A Note on Unitarity of Non-Relativistic Non-Commutative Theories

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Abstract

We analyze the unitarity of a non-relativistic non-commutative scalar field theory. We show that electric backgrounds spoil unitarity while magnetic ones do not. Furthermore, unlike its relativistic counterparts, unitarity can not be restored (at least at the level of one-to-one scattering amplitude) by adding new states to the theory. This is a signal that the model cannot be embedded in a natural way in string theory.

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1 Introduction

It has been proved [1, 2] that non-commutative (NC) field theories are non-unitary and acausal when the non-commutativity involves the time coordinate. The non-locality in time requires a more subtle hamiltonian formalism [3, 4] where one needs to add an extra time-like coordinate. Except for several special cases of light-like non-commutativity [5], this non-locality in time is responsible for spoiling unitarity.

From a string theoretical point of view, these problems seem reasonably well understood. Magnetic $B_{\mu\nu}$ backgrounds of string theory admit a decoupling limit of the massive modes [6, 7, 8] and lead to magnetic non-commutative field theories in the worldvolume of the D-branes. The unitarity of string theory is then inherited by the effective field theory.

On the other hand, electric backgrounds of string theory do not admit such a decoupling limit [9, 10, 11]. The lack of unitary of the effective field theory in the brane is just telling us that we should have taken into account all the massive string modes. In particular, it is found in [12] that adding new stringy states (the so-called $\chi$-particles) to a NC $\phi^4$ theory, one could understand and solve the unitarity failure at the level of one-to-one scattering amplitude.

Our work was doubly motivated. First of all, we wanted to check if these unitarity problems are present in non-relativistic theories too, where the treatment of space and time is completely different. In this paper, we study a non-relativistic NC $\phi^4$ theory in (2+1) dimensions, which can be nicely viewed as the realization of the Galileo group with two central extensions (the mass and the non-commutativity parameter $\theta$)[13]. We found an affirmative answer: only electric backgrounds break unitarity. This is the subject of chapter two.

Secondly, we wanted to understand whether unitarity could be restored by adding some new states. In such case, they would probably come from a non-decoupled non-relativistic theory of open strings, like that in [14, 15]. We found that such a procedure is not possible in our non-relativistic model, even for the simplest case of one-to-one scattering. Better understanding is therefore still required. We address this problem in section three.

2 Four Points Function and Unitarity

To set up our framework, let us start with a non-relativistic NC scalar field theory in $D = 2 + 1$ with quartic interactions and Lagrange density

$$L_{nr} = \phi^\dagger \left( i\partial_t + \nabla^2 \right) \phi - \frac{\lambda}{4} \phi^\dagger \phi^\dagger \phi^\dagger \phi.$$

Following the idea of [1], we will study the unitarity of the theory by checking whether the Optical Theorem is fulfilled at the level of two particles scattering. It should happen that

$$2 \text{ Im} \left( \begin{array}{c} p_1 \\ p_2 \\ p_{-k} \end{array} \right) = \begin{array}{c} p_1 \\ p_2 \\ q_1 \\ q_2 \end{array}$$

(2)

1The perturbative properties of this model in the magnetic case have been studied in [16] and exact results can be found in [17].

2It can be seen that having taken the other possible ordering of the vertex, i.e. $\phi^\dagger \phi^\dagger \phi^\dagger \phi$, would have lead to exactly the same unitarity problems.
The left hand side (LHS) and the right hand side (RHS) can be written as

\[ LHS = 2 \text{Im}\left(\frac{-i\lambda^2}{2} \cos^2 \tilde{\rho} p_1 p_2 \int \frac{d^2 k d\vec{k}}{(2\pi)^3} \left( \frac{\cos^2 \tilde{\rho}_k}{2} \right)^2 \delta\left(\frac{(\vec{k}^2 - \tilde{\rho}^2)(\vec{k}^2 - \hat{k}^2)}{2} + i\epsilon\right) \right) \]

\[ RHS = \frac{\lambda^2}{4\pi} \cos^2 \tilde{\rho} p_1 p_2 \int d^3 q_1 d^3 q_2 \delta(q_0^0 - \frac{q_0^2}{2})\delta(q_1^0 - \frac{q_1^2}{2})\delta(3)(p_1 + p_2 - q_1 - q_2)\cos^2 \tilde{\rho} q_1 q_2 \]

