directions. From the scaling of the coupling and (30), we see that we get a finite Yang-Mills coupling in two dimensions. All noncommutativity parameters vanish and we thus get an effective description in terms of a 2d YM theory on the strings in the 5-brane.

If we want to look for a six-dimensional S-dual of the NCOS, we therefore have to consider the case \( Q \neq 0 \), and then the open string data are
\[ \Theta^{01} = -\ell^2 k \sqrt{2} Q (Q^2 + 2k^4 q^2)^{-1}, \quad \Theta^{23} = \Theta^{45} = 0 \]
\[ G_{\alpha\beta} = \frac{1}{\sqrt{2} \epsilon} k^{-1} (Q^2 + 2k^4 q^2), \quad G_{0} = \frac{1}{\sqrt{2} k} k^{-1} \] (49)

We get space-time noncommutativity and finite effective open string tension. Even in this case we get an S-dual NCOS when the charges has values such that \( Q \) scales like \( k^\beta \), with \( \beta > 1 \). In particular, when \( \beta = 2 \), the open string data take the simple form
\[ \Theta^{01} = -\ell^2 \sqrt{2} Q (Q^2 + 2k^4 q^2)^{-1}, \quad \Theta^{23} = \Theta^{45} = 0 \] (50)
\[ G_{\alpha\beta} = \frac{1}{\sqrt{2} \epsilon} k (1 + 2q^2), \quad G_{0} = \frac{1}{\sqrt{2} k} k \] (51)

### 3.2.5 Magnetic rank 2

This case corresponds to \( \nu_1 = \nu_2 = 0 \). The background is a (D3,D5) bound state. In fixed coordinates the open string data are
\[ \Theta^{45} = -\ell^2 k \sqrt{2} Q (Q^2 + k^4 q^2 \Delta_-)^{-1}, \quad \Theta^{01} = \Theta^{23} = 0 \]
\[ G_{\alpha\beta} = G^{\alpha\beta} = \frac{1}{\sqrt{2} \epsilon} k^{-1} \Delta_-^{-\frac{1}{2}} (Q^2 + k^4 q^2 \Delta_+)^{\frac{1}{2}} \]
\[ G_{0} = k^{-2} \Delta_-^{-\frac{1}{2}} (Q^2 + k^4 q^2 \Delta_+)^{\frac{1}{2}} (Q^2 + k^4 q^2 \Delta_+)^{-\frac{1}{2}} \] (52)
The UV limit is the same as in the electric rank 2 case. For a \((1, 0)\) probe we get

$$
\Theta^{45} = -\ell^2 k \sqrt{2} \epsilon, \quad \Theta^{01} = \Theta^{23} = 0
$$

$$
G_{\mu \nu} = \frac{1}{\ell^2} k^{-1} \epsilon^{-\frac{1}{2}} \eta_{\mu \nu}, \quad G_o = k^{-2} \epsilon^{-\frac{1}{2}}
$$

(53)

We only have space-space noncommutativity and the effective open string tension diverges. From the scaling of \(G_o\), it is seen that we get a finite Yang-Mills coupling. We therefore have an effective description in terms of NCYM. For a \((p, q)\)-string probe with rational axion and \(Q = 0\), the open string data become

$$
\Theta^{01} = \Theta^{23} = \Theta^{45} = 0
$$

$$
G_{\mu \nu} = \frac{G_{\mu \nu}}{Q} = \frac{k q}{2} k q \epsilon^{-\frac{1}{2}}, \quad G_o = k q \epsilon^{-\frac{1}{2}}
$$

(54)

All \(\Theta\) parameters vanish, the effective open string tension diverges, the open string metric only blows up on the D3-brane and the open string coupling is finite so the result looks like an ordinary Yang-Mills theory on the D3-brane sitting in the D5-brane.

On the other hand, for arbitrary axion and \(Q \neq 0\), we get

$$
\Theta^{45} = -\ell^2 k \sqrt{2} v Q^{-1}, \quad \Theta^{01} = \Theta^{23} = 0
$$

$$
G_{\mu \nu} = \frac{G_{\mu \nu}}{Q} = \frac{k^{-1} (Q^2 + 2 k^4 q^2)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}}{Q}, \quad G_o = k^{-2} Q (Q^2 + 2 k^4 q^2)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}
$$

(55)

We get space-space noncommutativity, a diverging effective open string tension and finite Yang-Mills coupling. The resulting theory is therefore a NCYM on the D5-brane.

