I. INTRODUCTION

There are two ways to obtain classical solutions in supersymmetric theories: one can either solve the equations of motion derived from the effective action which, for bosonic fields, are of the second order in derivatives, or "spinor Killing equations" resulting from the requirement that supersymmetry variations of the fermionic fields vanish. The latter are of the first order in derivatives. The first method, in general, provides a larger set of solutions which can include non-supersymmetric ones. The second way leads to supersymmetric solutions with partially broken supersymmetry.

In this paper, we discuss the stringy effects in the black-hole solutions, namely, effects of higher genus topologies of the world sheet, by following the second approach.

The $4D$ string effective action is obtained by dimensional reduction of $6D, N = 1$ supersymmetric string effective action on the two-torus. For this class of compactifications, $4D$ theory is $N=2$ supergravity interacting with matter. As a concrete example of this construction, we have in view heterotic string theory compactified on the manifold $K3 \times T^2$ or its suitable orbifold limit, although we do not rely on any specific properties of this model.

Due to $N=2$ supersymmetry, prepotential of the theory receives only one-string-loop corrections (from the string world sheets of torus topology) [1,2]. There are explicit calculations of the loop-corrected prepotential [1–4], but for the present study only its general structure is important.

First, solving the string-tree-level "spinor Killing equations" for gravitino and gaugini, we obtain the known spherically-symmetric magnetic black hole solutions [5,6]. The tree-level gauge couplings are proportional to the inverse effective string coupling and decrease at small distances from the origin, so that loop corrections to the gauge couplings are important in this region. As a technical simplification, we consider tree-level solutions with equal magnetic charges, in which case the tree-level moduli are constants. Next, using the loop-corrected prepotential, in the first order in string coupling, we find the loop-corrected gauge couplings, solve the Maxwell equations for the gauge fields and the loop-corrected "spinor Killing equations" for the moduli.

We obtain a family of solutions for the loop corrections to the tree-level metric and dilaton of magnetic black hole which depend on one parameter. Requiring that the loop-corrected magnetic black hole is extremal BPS saturated, we fix the constant, which in this case is proportional to the Green-Schwarz function which enters the Kaehler potential for the moduli. The one-parameter set of supersymmetric solutions of the "spinor Killing equations" is contained in the two-parameter set of solutions of the Einstein-Maxwell equations and the equations of motion for the moduli derived from the loop-corrected effective action.

II. HETEROYTIC VERSUS $N = 2$ PICTURES

$4D$ effective string theories obtained by two-torus compactification of $6D N = 1$ string effective actions share a number of universal properties. The resulting theory is $N = 2$ supersymmetric dilatonic supergravity interacting with matter. The bosonic part of the universal sector of this theory written in a holomorphic section admitting the prepotential in the standard form of $N = 2$ special geometry [7–14] is
$$I_4 = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} R + (\bar{N}_{IJ} F^{-I} F^{-J} - N_{IJ} F^{+I} F^{+J}) + k_{ij} \partial_{\mu} z^i \partial^\mu z^j + \ldots \right]. \quad (1)$$

Here $N_{IJ}$ are the gauge coupling constants,

$$F_{\mu \nu}^\pm = \frac{1}{2} (F_{\mu \nu} \pm \frac{i}{2} \epsilon_{\mu \nu \rho \lambda} F^{\rho \lambda}) = \frac{1}{2} (F_{\mu \nu} \pm \frac{i}{2} \sqrt{-g} F_{\mu \nu}).$$

Here $^* F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \lambda} F^{\rho \lambda}$, where $\epsilon_{\mu \nu \rho \lambda}$ is the flat antisymmetric tensor.

The moduli $z_i$ are identified below, and dots stand for contributions from other moduli. Here and below $I, J = 0, \ldots, 3$ and $i, j = 1, 2, 3$. The moduli $z^i$ and the vector fields are identified by comparing the action (1) with that resulting from compactification of the universal sector of the 6D theory

$$F = -\frac{X^1 X^2 X^3}{X^0} - i X^{02} h^{(1)} (-i \frac{X^2}{X^0}, -i \frac{X^3}{X^0}) + \ldots \quad (2)$$

where

$$\frac{X^1}{X^0} = z^1 = iy_1 = i \left( e^{-\phi} + ia_1 \right),$$

$$\frac{X^2}{X^0} = z^2 = iy_2 = i \left( e^{\gamma + \sigma} + ia_2 \right),$$

$$\frac{X^3}{X^0} = z^3 = iy_3 = i \left( e^{\gamma - \sigma} + ia_3 \right), \quad (3)$$

and dots stand for contributions from other moduli. Here and below $I, J = 0, \ldots, 3$ and $i, j = 1, 2, 3$. The moduli $z^i$ and the vector fields are identified by comparing the action (1) with that resulting from compactification of the universal sector of the 6D theory

$$I_6 = \int d^6 x \sqrt{-G^{(6)}} e^{-\phi} \left[ R^{(6)} + (\partial \Phi)^2 - \frac{H^2}{12} \right] + \ldots \quad (4)$$

on the two-torus. Here

$$G^{(6)} = \begin{pmatrix} G_{\mu \nu} + A^{m}_{\mu} A^{N}_{\nu} C_{m n} & A^{m}_{\nu} C_{m n} \\ A^{n}_{\nu} C_{m n} & G_{m n} \end{pmatrix}, \quad (5)$$

where $\mu, \nu = 0, \ldots, 3$ and $m, n = 1, 2$. Here $A^{n}_{\mu} = G^{m n} G_{m \mu}$. The second pair of vector fields are the components $B_{m \mu}$ of the antisymmetric field $B$.