where \( \tilde{\rho}^\mu \equiv p_\mu \theta^\mu, \) \( P^\mu = p_1^\mu + p_2^\mu \) and the products are defined by \( pk \equiv p^0 k^0 - \vec{p} \cdot \vec{k}. \) Using the identity \( 2 \cos^2 x = 1 + \cos 2x \) for the cosinus involving integrated momenta, both sides can be written as a sum of a planar integral plus a non-planar one. It is straightforward to show that the planar parts are identical in both sides. Therefore, the only job left is to check for the non-planar ones. For the RHS it gives

\[ RHS|_{np} = \frac{\lambda^2}{4} \cos^2 \tilde{\rho} p_1 p_2 \int \frac{d^3 q_1}{2\pi} \frac{\cos \tilde{\rho} k}{2} \delta(P^0 - k^0 - \frac{\vec{P} - \tilde{\rho}}{2})\delta(k^0 - \frac{\vec{k}^2}{2}) \]

2.1 Magnetic Case

Take the non-commutativity only in the two spatial coordinates \([x, y] = i \theta\). In this case we have \( \vec{P}^0 = 0 \) and so we can take the cosinus of (3) out of the \( k^0 \) integral. Therefore, we can perform the \( k^0 \) integral using Cauchy’s theorem. We are left with

\[ LHS|_{np} = -\frac{\lambda^2}{2(2\pi)^2} \cos^2 \tilde{\rho} p_1 p_2 \int d^2 k \frac{\cos \tilde{\rho} k}{P^0 - \frac{\vec{k}^2}{2} - \frac{(\vec{P} - \tilde{\rho})^2}{2} + i\epsilon} \]

The imaginary part is extracted by using that \( (x + i\epsilon)^{-1} = \frac{P_\tau}{x} - i\pi \delta(x) \), and it is then straightforward to show that we obtain exactly (5). It can be easily seen that these last two steps are equivalent to replacing the internal propagators by delta functions. Indeed, this is nothing but a proof that the cutting rules are valid for the magnetic case. Notice that we have been able to check the Optical Theorem to all orders in \( \theta \).

2.2 Electric Case

Now, take non-commutativity between space and time, i.e. \([t, x] = i \theta\). The main difference with respect to the magnetic case is that now \( \vec{P}^0 \neq 0 \) and, therefore, the cosinus factor in (3) cannot be taken out of the \( k^0 \) integral. We will find that the order zero \( \theta \)-term is different from the one we obtain in expanding the RHS (5). Furthermore, a linear term arises, in contrast with the RHS, where the first \( \theta \) term is quadratic.

Here one needs to go through Feynman parameters and residue integrals. We arrive at:

\[ LHS|_{np} = \frac{i\lambda^2}{16\pi} \int_0^1 dx \frac{1}{|1 - 2x|} \left( e^{i f(P, \theta, x)} \Omega(P^0, \theta) + e^{i f(P, -\theta, x)} \Omega(P^0, -\theta) \right) \]

being

\[ f(P, \theta, x) \equiv \frac{|\tilde{P}_0|}{2|1 - 2x|} (2P^0(1 - x) - \tilde{P}^2 x(1 - x)) + \frac{\tilde{P}^2}{2|\tilde{P}_0|} |1 - 2x| - \tilde{P} \cdot \tilde{P}(1 - x) \]

\[ \Omega(P^0, \theta) \equiv \Theta(\tilde{P}^0) \Theta(x - \frac{1}{2}) + \Theta(\tilde{P}^0) \Theta(1 - x - \frac{1}{2}) \]

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where we have chosen the symbol $\Theta(x)$ to name the step function, not to be confused with the non-commutative parameter $\theta$.

