We show how the motion of a charged particle near the horizon of an extreme Reissner-Nordström black hole can lead to different forms of conformal mechanics, depending on the choice of the time coordinate.
Recently, it has been shown that the motion of a charged particle near the horizon of an extremal Reissner-Nordström black hole can be described (through dimensional reduction to adS$_2$) by a model of conformal mechanics \cite{1}, which, for great black hole mass reduces to the model of De Alfaro, Fubini and Furlan (DFF) \cite{2}. When quantized, the DFF model has a continuous spectrum of positive energy eigenstates, but no normalizable ground state.

In \cite{3,4}, the absence of a ground state was interpreted as due to a wrong choice of coordinates in the black hole metric, which do not cover the entire manifold, and a different choice was proposed, that solves the problem. The new time coordinate corresponds to a different choice of conformal generators as Hamiltonian for the conformal mechanics, and gives rise to a ‘regularized’ version of the DFF model, possessing a normalizable ground state \cite{2}.

Although an algebraic proof of this fact was given in \cite{3,4}, it was not shown how to derive the new Hamiltonian from the motion of a charged particle in the near-horizon Reissner-Nordström background, and in particular the relation between the parameters of the black hole and those of the regularized DFF Hamiltonian remained obscure.

In this letter, we derive explicitly the regularized DFF Hamiltonian from the motion of a charged particle in a suitably parametrized adS$_2$ spacetime, in the ‘non-relativistic’ limit, by taking into account higher order corrections in the inverse mass of the black hole. We also discuss a further possible choice of time coordinate, which however leads to a spectrum of energy unbounded from below.

It is well known that the extreme Reissner-Nordström metric in the near-horizon limit, $r/M \gg 1$ can be put in the Bertotti-Robinson form:

$$ds^2 = - \left( \frac{r}{M} \right)^2 dt^2 + \left( \frac{M}{r} \right)^2 dr^2 + M^2 d\Omega^2,$$
\hspace{1cm} (1)

which is a direct product adS$_2 \times S^2$. The motion of a test particle in this background can be studied by considering the 2-dimensional anti-de Sitter section \cite{1}, namely

$$ds^2 = - \left( \frac{r}{M} \right)^2 dt^2 + \left( \frac{M}{r} \right)^2 dr^2.$$
\hspace{1cm} (2)

This provides a model of conformal mechanics in which the SO(1, 2) isometry of the background spacetime is realized as a one-dimensional conformal symmetry. The $so(1, 2)$ algebra is generated by the Killing vectors $h = \partial_t$, $d = t\partial_t - r\partial_r$, $k = (t^2 + M^4/r^2)\partial_t - 2tr\partial_r$, which obey the commutation relations,

$$[d, h] = -h, \quad [d, k] = k, \quad [h, k] = 2d.$$
\hspace{1cm} (3)

Defining a new coordinate $q = -2M \sqrt{\frac{M}{r}}$ the metric (2) transforms into

$$ds^2 = - \left( \frac{2M}{q} \right)^4 dt^2 + \left( \frac{2M}{q} \right)^2 dq^2.$$
\hspace{1cm} (4)
The Hamiltonian of a particle of mass $m$ and charge $Q$ in this background is \[ H = \left( \frac{2M}{q} \right)^2 \left( \sqrt{m^2 + \frac{q^2 p_q^2}{4M^2}} - Q \right). \] (5)
where $p_q$ is the momentum conjugate to $q$ and we have restricted our attention to the radial motion. In the 'non-relativistic' limit \[ M \to \infty, \ m - Q \to 0, \] with $(m - Q)M^2$ fixed, the Hamiltonian (5) reduces to that of DFF quantum mechanics \[ H = \frac{1}{2} \left( \frac{p_q^2}{m} + \frac{g}{q^2} \right), \] (6)
where the coupling constant $g$ is given by $8M^2(m - Q)$.

It is known that the DFF model has no ground state and its spectrum is continuous. This is due to the fact that the $so(1,2)$ generator $h$ associated to the Hamiltonian (6) is noncompact, and can also be understood as a consequence of the scaling invariance of the DFF model which does not allow a choice of a scale for the energy. From the black hole point of view, the absence of a ground state can be interpreted as due to the existence of a fixed set for the Killing vector $\partial_t$, corresponding to its Killing horizon \[ 3. \] This problem can be remedied by adopting globally well defined coordinates,

\[ u = \arctan \frac{2rt}{r + r^{-1} - rt^2}, \quad v = \frac{1}{2}(r - r^{-1} + rt^2), \]
in terms of which the metric takes the form

\[ ds^2 = - \left[ \left( \frac{v}{M} \right)^2 + 1 \right] du^2 + \left[ \left( \frac{v}{M} \right)^2 + 1 \right]^{-1} dv^2. \] (7)

In this parametrization, the timelike Killing vector $\partial_u$ corresponds to the compact generator of $so(1,2)$, $\partial_u = h + k$. It is known that this generator admits a discrete spectrum with well-defined ground state \[ 2. \]

In order to obtain the Hamiltonian for a charged particle which is conformal in the appropriate limit, we make another change of coordinates, which casts the metric in the form, inspired by (4),

\[ ds^2 = -A^2(x) du^2 + A(x) dx^2. \]