Dimensional reduction of the action (4) on the two-torus yields the 4D action [15]

$$I_4 = \int d^4 x \sqrt{-G e^{-\phi}} \left[ R + (\partial \phi)^2 - \frac{(H)^2}{12} - \frac{1}{4} F(LML) F + \frac{1}{8} Tr(\partial ML \partial ML) \right], \quad (6)$$

where
\[ M = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G \end{pmatrix}, \quad L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \] (7)

The metric of the two-torus is parametrized as [16]
\[ G_{mn} = e^{2\sigma} \begin{pmatrix} e^{2\gamma-2\sigma} + a_3^2 - a_3 & -a_3 \\ -a_3 & 1 \end{pmatrix} \] (8)

and
\[ \phi = \Phi - \frac{1}{2} \ln \det(G_{mn}). \]

The dilaton \( \phi \) can be split into the sum of the constant part and a term vanishing at spatial infinity \( \phi = \phi_0 + \phi_1 \). In string perturbation theory, higher order contributions enter with the factor \( e^{\chi \phi} \), where \( \chi \) is the Euler characteristic of the string world sheet. The exponent \( e^{\phi_0} \equiv \epsilon \) can be considered as a string-loop expansion parameter. In the following, we include the factor \( \epsilon \) in string-loop corrections, and use the notation \( \phi \) for the non-constant part of the dilaton.

The moduli (2) are equal to conventional moduli \( S, T, U \):
\[ (y_1, y_2, y_3) = (S = e^{-\phi} + ia_1, T = \sqrt{G} + iB_{12}, U = \frac{(\sqrt{G} + iG_{12})}{G_{22}}). \]

\( a_1 \) is the axion. The antisymmetric tensor is \( B_{mn} = a_2 \epsilon_{mn} \). The tree-level magnetic black hole solutions we discuss in this paper have \( a_i = 0 \).

The gauge part of the action (6) with \( G_{12} = 0 \) and \( B_{12} = 0 \) is
\[ -\frac{1}{4} G_{11}(F^{(1)})^2 - \frac{1}{4} G_{22}(F^{(1)})^2 - \frac{1}{4} G_{11}(F^{(2)})^2 - \frac{1}{4} G_{22}(F^{(2)})^2 \] (9)

It is convenient to relabel the vector fields in correspondence with the moduli with which they form the superfields
\[ A_1^\mu = \hat{A}_0^\mu, \quad B_1^\mu = \hat{A}_1^\mu, \quad A_2^\mu = \hat{A}_2^\mu, \quad B_2^\mu = \hat{A}_3^\mu. \] (10)

Let us turn to the \( N = 2 \) supersymmetric action (1). In sections which admit the prepotential, the coupling constants in the action (1) are calculated using the formula
\[ N_{IJ} = \bar{F}_{IJ} + 2i \frac{(\text{Im} F_{IK} X^K)(\text{Im} F_{IJ} X^L)}{(X^K \text{Im} F_{IJ} X^L)}, \] (11)

where \( F_I = \partial_X F, F_{IJ} = \partial^2_{XY} F \), etc. In sections which do not admit a prepotential (including that which naturally appears in compactification of the heterotic string action), the gauge couplings are calculated by making a symplectic transformation of the couplings in a section with the prepotential. Specifically, the couplings in the section connected with the compactification of the 6D action on the two-torus are calculated by using the couplings obtained from the prepotential (2) via the symplectic transformation [2,12]
\[ O = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \] (12)
where
\[ A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = 1. \]
of the form
\[ A = \text{diag}(1, 0, 1, 1), \quad B = \text{diag}(0, 1, 0, 0), \quad C = \text{diag}(0, -1, 0, 0), \quad D = \text{diag}(1, 0, 1, 1) \] (13)
as
\[ \hat{N} = (C + D N)(A + B N)^{-1}. \] (14)

The Kähler potential is invariant under symplectic transformations and its part which depends on the moduli \( y_i \) is given by
\[ K = -\ln[(y_1 + \bar{y}_1 + V)(y_2 + \bar{y}_2)(y_3 + \bar{y}_3)], \] (15)

where the Green-Schwarz function \( V \) is
\[ V(y_2, \bar{y}_2, y_3, \bar{y}_3) = \frac{Re h^{(1)} - Re y_2 Re \partial_y h^{(1)} - Re y_3 Re \partial_y h^{(1)}}{Re y_2 Re y_3}. \] (16)

The field equations and Bianchi identities for the gauge field strengths are
\[ \partial_\mu (\sqrt{-g} Im G^{-\mu}) = 0, \]
\[ \partial_\mu Im F^{-J} = 0, \] (17)

where \( G^{-\mu} = \hat{N}_{IJ} F^{-J\mu} \). Eqs. (17) and are invariant under the symplectic transformations (12) of general form.

For the symplectic transformation (12),(13), the coupling constants in the heterotic basis are expressed via the couplings in the basis with a prepotential as
\[ \hat{N}_{IJ} = \begin{pmatrix}
N_{00} & \frac{N_{01}}{N_{11}} & \frac{N_{02}}{N_{11}} & \frac{N_{03}}{N_{11}} \\
\frac{N_{10}}{N_{11}} & N_{11} & \frac{N_{12}}{N_{11}} & \frac{N_{13}}{N_{11}} \\
\frac{N_{20}}{N_{11}} & \frac{N_{21}}{N_{11}} & N_{22} & \frac{N_{23}}{N_{11}} \\
\frac{N_{30}}{N_{11}} & \frac{N_{31}}{N_{11}} & \frac{N_{32}}{N_{11}} & N_{33}
\end{pmatrix}. \] (18)

