4D, $N = 1$ Born-Infeld Supergravity

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Abstract

We propose the 4D, $N = 1$ supergravitational analogues (avatars) of the 4D, $N = 1$ supersymmetric Born-Infeld action in four dimensions for the first time, by using superspace. In particular, a new Born-Infeld type generalization of the Weyl supergravity action is given. A natural new Born-Infeld type generalization of the Einstein supergravity is found as well. We also briefly discuss a construction of the four-dimensional Born-Infeld-Einstein supergravity from the AdS supergravity in five dimensions, which seems to be very natural in our approach.

¹Supported in part by NSF grant # PHY-98–02551
²Supported in part by the ‘Deutsche Forschungsgemeinschaft’
1 Introduction

The Born-Infeld (BI) electrodynamics [1] is the non-linear (gauge- and Poincaré-invariant) generalization of the Maxwell electrodynamics. The BI theory shares with the Maxwell theory electric-magnetic self-duality [2] and physical propagation (no shock waves) [3], which are quite non-trivial in the non-linear case. Similarly to the Maxwell Lagrangian, the BI Lagrangian is independent upon the spacetime derivatives of the Maxwell field strength,

\[ \mathcal{L}_{\text{BI}}(F) = \frac{1}{b^2} \left\{ 1 - \sqrt{-\det(\eta_{\mu\nu} + bF_{\mu\nu})} \right\}, \]

(1.1)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \mu, \nu = 0, 1, 2, 3 \), and \( b \) is the dimensional coupling constant. The BI action is also known to be the low-energy bosonic part of the (gauge-fixed) effective action of a D3-brane filling in four spacetime dimensions (see ref. [4] for a recent review). The corresponding \( N = 1 \) supersymmetric abelian BI action in four dimensions is the Goldstone-Maxwell action associated with Partial (1/2) Spontaneous Supersymmetry Breaking (PSSB) \( N = 2 \) to \( N = 1 \), whose Goldstone fields belong to a (Maxwell) vector supermultiplet with respect to unbroken \( N = 1 \) supersymmetry [5, 6].\(^3\) The similar \( N = 2 \) supersymmetric abelian BI action [9] is the relevant (low-energy) part of the effective worldvolume action of a D3-brane in six dimensions [10], since the \( N = 2 \) BI action [9] is the most relevant part of the \( N = 2 \) Goldstone-Maxwell action associated with the PSSB \( N = 4 \) to \( N = 2 \), modulo terms with spacetime derivatives of the \( N = 2 \) Maxwell superfield strength [9, 10, 11, 12].

It is of considerable interest to construct possible gravitational analogues of the BI action (see, e.g., refs. [13, 14] for earlier discussions without supersymmetry). A supersymmetric BI action possesses more physically important features when compared to its purely bosonic BI part (e.g. PSSB), while supersymmetry also implies more constraints on a BI-type (non-linear in the curvature) supergravity when compared to the purely bosonic theory. This is the main idea of this paper, which is based on manifest local \( N = 1 \) supersymmetry as the sole construction tool. Our investigation may be considered as part of the more ambitious programm of summing up gravitational corrections in string theory. To the best of our knowledge, no reasonable proposal for a BI-supergravity action was ever made. Further constraints like PSSB, ghost freedom and duality on the top of manifest local \( N = 1 \) supersymmetry will be considered elsewhere.

Supersymmetry does not necessarily prefer the standard determinantal form of the BI action in eq. (1.1). Moreover, eq. (1.1) is not even the most compact (and,