In the magnetic rank 2 case, open D3-branes should be light, due to the presence of the critical electric 4-form potential. The proper description of the theory on the 5-brane is therefore not in terms of the open strings but instead the open D3-branes, yielding the OD3 theory.

4 Discussion

In this paper we have constructed an SL\((2, \mathbb{Z})\)-covariant generalisation of NCOS and NCYM on the D3-brane and NCOS and OD1 on the 5-branes in type IIB. In particular, we have obtained SL\((2, \mathbb{Z})\)-covariant expressions for the effective open string metric, the noncommutativity parameter and the effective open string coupling. We then inserted the fields of the relevant D3-brane and D5-brane supergravity backgrounds in these expressions, yielding a holographic picture of the theories. As a result we get noncommutative open \((p, q)\)-string theories, with the string charges appearing explicitly in the open string data. As mentioned, it is the relative angle between the background 2-form doublet and the doublet of charges of the strings on the probe which determines what kind of theory we get. Starting with ordinary NCOS, we get an SL\((2, \mathbb{Z})\) orbit of equivalent \((p, q)\)-string theories by transforming both the background and the strings on the probe. On the other hand, we get
inequivalent theories, if we just rotate the strings on the probe or the background. The duality properties of the theories were also examined and we get agreement with previous results for the D3-brane [1], namely for rational axion, NCOS is S-dual to a NCYM, described by a string with charges fulfilling $p - \chi q = 0$. For arbitrary axion and charges such that $p - q \chi \neq 0$, we always get an NCOS and for certain charges $(p, q)$, this NCOS is S-dual to ordinary NCOS.

Similarly, on the 5-branes, NCOS and the OD1 theory are equivalent and are part of an SL$(2, \mathbb{Z})$ orbit of equivalent open $(p, q)$-string theories, which can be obtained by transforming both the background and the strings on the probe. The ordinary NCOS and the OD1 are obtained in the electric rank 2 case. We can get an S-dual NCOS for arbitrary values of the axion and charges such that $p - q \chi \neq 0$. A property which holds for most of the NCOS theories considered in this paper. All other 2-form configurations were also analysed. In the rank 6 case, a $(1, 0)$ NCOS with rational axion is S-dual to a $(p, q)$ NCOS when $p - q \chi = 0$. For arbitrary axion and $p - q \chi \neq 0$, the dual theory has an effective description in terms of NCYM. In the electric rank 4 case with rational axion, $(1, 0)$ NCOS has an SL$(2, \mathbb{Z})$-dual description in terms of NCYM on the D3-brane in the D5-brane. This theory is described by strings with charges fulfilling $p - q \chi = 0$. In the magnetic rank 4 case, NCYM is S-dual to NCOS for arbitrary values of the axion. In the magnetic rank 2 case, we only have NCYM descriptions. In the latter case, the strings are not the proper objects to describe the theory on the brane, since they are not the lightest objects. This role is instead played by D3-branes. The NCYM theories obtained in the magnetic rank 2 case are therefore only effective descriptions of the OD3-theory. A proper description of this theory would require quantising the D3-branes, something which is not yet possible. A first step could instead be to generalise the open string data to the case of 5-branes.

When we have a critical field, the asymptotic closed string metric gives a hint as to which objects become light. In the magnetic rank 2 case, the metric corresponds to a smeared out 3-brane [3]. In this case we get light D3-branes and the proper theory is OD3. Actually, this is the only case where the asymptotic closed string metric corresponds to a smeared out 3-brane. In all other cases, the metric is that of a smeared out string. This is in agreement with the fact that we can get light strings in these cases. According to this, there should not be light D3-branes in the higher rank cases, a statement which is confirmed by the behaviour of the 4-form. In the critical field limit, the 4-form diverges logarithmically in the UV [7], and therefore we cannot get a cancellation of the divergence of the usual D3-brane tension.