From the symplectic transformation of the field strengths with the matrices (12),(13) we have
\[ \begin{pmatrix}
\hat{F}^- \\
\hat{G}^{-}
\end{pmatrix} = O \begin{pmatrix}
F^- \\
G^{-}
\end{pmatrix}, \] (19)

where the field strengths without hats refer to the holomorphic section with the prepotential, we obtain the relations between the field strengths
\[ F^0 = \hat{F}^0, \quad F^2 = \hat{F}^2, \quad F^3 = \hat{F}^3 \]
\[ F^{-1} = -\frac{\hat{N}_{10}}{N_{11}} \hat{F}^0 + \frac{1}{N_{11}} \hat{F}^{-1} - \frac{\hat{N}_{12}}{N_{11}} \hat{F}^{-2} - \frac{\hat{N}_{13}}{N_{11}} \hat{F}^{-3}. \] (20)
To write the supersymmetry transformations, one introduces the expressions (for example, [9–13])

\[ T_{\mu\nu}^- = 2ie^{K/2}X^I I_m N_{IJ} F_{\mu\nu}^- \]  \hspace{1cm} (21)

and

\[ G_{\mu\nu}^{-i} = -k^{ij} f_j^I I_m N_{IJ} F_{\mu\nu}^- . \]  \hspace{1cm} (22)

Here \( k^{ij} \) is the inverse Kaehler metric, and

\[ f_j^I = (\partial_j + \frac{1}{2} \partial_j K) e^{K/2} X^I . \]

Supersymmetry transformations of the chiral gravitino \( \psi_{\alpha\mu} \) and gaugino \( \lambda^{i\alpha} \) are (for example, [9–11,13])

\[ \delta \psi_{\alpha\mu} = D_\mu \epsilon_\alpha - T_{\mu\nu}^- \gamma^\nu \epsilon_{\alpha\beta} \epsilon_\beta , \]  \hspace{1cm} (23)

\[ \delta \lambda^i_{\alpha} = i \gamma^\mu \partial_\mu \hat{z}^i \epsilon_\alpha + G_{\mu\nu}^{-i} \gamma^\mu \epsilon_{\alpha\beta} \epsilon_\beta , \]  \hspace{1cm} (24)

where

\[ D_\mu \epsilon_\alpha = (\partial_\mu - \frac{1}{4} w^{\hat{a}\hat{b}} \gamma_a \gamma_b + i \frac{Q_\mu}{2}) \epsilon_\alpha . \]

Here \( w^{\hat{a}\hat{b}} \) is the spin and \( Q_\mu \) Kaehler connections. Here \( \hat{a}, \hat{b}, ... \) are the tangent space indices, \( a, b, ... \) are the curved space indices.

Requiring that supersymmetry variations of spinors vanish, we obtain a system of supersymmetric Killing equations for the moduli. We look for a solution of this system with supersymmetry parameter satisfying the relation \( \epsilon_\alpha = \gamma_0 \epsilon^{\alpha\beta} \epsilon_\beta \). The \( \mu = 0 \) component of equation \( \delta \psi_{\alpha0} = 0 \) takes the form

\[ \left( \frac{1}{2} w^{\hat{a}\hat{b}} \gamma_0 \gamma_b \gamma_0 - T_{0m} \epsilon^{\hat{b}n} \gamma_b \right) \epsilon_{\alpha\beta} \epsilon_\beta = 0 . \]  \hspace{1cm} (25)

Here \( e^{\hat{a}\hat{b}} \) is the inverse Vielbein.

In this paper we are interested in static spherically-symmetric solutions of the field equations. The metric is

\[ ds^2 = -e^{2U} dt^2 + e^{-2U} dx^i dx_i . \]  \hspace{1cm} (26)

The only non-vanishing components of the spin connection \( w^{\hat{a}\hat{b}} \) are \( w^{\hat{a}0} = \frac{1}{2} \partial_\hat{a} e^{2U} \). The Vielbein \( e^b_\mu \) is \( e^b_\mu = \delta^b_\mu e^U \). To have a nontrivial solution for the supersymmetry parameter, we must require that

\[ \frac{1}{2} w^{\hat{a}0} - e^U T_{0n}^- = 0 . \]  \hspace{1cm} (27)

Using the relations

\[ G_{mn}^- = i \epsilon_{mn0} G^{-\mu0} \]
and
\[ G^{-\mu\nu} \gamma^\mu \gamma^\nu \epsilon_\alpha = 4G_{0n}^{-i} \gamma^0 \gamma^n \epsilon_\alpha \]
valid for any self-dual tensor and chiral spinor, the condition the gaugino supersymmetry transformation to vanish is written as
\[ (i \gamma^n \partial_n z^i \gamma^0 + 4G_{0n}^{-i} \gamma^n) \epsilon^{\alpha\beta} \epsilon_\beta = 0. \]  
(28)

There is a nontrivial solution provided
\[ i\partial_n z^i + 4e^{-U}G_{0n}^{-i} = 0. \]  
(29)

The factor \( e^{-U} \) is due to the relation \( \gamma_0 = -\gamma^0 = -e^{0} \gamma^0 = -e^{U} \gamma^0 \). Convoluting the equation (29) with the functions \( f_i \) and using the relation of special \( N = 2 \) geometry
\[ k^{ij} f_i \bar{f}_j = -\frac{1}{2} (ImN)^{IJ} - e^K X^I X^J, \]
it is finally obtained in the form (cf. [19,18])
\[ if_i \partial_n z^i + 4e^{-U} \left( \frac{1}{2} \mathcal{F}_{0n}^{-i} + e^K X^I (X^J Im N_{JL} \mathcal{F}_{0n}^{-L}) \right) = 0. \]  
(30)

III. SOLUTION OF SPINOR KILLING EQUATIONS FOR MAGNETIC BLACK HOLE

Next, we solve the combined system of the equations for the gauge field strengths and the moduli. We look for a string-tree-level solution with the metric in the form (26), two magnetic fields \( \hat{F}^0_{\mu\nu} \) and \( \hat{F}^1_{\mu\nu} \) and purely real moduli \( y_i \) (3). This means, that we consider configurations with diagonal metrics (8), vanishing tensor \( B_{mn} \) and vanishing axion \( a_1 \). Such configurations appear as solutions of the “chiral null models” [5,6].