\(^3\)See also refs. [7, 8] for an earlier construction of the \( N = 1 \) supersymmetric BI actions.)
hence, most elegant and simple) form of the BI theory! The bosonic variable having the most natural $N = 1$ supersymmetric extension (with linearly realized $N = 1$ supersymmetry in superspace) is given by [5]

$$\omega = \alpha + i\beta , \quad \text{where} \quad \alpha = \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \equiv \frac{1}{4} F^2 \quad \text{and} \quad \beta = \frac{1}{4} F^{\mu \nu} \tilde{F}_{\mu \nu} \equiv \frac{1}{4} F \tilde{F} , \quad (1.2)$$

and $\tilde{F}_{\mu \nu}$ is the dual tensor, $\tilde{F}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \lambda \rho} F^{\lambda \rho}$. The BI Lagrangian (1.1) can be rewritten to the form ($b = 1$)

$$\mathcal{L}_{\text{BI}}(\omega, \bar{\omega}) = 1 - \sqrt{1 + (\omega + \bar{\omega}) + \frac{1}{4}(\omega - \bar{\omega})^2} , \quad (1.3)$$

or, equivalently,

$$\mathcal{L}_{\text{BI}}(\omega, \bar{\omega}) = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int.}} \equiv - \frac{1}{2} (\omega + \bar{\omega}) + \omega \bar{\omega} \mathcal{Y}(\omega, \bar{\omega}) , \quad (1.4)$$

with the structure function

$$\mathcal{Y}(\omega, \bar{\omega}) \equiv \frac{1}{1 + \frac{1}{2} (\omega + \bar{\omega}) + \sqrt{1 + (\omega + \bar{\omega}) + \frac{1}{4}(\omega - \bar{\omega})^2}} . \quad (1.5)$$

The remarkably simple equivalent form of the BI action [5, 6],

$$\mathcal{L}_{\text{BI}}(\omega, \bar{\omega}) = - \frac{1}{2} (\chi + \bar{\chi}) = - \text{Re} \chi , \quad (1.6)$$

arises as a solution to the non-linear constraint

$$\chi = - \frac{1}{2} \chi \bar{\chi} + \omega . \quad (1.7)$$

This Non-Linear Sigma-Model (NLSM) form of the BI action is quite natural from the viewpoint of PSSB [6, 15]. Indeed, to spontaneously break any rigid symmetry, one may start with a free action that is invariant under the linearly realized symmetry, and then impose an invariant non-linear constraint that gives rise to the NLSM whose solutions break the symmetry.

Unlike the Maxwell action minimally coupled to gravity,

$$S_M = - \frac{1}{4} \int d^4 x \sqrt{-g} g^{\mu \lambda} g^{\nu \rho} F_{\mu \nu} F_{\lambda \rho} , \quad (1.8)$$

which is invariant under the local Weyl transformations,

$$g_{\mu \nu} \rightarrow e^{2\lambda(x)} g_{\mu \nu} \quad A_{\mu} \rightarrow A_{\mu} , \quad (1.9)$$

the BI action minimally coupled to gravity is obviously not Weyl-invariant. Nevertheless, the BI action can be made Weyl-invariant by using a conformal compensator $\phi(x)$ with the Weyl transformation law

$$\phi \rightarrow e^{-\lambda(x)} \phi . \quad (1.10)$$
The modified BI action

\[ S = \int d^4x \left\{ \sqrt{\det(\phi^2 g_{\mu\nu})} - \sqrt{\det(\phi^2 g_{\mu\nu} + F_{\mu\nu})} \right\} \quad (1.11) \]

is obviously invariant under the transformations (1.9) and (1.10). Equation (1.11) can be further generalized to [13]

\[ S_{DG} = \int d^4x \left\{ \sqrt{\det(\phi^2 g_{\mu\nu})} - \sqrt{\det(\phi^2 g_{\mu\nu} + \phi^{-2} D_{\mu} \phi D_{\nu} \phi + F_{\mu\nu})} \right\} , \quad (1.12) \]

which is also Weyl-invariant when using the Weyl-covariant derivative \( D_{\mu} = \partial_{\mu} \phi + A_{\mu} \) and the abelian gauge transformation law \( A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \lambda \). The use of Weyl (conformal) compensators is the standard tool in supergravity [16].