The integral (7) cannot be solved exactly. Anyway, every term in (8) is linear in the non-commutativity parameter $\theta$, since $\vec{P}^0 = \theta P^1$ and $\vec{P} = (\theta P^0, 0)$. Therefore, we can expand the exponentials of (7) in order to obtain a power series in $\theta$ of the LHS. However, due to the singular behaviour of (8) about $x = \frac{1}{2}$, one must expand only the exponential of the non-singular terms. Taking all this into account we finally obtain

$$LHS|_{np} = \frac{\lambda^2}{16} + |\theta| \frac{\lambda^2}{32\pi} \left(|P^1|^2 + \frac{2\langle P^0 \rangle^2}{|P^1|}\right) + ... \quad (10)$$

The first term arises from expanding a gamma function with imaginary argument, in contrast with the logarithms one finds in the relativistic case [1]. Its value is exactly half of its RHS counterpart (5), and so unitarity is violated. The linear term is not present in (5) either. The non-analicity in $P^1$ should not come as a surprise, it is associated to the new infinite uncertainty relations introduced by the non-commutativity. Notice too that only the absolute value of $\theta$ appears in (10), in agreement with the original symmetry $\theta \rightarrow -\theta$ in (3).

### 3 Two Points Function and Possible New States

Once the non-unitarity of theory has been shown, we could try to fix it perturbatively by introducing some new states in the theory with the right couplings. In [12] it was shown that in the relativistic case one could introduce the so called $\chi$-particles to fix the unitarity failure. A similar analysis will show that there does not seem to be such a procedure for the non-relativistic case.

The optical theorem for the one-to-one scattering amplitude states that

$$2 \text{Im}[i A(p)] = 2 \text{Im} \left( \begin{array}{c} k \\ p \\ 0 \end{array} \right) = 0 \quad (11)$$

A short calculation shows that

$$A(p) = -i\lambda \int \frac{dk^0d^2k}{(2\pi)^3} \frac{\cos^2 \frac{j}{2} k}{k^0 - \vec{k}^2 + i\epsilon} = -i\lambda \frac{\Lambda^2}{16\pi} \frac{\exp\left(\frac{\vec{p}^2}{2\theta^2}\right)}{2\pi^2 |\vec{p}^0|} \quad (12)$$

In obtaining this result, we have introduced a hard cutoff for the planar integral (it diverges as in any commutative theory). For magnetic cases, we have $\vec{p}^0 = 0$. If we take this limit in our result (12), we recover the result of [16], i.e. $A(p) = \lambda \delta^{(2)}(\vec{p})/4\theta^2$. It has no imaginary part, and so unitarity is preserved. On the contrary, for electric cases $\vec{p}^0$ is finite, and there is always an imaginary contribution

$$2 \text{Im} [A(p)] = \frac{\lambda}{4\pi} \frac{\cos \frac{\vec{p}^2}{2\theta^2}}{|\vec{p}^0|} \quad (13)$$

The analog of this unitarity breakdown in the relativistic case [12] was a delta function. There, the authors reexpressed it as

$$2 \text{Im} [A(p)]_{\text{relativistic}} = \int \frac{d^3k}{2(2\pi)^3} \rho(\lambda, \theta)\delta^{(4)}(p - k) \quad (14)$$
and so they interpreted it as a term coming from the contribution of new states in the theory, with coupling $\rho(\lambda, \theta)$ to our $\phi$ field. It can be seen that this cannot be done in our case even if we allow for the coupling to depend on the momenta. This is due to the fact that our result (13) is a smooth function of the momenta, and so it can never be written as a delta function times a coupling.

4 Conclusions

We have shown that scalar non-relativistic non-commutative theories suffer from the same unitarity problems as the relativistic ones. Magnetic cases have passed the test to all orders in $\theta$, while electric ones failed in both the two and the four points functions. The only relativistic theories in which non-locality in time was compatible with unitary evolution were some light-like $\theta^{\mu\nu}$ backgrounds [5]. Since there are no such configurations for non-relativistic theories (light-cone coordinates are nonsense), non-locality in time always destroys unitarity.

We have also shown that the attempt of introducing new particles to our Lagrangian will never solve unitarity problems, not even at the level of the one-to-one scattering. The interpretation of the $\chi$-particles as playing the role of some string modes that we have been missing in our original effective field theory fails in our non-relativistic model, which is nothing but a signal that this model cannot be embedded in string theory.

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