In the basis with the prepotential (2), the non-zero tree-level couplings \( N_{IJ} \) are
\[ N_{00} = -iy_1 y_2 y_3, \quad N_{11} = -\frac{iy_2 y_3}{y_1}, \quad N_{22} = -\frac{iy_1 y_3}{y_2}, \quad N_{33} = -\frac{iy_1 y_2}{y_3}. \]  
(31)

In the heterotic basis, the couplings are calculated via (18):
\[ \hat{N}_{00} = -iy_1 y_2 y_3, \quad \hat{N}_{11} = -\frac{iy_1}{y_2 y_3}, \quad \hat{N}_{22} = -\frac{iy_1 y_3}{y_2}, \quad \hat{N}_{33} = -\frac{iy_1 y_2}{y_3}. \]

Solving the system of Maxwell equations and Bianchi identities, we have
\[ \hat{F}_{0n}^{-0} = P^0 i \frac{2U x^n}{r^3}, \quad \hat{F}_{0m}^{-1} = P^1 i \frac{2U x^m}{r^3}. \]  
(32)

The tree-level Kaehler potential is
\[ K = -\ln 8y_1 y_2 y_3. \]  
(33)
Using the field strengths (20) and the constants (31), we obtain the tree-level expression for the combination (21) \( T_{0n} = 2ie^{K/2} S_{0n} \) which enters the supersymmetry transformations

\[
T_{0n} = 2ie^{K/2} S_{0n} = 2ie^{K/2}(IMN_{0n}F^{-0}_{0n} + iy_1IMN_{11}F^{-1}_{0n})
\]

\[
= 2ie^{K/2}(-y_1y_2y_3F^{-0}_{0n} - y_1\hat{F}^{-1}_{0n}) = \left(\frac{y_1y_2y_3}{8}\right)^{1/2} \left(P^0 + \frac{P^1}{y_2y_3}\right) e^{2U} \frac{x^n}{r^3}.
\]  (34)

Eq.(27) for the gravitini takes the form

\[
\frac{1}{4} \partial_n e^{2U} - \left(\frac{y_1y_2y_3}{8}\right)^{1/2} e^{3U} \left(P^0 + \frac{P^1}{y_2y_3}\right) \frac{x^n}{r^3} = 0.
\]  (35)

The tree-level gaugini equations (30) are

\[
I = 0 : \quad \frac{ie^{K/2}}{2} \partial_n \ln y_1y_2y_3 - 4e^{-U} \left(\frac{1}{2} F^{-0}_{0n} + e^K S_{0n}\right) = 0,
\]

\[
I = 1 : \quad \frac{y_1e^{K/2}}{2} \partial_n \ln \frac{y_2y_3}{y_1} + 4e^{-U} \left(\frac{F^{-1}_{0n}}{2N_{11}} - iy_1e^K S_{0n}\right) = 0,
\]

\[
I = 2 : \quad \frac{y_2e^{K/2}}{2} \partial_n \ln \frac{y_1y_3}{y_2} + 4e^{-U} (-iy_2e^K S_{0n}) = 0,
\]

\[
I = 3 : \quad \frac{y_3e^{K/2}}{2} \partial_n \ln \frac{y_1y_2}{y_3} + 4e^{-U} (-iy_3e^K S_{0n}) = 0.
\]  (36)

Substituting the explicit expressions for the field strengths and couplings, we have

\[
I = 0 : \quad \partial_n \ln y_1y_2y_3 - 2(8y_1y_2y_3)^{1/2} e^{2U} \left(P^0 - \frac{1}{4} \left(P^0 + \frac{P^1}{y_2y_3}\right)\right) \frac{x^n}{r^3} = 0,
\]

\[
I = 1 : \quad \partial_n \ln \frac{y_1y_2y_3}{y_1} - 2(8y_1y_2y_3)^{1/2} e^{2U} \left(\frac{P^1}{y_2y_3} - \frac{1}{4} \left(P^0 + \frac{P^1}{y_2y_3}\right)\right) \frac{x^n}{r^3} = 0,
\]

\[
I = 2 : \quad \partial_n \ln \frac{y_1y_2y_3}{y_2} - 2(8y_1y_2y_3)^{1/2} e^{2U} \left(\frac{1}{4} \left(P^0 + \frac{P^1}{y_2y_3}\right)\right) \frac{x^n}{r^3} = 0,
\]

\[
I = 3 : \quad \partial_n \ln \frac{y_1y_2y_3}{y_3} - 2(8y_1y_2y_3)^{1/2} e^{2U} \left(\frac{1}{4} \left(P^0 + \frac{P^1}{y_2y_3}\right)\right) \frac{x^n}{r^3} = 0.
\]  (37)

The system of equations (35) and (37) is solved by a general configuration of magnetic black hole with two arbitrary magnetic charges consisting of a metric, dilaton and moduli [5,6]. In the following, we shall consider a particular extremal solution

\[
e^{-2U} = 1 + \frac{P}{r}, \quad y_1 = e^{-\phi} = (1 + \frac{P}{r})^{-1},
\]  (38)

The charges \( P^0 \) and \( P^1 \) are expressed via a charge \( P \)

\[
P^0 = \frac{P^1}{y_2y_3}, \quad P = \sqrt{8y_2y_3P^0}
\]  (39)
and the moduli $y_2, y_3$ are arbitrary real constants. The metric components of the torus $T^2$ are

$$G_{11} = y_2 y_3 = e^{2\gamma}, \quad G_{22} = y_2 / y_3 = e^{2\sigma}.$$ 

Normalizations of the gauge terms in the actions (1) and (6) differ by the factor $8y_2 y_3$ resulting in this factor in (39).