It also follows from the above that the deformed 5-brane solution cannot describe the OD5 theory, since we never get an asymptotic closed string metric which blows up on the entire 5-brane. The only way to get SO$(1, 5)$ isometry on the 5-brane is by setting all the deformation parameters $\nu_i$ to zero. Indeed, it can be shown that the NS5 supergravity solution with a 6-form RR-potential on the brane, which can be obtained by a double T-duality on the (NS5,D3) bound state, corresponds to an ordinary $(p, q)$ 5-brane [8]. The 6-form is just the potential for the 7-form which is Hodge dual to the 3-form giving the D5-brane charge. The critical 6-form is obtained by choosing the integration constant in the harmonic function to be

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[8] This was also noted in [4].
zero. By doing this, one also ends up with an asymptotic metric which instead of becoming minkowskian, blows up on the entire 5-brane. It can also be seen from the supergravity solution that we get a finite effective tension for the D5-branes which therefore become light. Similarly, for the ordinary undeformed Dp-brane solutions, the Dp-branes themselves become light due to the compensating electric \((p + 1)\)-form potential, which becomes critical in the UV if we choose the constant in the harmonic function to be zero. By doing this, the full metric becomes \(\text{AdS}_{p+2} \times S^{8-p}\), and therefore the asymptotic metric blows up on the entire Dp-brane. It is not clear to us, though, how these light branes should be interpreted.

Recently, ideas related to our open string theories have been discussed in [24] from the world volume point of view and in [23] from the supergravity duals. In these papers new theories of open D-branes are proposed, e.g., on the D3-brane, a theory of open D-strings should be S-dual to NCOS. In the latter paper, the definition of the theories also involves considering large noncommutativity parameters. It remains to be seen how our results are related to the work of Lu [24] and Larsson and Sundell [23].

There is another important aspect which should be mentioned. It is known that little strings live on the 5-branes [22], and that these will dominate in certain regimes of the noncommutative theories, at least in the rank 2 case [24]. One could speculate that the D-strings in [23] play the same role on the D3-brane as the little strings on the 5-branes, since both strings can be regarded as solitons of theNCYM on the brane. In the present work the little strings have not been considered. An analysis including the little strings might modify some of the conclusions reached in this paper.

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A Type IIB supergravity

Type IIB supergravity in ten dimensions has an \(\text{SL}(2,\mathbb{R})\) invariance (which is broken to \(\text{SL}(2,\mathbb{Z})\) by quantum effects) and contains the following fields: the metric, 2 scalars (the dilaton \(\phi\) and the axion \(\chi\)), the NS-NS 2-form potential \(B\), the R-R 2-form potential \(C\) and the R-R 4-form potential \(C_{(4)}\). The 2-forms can be collected in an \(\text{SL}(2,\mathbb{R})\) doublet \(C_r\), where \(r=1,2\) corresponds to the NS-NS and R-R 2-forms respectively. There exists a formulation with manifest \(\text{SL}(2,\mathbb{R})\) covariance [28, 29]. Here we use the notation of [21, 30]. The scalars can be described by a complex doublet \(U^r\), with \(\tau = U^1/U^2 = \chi + i e^{-\phi}\). The scalar doublet fulfills the \(\text{SL}(2,\mathbb{Z})\)-invariant constraint

\[
\frac{1}{2} \epsilon_{rs} U^r \bar{U}^s = 1 \tag{56}
\]

The 2-form doublet has a 3-form doublet of field strengths \(H_{(3)r} = dC_r\), which can be combined with the scalar doublet into a complex 3-form

\[
\mathcal{H}_{(3)} = U^r H_{(3)r}, \quad H_{(3)r} = \epsilon_{rs} \text{Im}(U^s \bar{H}_{(3)}}
\]

18
From the scalar doublet we can construct the Mauher-Cartan 1-forms $P$ and $Q$

$$Q = \frac{i}{2} \epsilon_{r s} dU^r U^s, \quad P = \frac{i}{2} \epsilon_{r s} dU^r U^s$$

(57)

The equations of motion can now be written as

\begin{align*}
D \ast P + i \frac{1}{4} H_3 \wedge \ast H_3 &= 0 \\
D \ast H_3 + i P \wedge \ast H_3 - i H_5 \wedge H_3 &= 0 \\
D H_3 + i \ast H_3 \wedge P &= 0 \\
d H_5 - \frac{i}{4} H_3 \wedge \ast H_3 &= 0 \\
R_{M N} = 2 \tilde P_{(M} P_{N)} + \frac{i}{2} \tilde H_{(M}^{\ RST} H^{N) RST} - \frac{1}{4 \pi} g_{M N} \tilde H_{RST} H^{RST} + \frac{1}{4 \pi} H_{(RST} H^{N) RSTU}
\end{align*}

(58)

The first two equations are the equations of motion for $P$ and $H_3$, respectively. The following two are the Bianchi identities for the 3-forms and the 5-form. The last line is the Einstein equations.
References


[18] R. Cai and N. Ohta, (F1, D1, D3) bound state, its scaling limits and SL(2,Z) duality, hep-th/0007106.