This is obtained by defining

\[ x = \int \frac{dv}{\left[ \left( \frac{v}{M} \right)^2 + 1 \right]^{3/4}}. \]

Unfortunately, this is an elliptic integral. In order to obtain a closed form for the metric, we expand the integral for $M/v \ll 1$ (near-horizon limit), keeping track of the first order corrections in $M/v$. It results

\[ x \approx -2M \sqrt{\frac{M}{v}} \left( 1 - \frac{3M^2}{20v^2} \right), \quad \text{i.e.} \quad v \approx \frac{4M^3}{x^2} \left[ 1 - \frac{3}{10} \left( \frac{x}{2M} \right)^4 \right], \]
and then
\[ ds^2 \approx -\left(\frac{2M}{x}\right)^4 \left[ 1 + \frac{2}{5} \left(\frac{x}{2M}\right)^4 \right] du^2 + \left(\frac{2M}{x}\right)^2 \left[ 1 + \frac{1}{5} \left(\frac{x}{2M}\right)^4 \right] dx^2. \] (8)

The Hamiltonian for a charged particle moving in this background is given by
\[ H \approx \left(\frac{2M}{x}\right)^2 \left[ 1 + \frac{1}{5} \left(\frac{x}{2M}\right)^4 \right] \left[ \sqrt{m^2 + x^2 p_x^2} \left[ 1 - \frac{1}{5} \left(\frac{x}{2M}\right)^4 \right] - Q \right], \] (9)
which, in the 'non-relativistic' limit \( M \to \infty, m - Q \to 0 \), reduces to
\[ H = \frac{1}{2} \left( \frac{p_x^2}{m} + \frac{g}{x^2} + \omega^2 x^2 \right), \] (10)
where \( g = 8M^2(m - Q), \omega^2 = (m - Q)/10M^2 \). This has the form of the regularized DFF Hamiltonian, which was introduced in [2] in order to obtain a discrete energy spectrum with normalizable ground state. The parameter \( \omega \) acts as an infrared cutoff, which breaks the scaling invariance, setting the scale for the energy. In the approximation of great \( M \), \( \omega^2 \ll g \). In [3,4] was argued that (10) should be related to the motion of a charged particle in the background (7), but no explicit derivation was given, and in particular the value of \( \omega \) was left undetermined. The spectrum of energy can now be obtained from the results of [2] and reads:
\[ E_n = \sqrt{\frac{2m(m - Q)}{5}} \left( 2n + 1 + \sqrt{2 \frac{m - Q}{m} + \frac{1}{4}} \right), \] which is independent from \( M \). The separation of levels tends to zero for \( m \to Q \).

To complete our discussion, we must consider another natural parametrization of the two-dimensional anti-de Sitter space, which is obtained through the change of coordinates [5]
\[ \rho = rt, \quad \tau = \arctanh \frac{r + r^{-1} - rt^2}{r - r^{-1} + rt^2}. \]

In these coordinates the metric takes the form
\[ ds^2 = - \left[ \left(\frac{\rho}{M}\right)^2 - 1 \right] d\tau^2 + \left[ \left(\frac{\rho}{M}\right)^2 - 1 \right]^{-1} d\rho^2, \] (11)
and the timelike Killing vector \( \partial_\tau \) corresponds to the non-compact \( so(1,2) \) generator \( d \). This parametrization has not been considered previously in this context.

As before, one can define a new coordinate
\[ \sigma = \int \frac{d\rho}{\left[ \left(\frac{\rho}{M}\right)^2 - 1 \right]^{3/4}} \approx -2M \sqrt{\frac{M}{\rho}} \left( 1 + \frac{3M^2}{20\rho^2} \right), \]
in terms of which the metric takes the form
\[ ds^2 \approx -\left( \frac{2M}{\sigma} \right)^4 \left[ 1 - \frac{2}{5} \left( \frac{\sigma}{2M} \right)^4 \right] d\tau^2 + \left( \frac{2M}{\sigma} \right)^2 \left[ 1 - \frac{1}{5} \left( \frac{\sigma}{2M} \right)^4 \right] d\sigma^2. \] (12)

The Hamiltonian for a charged particle moving in this background is given by
\[ H \approx \left( \frac{2M}{\sigma} \right)^2 \left[ 1 + \frac{1}{5} \left( \frac{\sigma}{2M} \right)^4 \right] \left[ \sqrt{m^2 + \frac{\sigma^2 p^2}{4M^2}} \left[ 1 - \frac{1}{5} \left( \frac{\sigma}{2M} \right)^4 \right] - Q \right], \] (13)

which, in the 'non-relativistic' limit, reduces to
\[ H = \frac{1}{2} \left( \frac{p_\sigma^2}{m} + \frac{g}{\sigma^2} - \omega^2 \sigma^2 \right), \] (14)

where \( g \) and \( \omega \) are defined as before. We have again a regularized DFF model, but now the harmonic potential has the wrong sign, leading again to the absence of a ground state and to a spectrum unbounded from below [2]. This is in agreement with the fact that the Hamiltonian corresponds in this case to a non-compact generator \( d \). From the black hole point of view, this can again be related to the presence of a horizon at \( \rho = M \).

References