IV. SOLUTION OF THE LOOP-CORRECTED SPINOR KILLING EQUATIONS

Our next aim is to solve the Maxwell equations and the loop-corrected spinor Killing equations. We look for a solution in the first order in string coupling constant. In the $N = 2$ supersymmetric theory, the prepotential has no loop corrections beyond one loop. The loop corrections are calculated by substituting the tree-level moduli. For the constant moduli $y_2, y_3$, the loop correction to the prepotential and its derivatives are also independent of coordinates, resulting in considerable technical simplifications in solution of the loop-corrected spinor Killing equations.

Using the loop-corrected prepotential and the formula (11), we obtain the gauge couplings $\hat{N}_{IJ}$

$$N_{00} = iy^3 \left( -1 + \frac{n}{4y^3} \right), \quad N_{01} = -\frac{n + 2v}{4y_2} + ia_1 \frac{y_2 y_3}{y_1},$$

$$N_{02} = -\frac{n + 2v - 2y_2 h y + 4y_2 h_2}{4y_2} + ia_2 \frac{y_3}{y_2},$$

$$N_{03} = -\frac{n + 2v + 2y_3 h y + 4y_3 h_3}{4y_3} + ia_3 \frac{y_2}{y_3},$$

$$N_{11} = -i \frac{y^3}{y_1} \left( 1 + \frac{n}{4y^3} \right), \quad N_{12} = i y_3 \frac{2y_2 h y - n}{4y^3}, \quad N_{13} = i y_2 \frac{2y_3 h y - n}{4y^3},$$

$$N_{22} = -i \frac{y^3}{y_2} \left( 1 - \frac{y_2 h_3 y_3}{y^3} + \frac{n}{4y^3} \right), \quad N_{33} = -i \frac{y^3}{y_3} \left( 1 - \frac{y_3 h_2 y_3}{y^3} + \frac{n}{4y^3} \right).$$

(40)

Here we used the notations: $y^3 = y_1 y_2 y_3$, $h y = h_a y_a = h_2 y_2 + h_3 y_3$, $h_a = \partial_{y_a} h$, $h_{ab} = \partial_{y_a} \partial_{y_b} h$ and

$$v = h - y_a h_a, \quad n = h - h_a y_a + y_a h_{ab} y_b, \quad y_2 h y = y_2 h_2 y_a.$$ (41)

We introduced the imaginary parts of the moduli $y_i$ (3) which are of the first order in the string coupling constant, retaining notations $y_i$ for the real parts of the moduli $y_i$. Only the couplings $\hat{N}_{0i}$ depend on $a_i$.

The couplings in the heterotic basis are calculated by using (18). The terms of the form $N_{KJ} N_{LJ}$ are of the next order in the string coupling and do not contribute. The tree-level gauge field strengths acquire corrections of the first order in string coupling, and also the gauge fields, absent at the tree level, are generated.

The Maxwell equations (17) which we rewrite as

$$\partial_{\mu}(\sqrt{-g} \, Im \hat{N}_{IJ} \hat{F}^J + Re \hat{N}_{IJ} \hat{F}^J)^{\mu\nu} = 0,$$ (42)
with the required accuracy have the form

\[ J = 0: \quad \partial_t [\sqrt{-g} \, \text{Im} \hat{N}_{00} \hat{F}^0 + \text{Re} \hat{N}_{01} \hat{F}^1] = 0, \]  
\[ J = 1: \quad \partial_t [\sqrt{-g} \, \text{Im} \hat{N}_{11} \hat{F}^1 + \text{Re} \hat{N}_{11} \hat{F}^0] = 0, \]  
\[ J = 2: \quad \partial_t [\sqrt{-g} \, \text{Im} \hat{N}_{22} \hat{F}^2 + \text{Re} \hat{N}_{20} \hat{F}^0 + \text{Re} \hat{N}_{21} \hat{F}^1] = 0, \]  
\[ J = 2: \quad \partial_t [\sqrt{-g} \, \text{Im} \hat{N}_{33} \hat{F}^2 + \text{Re} \hat{N}_{30} \hat{F}^0 + \text{Re} \hat{N}_{31} \hat{F}^1] = 0. \]

Here \( *F^{0r} = F_{\vartheta \varphi} \) and \( *F_{\vartheta \varphi} = -F^{0r} \). Only the diagonal gauge couplings \( \hat{N}_{II} \) contain terms of zero-order in string coupling.

First, we solve the equations (46) and (46):

\[ \hat{F}^{20r} = \frac{C_2(\vartheta, \varphi) - \text{Re} \hat{N}_{20} \hat{F}^0_{\vartheta \varphi} - \text{Re} \hat{N}_{21} \hat{F}^1_{\vartheta \varphi}}{\sqrt{-g} \, \text{Im} \hat{N}_{22}}, \]  
\[ \hat{F}^{30r} = \frac{C_3(\vartheta, \varphi) - \text{Re} \hat{N}_{30} \hat{F}^0_{\vartheta \varphi} - \text{Re} \hat{N}_{31} \hat{F}^1_{\vartheta \varphi}}{\sqrt{-g} \, \text{Im} \hat{N}_{33}}. \]

Here \( \hat{F}^{0,1}_{\vartheta \varphi} = P^{0,1} \sin \vartheta \), and, using (18) we expressed all the couplings \( \hat{N}_{II} \) through those in the basis with the prepotential. \( C_{2,3}(\vartheta, \varphi) \) are arbitrary functions of the first order in string coupling. Bianchi identities (17) show that the functions \( C_{2,3}(\vartheta, \varphi) \) are of the form \( C_{2,3}(\vartheta, \varphi) = C_{2,3} \sin \vartheta \), where \( C_{2,3} \) are constants. Electric fields \( \hat{F}^{2,3} \) are of the first order in string coupling.

Eqs. (44) and (45) yield

\[ \hat{F}^{00r} = \frac{C_0 + a_1 P^1}{\sqrt{-g} \, \text{Im} \hat{N}_{00}}, \]  
\[ \hat{F}^{10r} = \frac{C_1 + a_1 P^0}{\sqrt{-g} \, \text{Im} \hat{N}_{11}}, \]

where \( \sqrt{-g'} = e^{-2u} r^2 \) and \( C_0, C_1 \) are arbitrary constants of the first order in string coupling.

One can also introduce magnetic fields \( *F^2 \) and \( *F^3 \) with the charges of the first order in string coupling. However, since these fields enter the Maxwell equations multiplied by the coupling constants of the first order in string coupling, at the required level of accuracy, they are omitted from the equations.

Let us calculate the the expression

\[ S_{\vartheta \varphi} = (X^I \hat{F}_{\vartheta \varphi}^I). \]

in the first order in string coupling constant.

In the basis with the prepotential, the field strengths are expressed through the field strengths in the heterotic basis by using the relations (20). Substituting solutions for the field strengths (47) and (48) and using the expressions (40) for the gauge couplings, we have

\[ S_{\vartheta \varphi} = \hat{F}^{0r}_{\vartheta \varphi}[\text{Im} N_{00} + y_i \text{Re} N_{i0} + i(a_1 y_2 y_3 + a_2 y_1 y_3 + a_3 y_1 y_2)] - \hat{F}^{1r}_{\vartheta \varphi} y_1 \]
\[ + \frac{1}{2} [y_2 C_2 + y_3 C_3] + i(C_0 + C_1 y_2 y_3) \sin \vartheta. \]
Only the the couplings $N_{00}$ and $N_{0i}$, $i = 1, 2, 3$ enter the final expression (50) yielding

$$\text{Im } N_{00} + y_i \text{Re } N_{0i} = -(y_1 y_2 y_3 + 2v + h_a y_a).$$

Note that all the terms containing second derivatives of the prepotential have canceled. In the first order in string coupling, the Kaehler potential (15) is independent of the functions $a_i$.

Using the Kaehler potential (15), we calculate the combinations $B_n^I = f_I^i \partial_n z^i$ which enter the spinor Killing equations (30) for the moduli $z_i$. We have

$$B_n^0 = -\frac{1}{2} e^{K/2} \left( 1 - \frac{V}{2y_1} \right) \partial_n \ln y^3,$$

$$B_n^i = i y_i \left( B_n^0 + e^{K/2} \partial_n \ln y_i \right), \quad i = 1, 2, 3.$$  

(51)

Here we used the definitions

$$V = e^{-2\gamma_0},$$

and $y^3 = y_1 y_2 y_3$. All the expressions are calculated in the first order in string coupling. In particular, all the factors multiplying the Green-Schwarz function $V$ are taken in the leading order in string coupling.

Let us introduce notations for the loop-corrected metric and moduli. We shall split the functions $\phi, \gamma$ and $\sigma$ which appear in the moduli (3) into the tree-level parts $\phi_0, \gamma_0$ and $\sigma_0$ and the those of the first order in string coupling: $\phi_1, \gamma_1$ and $\sigma_1$. We have $\phi = \phi_0 + \phi_1$, etc.. The function $2U$ in the metric will be written as $2U_0 + u_1$. At the tree level (see (38),

$$e^{-2U_0} = e^{\phi_0} = f_0 = 1 + \frac{P}{r}$$

From (39), we have

$$P_0 = \frac{P e^{-\gamma_0}}{\sqrt{8}}, \quad P_1 = \frac{P e^{\gamma_0}}{\sqrt{8}}.$$  

(52)

For the Kaehler potential we obtain

$$e^K = \frac{f_0 e^{-2\gamma_0}}{8} \left[ 1 + \left( \phi_1 - 2\gamma_1 - \frac{V f_0}{2} \right) \right].$$  

(53)

Let us turn to the function $S_{\theta \phi}$ (50). The functions $a_i$ are of the first order in the string coupling constant. The terms containing the factors $a_i$ are imaginary. Because the spinor Killing equations for the moduli (29) are linear in derivatives of the moduli, the equations for the imaginary parts of the moduli decouple from those for the real parts. In this section, solving the equations for the real parts of the moduli, we shall not write the imaginary parts of the moduli explicitly.

Because the tree-level moduli are constants, the terms $2v + h_a y_a$ and $y_a C_a$, which are of the first order in string coupling, are also constants.

The function $T_{0n}^- = 2i e^{K/2} S_{0n}$, which enters the gravitini equation (27), we write in the form
\[ T_{0n} = e^{K/2} y_1 \left[ P^0 e^{2\gamma} \left( 1 + (2V + h_ay_a e^{-2\gamma_0}) f_0 \right) + P^1 + C_a y_a f_0 \right] e^{2U x^n / r^3}. \]  

(54)

The factors \( f_0 \) appear because the modulus \( y_1 \), when it multiplies an expression of the first order in string coupling, can be substituted by its tree-level value \( f_0^{-1} \).

Using the expression for the Kaehler potential (53) and expanding in (54) all the terms to the first order in string coupling, we finally obtain

\[ T_{0n}^- = f_0^{-3/2} P \frac{4}{4} \left[ 1 + \left( \frac{\phi_1}{2} + u_1 + \left( \frac{3V}{4} + C \right) f_0 \right) \right] x^n / r^3, \]  

(55)

where the constant \( C \) is

\[ C = \frac{1}{2} \left( h_a y_a + \frac{C_a y_a P^0}{P^0} \right) e^{-2\gamma_0}. \]  

(56)

Substituting (55), we obtain the gravitino spinor Killing equation in the form

\[ \frac{1}{4} \partial_n [f_0^{-1}(1 + u_1)] - \frac{P}{4} f_0^{-2} \left[ 1 + \left( \frac{3u_1}{2} - \frac{\phi_1}{2} + \left( \frac{3V}{4} + C \right) f_0 \right) \right] x^n / r^3 = 0. \]  

(57)

In this equation the sum of the tree-level terms vanishes; the remaining part of the first order in string coupling is

\[ \frac{u_1'}{q'} + \frac{u_1 - \phi_1}{2} + \left( \frac{3V}{4} + C \right) f_0 = 0. \]  

(58)

Here \( q' = f_0' / f_0 \).

Let us turn to the gaugini spinor Killing equations (30). Using the expression

\[ e^K S_{0n} = -\frac{1}{4} \left( 1 - \gamma_1 + \left( \frac{V}{2} + C \right) f_0 \right) \left( P e^{-\gamma_0} i \frac{e^{2U x^n}}{\sqrt{8} 2 e^{2U x^n / r^3}} \right), \]  

(59)

for the combination \( \frac{1}{2} F_{0n}^- + e^K S_{0n} \) we have

\[ \frac{1}{2} F_{0n}^- + e^K S_{0n} = \frac{1}{4} \left( 1 + \gamma_1 - \left( \frac{V}{2} + C \right) f_0 \right) \left( P e^{-\gamma_0} i \frac{e^{2U x^n}}{\sqrt{8} 2 e^{2U x^n / r^3}} \right). \]  

(60)

To calculate the combination \( \frac{1}{2} F_{0n}^- - iy_1 e^K S_{0n} \), by using (20), we express the field strength \( F_{0n}^- \) in terms of the field strengths in the heterotic basis.

\[ F_{0n}^- = \frac{\hat{F}_{0n}^- - N_{00} \hat{F}_{0n}^0}{N_{11}} = -iy_1 \left( 1 - 2\gamma_1 + \frac{V f_0}{2} \right) \left( P e^{-\gamma_0} i \frac{e^{2U x^n}}{\sqrt{8} 2 e^{2U x^n / r^3}} \right). \]  

(61)

We obtain

\[ \frac{1}{2} F_{0n}^- - iy_1 e^K S_{0n} = -\frac{iy_1}{4} \left( 1 - 3\gamma_1 + \left( \frac{V}{2} + C \right) f_0 \right) \left( P e^{-\gamma_0} i \frac{e^{2U x^n}}{\sqrt{8} 2 e^{2U x^n / r^3}} \right). \]  

(62)
With the accuracy of the first order in string coupling, the one-loop-corrected expressions for $B^0_n$ and $B^1_n$ are

$$B^0_n = \frac{q' f_0 1/2 e^{-\gamma_0}}{2\sqrt{8}} \left[ 1 + \frac{\phi_1' - 2 \gamma_1'}{q'} + \frac{\phi_1 - u_1}{2} - \frac{3 V f_0}{4} \right] x^n, \frac{r}{r},$$

$$B^1_n = -i \frac{q' f_0 1/2 e^{-\gamma_0}}{2\sqrt{8}} \left[ 1 + \frac{\phi_1' + 2 \gamma_1'}{q'} - \frac{\phi_1 + u_1}{2} + \frac{V f_0}{4} \right] x^n. \frac{r}{r}. \tag{63}$$

Using the expressions (60)-(63), we verify that in Eqs.(30) with $I = 0$ and $I = 1$ the leading-order terms cancel, and the remaining equations for the terms of the first-order in string coupling are

$$I = 0 : \frac{\phi_1' - 2 \gamma_1'}{q'} + \frac{\phi_1 - u_1}{2} - 2 \gamma_1 - \left( \frac{V}{4} - C \right) f_0 = 0, \tag{64}$$

$$I = 1 : \frac{\phi_1' + 2 \gamma_1'}{q'} + \frac{\phi_1 - u_1}{2} + 2 \gamma_1 - \left( \frac{V}{4} - C \right) f_0 = 0.$$

Eqs. (64) split into the following system

$$\frac{\phi_1'}{q'} + \frac{\phi_1 - u_1}{2} - \left( \frac{V}{4} - C \right) f_0 = 0 \tag{65}$$

$$\gamma_1' + q' \gamma_1 = 0$$

Let us consider the remaining equations with $I = 2$ and $I = 3$. Substituting in the expressions for the loop-corrected couplings (40) and the field strengths (47), we have

$$\hat{F}^{-2}_{0n} = \frac{P^0}{y_1 y_3} \left( \frac{v}{2} + h_2 y_2 + \frac{C_2 y_2}{P^0} \right) 1/2 e^{2U} x^n \frac{r}{r^3} = y_2 P^0 f_0 \left( \frac{V}{2} + L_2 \right) 1/2 e^{2U} x^n \frac{r}{r^3} \tag{66}$$

and similar expression for $\hat{F}^{-3}_{0n}$ obtained by substitution $2 \to 3$. The field strengths $\hat{F}^{-2,3}$, absent at the string tree level, are of the first order in the string coupling. Here we introduced

$$L_2 = \left( h_2 y_2 + \frac{C_2 y_2}{P^0} \right) e^{-2 \gamma_0}, \quad L_3 = \left( h_3 y_3 + \frac{C_3 y_3}{P^0} \right) e^{-2 \gamma_0}. \tag{67}$$

Subtracting Eq.(40) with $I = 2$ from that with $I = 0$ (the same for $I = 3$ ), and using the expressions (51) for the combinations $B^0_i$, we have

$$i e^{K/2} \frac{\partial n y_2}{y_2} + 4 e^{-U} \left( \frac{\hat{F}^{-2}_{0n}}{2iy_2} - \frac{1}{2} \hat{F}^{-0}_{0n} - 2 e^{K} S_{0n} \right) = 0. \tag{68}$$

Substituting the expressions for the field strengths $\hat{F}^{-0}_{0n}, \hat{F}^{-2}_{0n}$ and Eq.(59) for $e^K S_{0n}$, and keeping the terms of the first order in the string coupling, we obtain

$$\gamma_1' + \sigma_1' + (C - L_2 - \gamma_1 f_0^{-1}) \frac{P}{r^2} = 0 \tag{69}$$

$$\gamma_1' - \sigma_1' + (C - L_3 - \gamma_1 f_0^{-1}) \frac{P}{r^2} = 0.$$
The sum of the Eqs. (69) is
\[ \gamma_1' + \gamma_1 q' + (2C - L_2 - L_3)f_0' = 0. \] (70)
Substituting the expressions (56) and (67) for \( C \) and \( L_a \), we find that
\[ 2C - L_2 - L_3 = 0, \] (71)
so that Eq.(70) coincides with the second Eq.(65).

Let us solve the system of the gravitini Eq.(58) and the first Eq.(65). Adding and subtracting the equations, we obtain the solution
\[ u_1 + \phi_1 = c_1 - \left( \frac{V}{2} + 2C \right) f_0 \]
\[ u_1 - \phi_1 = \frac{c_2}{f_0} - \frac{Vf_0}{2}, \] (72)
where \( c_{1,2} \) are arbitrary constants. Requiring that at large distances from the center of the black hole the metric and dilaton are asymptotic to the Lorentzian metric and constant dilaton equal to unity, we have
\[ c_1 = \frac{V}{2} + 2C, \quad c_2 = \frac{V}{2}, \] (73)
and we obtain
\[ u_1 = -\left( \frac{V}{2} + C \right) \frac{P}{r} - \frac{V}{4} \frac{P}{r + P}, \quad \phi_1 = -C \frac{P}{r} + \frac{V}{4} \frac{P}{r + P}. \] (74)

At the tree level, magnetic black hole solution is the extremal BPS saturated configuration [5,6]. Provided supersymmetry is unbroken in perturbation theory, the loop-corrected solution should have the same properties. At the one-loop level, besides the magnetic charges \( P^0 \) and \( P^1 \) present at the tree level, there appear electric and magnetic fields (cf. Eqs.(47) and (48)) with the charges of the first order in string coupling. Each of these charges is defined up to an arbitrary constant. Thus, there is enough freedom to make the ADM mass, determined from the loop-corrected metric, equal to that determined in the framework on \( N = 2 \) supersymmetric theory as the asymptotic value of the central charge [12].

A particular possibility to fix this freedom and to obtain the BPS saturated solution, is to require that the sum of contributions to the central charge of electric and magnetic charges of the first order in string coupling vanishes, and the central charge retains the tree-level value. Then, to make the ADM mass calculated with the loop-corrected metric with the loop correction (74) equal to its tree-level value, we must set \( C = -\frac{3V}{4} \), and in this case we have
\[ u_1 = \frac{V}{4} \left( \frac{P}{r} - \frac{P}{r + P} \right). \] (75)
V. DISCUSSION

In our previous study [17], solving the system of the loop-corrected Einstein and Maxwell equations, we obtained a two-parameter set of solutions for the loop corrections to the metric and dilaton

\[ u_1 = A_1 \frac{P}{r} - A_2 \frac{P}{r + P}, \quad \phi_1 = (A_1 + \frac{V}{2}) \frac{P}{r} + A_2 \frac{P}{r + P}. \tag{76} \]

The one-parameter family of solutions (74) is contained in (76). A nontrivial check of consistency of both calculations is that in both cases the coefficients at the terms \( \frac{P}{r} \) in the expressions for the metric and dilaton differ by \( \frac{V}{2} \).

Near the locations of the enhanced symmetry points in the moduli space, the second derivatives of the prepotential have logarithmic singularities [1,2]. In particular, for \( y_2 \sim y_3 \),

\[ h^{(1)}(y_2, y_3) = (y_2 - y_3)^2 \log(y_2 - y_3)^2. \]

Although the loop-corrected gauge couplings (40) contain second derivatives of the prepotential, the final expressions for the metric and moduli depend on the Green-Schwarz function \( V \) which contains only the first derivatives of the prepotential and thus is regular at the points of enhanced symmetry. Note also, that the Green-Schwarz function is positive [3] (this can be verified by explicit calculations) as can be seen from the form of the Kaehler potential for the moduli which is a regular function at finite values of the moduli.

Our solution for the loop corrections is valid for all \( r \) for which is valid the perturbation expansion in string coupling. In particular, since the dilaton increases at small distances, we can use both the tree-level and a loop-corrected solution for \( \frac{r}{P} > \epsilon V \). However, if we extrapolate the loop-corrected metric of the extremal black hole to the region of small \( r \), it can be interpreted as a metric

\[ g_{ii} = -g^{00} = 1 + \frac{P}{r + \epsilon V}. \]

with the smeared singularity at the origin.

Our approach is different from that in papers [18] based on the assumption that there is a “small” modulus which can be used as an expansion parameter for the loop-corrected action. In string-loop perturbative expansion, a natural expansion parameter is associated with the dilaton, and the loop correction to the tree-level prepotential is independent of the modulus \( y_1 \equiv S \).

Our treatment of spinor Killing equations is similar in spirit to [19]. However, in this paper were discussed only tree-level spinor Killing equations. Another distinction is that in [19] the emphasis was made on the form of solution at the stabilization point [20], whereas we were interested in coordinate dependence of solution.

Finally, in perturbative approach, we neglect the terms of the form \( O(e^{2\pi S}) \), and the duality properties of the full theory [21] cannot be checked in this setting.

ACKNOWLEDGMENTS

I thank Renata Kallosh for helpful correspondence.

This work was partially supported by the RFFR grant No 00-02-17679.
REFERENCES