There are many ways to add a non-minimal coupling to gravity in the BI action as well as to build new ‘pure BI gravity’ actions by using curvature-dependent terms. For example, if one insists on the determinantal form of the bosonic BI gravity, one can simply substitute the metric \( g_{\mu\nu} \) under the square root of the determinant by \( (g_{\mu\nu} + \kappa^2 R_{\mu\nu}) \), where \( R_{\mu\nu} \) is the Ricci tensor of the metric \( g_{\mu\nu} \) and \( \kappa \) is a constant of dimension of length,

\[ S = \frac{1}{\kappa^4} \int d^4x \left\{ \sqrt{\det(g_{\mu\nu})} - \sqrt{\det (g_{\mu\nu} + \kappa^2 R_{\mu\nu})} \right\} . \quad (1.13) \]

The expansion of this action in powers of \( \kappa^2 \) yields the Einstein-Hilbert action as the leading contribution. The equations of motion for the action (1.13) are satisfied by any Ricci-flat metric, e.g. by the Schwarzschild black hole metric. Hence, any BI-type action merely depending upon the Ricci tensor is not going to remove the black hole singularity at the origin (no taming). In other words, the Weyl tensor should also enter the BI-type gravity action, and we now need a reasonable proposal for it. Of course, one could simply extend the determinantal prescription by using the most general substitution under the square root of the determinant, \( \text{viz} \)

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa^2 R_{\mu\nu} + \zeta X_{\mu\nu} , \quad (1.14) \]

where \( X_{\mu\nu} \) is any tensor quadratic or higher in the full curvature, and \( \zeta \) is yet another dimensional constant [13]. Unfortunately, all these prescriptions do not have natural supersymmetric extensions and, therefore, they are ignored in what follows.

The gravitational analogues of the Maxwell variables in eq. (1.2) are given by

\[ \alpha_G = R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} \quad \text{and} \quad \beta_G = R^{\mu\nu\lambda\rho} \tilde{R}_{\mu\nu\lambda\rho} , \quad (1.15) \]

where \( R_{\mu\nu\lambda\rho} \) is the full (Riemann-Christoffel) curvature tensor, \( \tilde{R}_{\mu\nu\lambda\rho} \) is the dual (in curved spacetime) curvature, while all indices are raised and lowered by the use of
$g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ as usual. The BI-NLSM prescription (1.7) gives rise to the non-linear (in the curvature) Born-Infeld-Gravity Lagrangian

$$\mathcal{L}_{\text{BI2G}} = -\sqrt{-g} \text{Re} \chi_G , \quad \chi_G = -\frac{1}{2} \chi G \chi G + \omega_G , \quad \text{and} \quad \omega_G = \alpha_G + i \beta_G , \quad (1.16)$$

which appears to be the BI-type extension of the quadratically-generated gravity with $\mathcal{L}_{2G} = -\sqrt{-g} R^2_{\mu\nu\lambda\rho}$. Requiring the existence of a locally $N = 1$ supersymmetric extension rules out the bosonic BI action (1.16), though it becomes possible after replacing the curvature tensor by the Weyl tensor above (sect. 3).

The action (1.16) does not have the Einstein-Hilbert (linear in the curvature) term. Of course, in the bosonic theory this drawback can be easily corrected, e.g. by combining the two prescriptions above,

$$\mathcal{L}_{\text{BIG}} = \left[ \sqrt{-\det(g_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu} + \kappa^2 R_{\mu\nu})} \right] \left( \kappa^{-4} + \text{Re} \chi_G \right) , \quad (1.17)$$

in order to get the Einstein-Hilbert term as the leading contribution. Of course, this is rather artificial and, in fact, it does not have a supersymmetric extension too. Some possible resolutions to this problem are given in the next sections.

To the end of this section we give a brief comment about ghosts. As is well-known, the Gauss-Bonnet density $\sqrt{-g} \beta_G$ or

$$E = \sqrt{-g} \left[ R^2_{\mu\nu\lambda\rho} - 4 R^2_{\mu\nu} + R^2 \right] \quad (1.18)$$

is a total derivative in four dimensions, so that any quadratically-generated gravity is actually proportional to a linear combination of $R^2_{\mu\nu}$ and $R^2$, which always results in the presence of ghosts in its free part. The ghost-freedom to this order can be obtained by arranging the ghost-free combination, $\sqrt{-g} \left( c_1 - \frac{1}{2} R \right) + c_2 E$ with some constants $(c_1, c_2)$ as the leading contribution (modulo curvature-cubed terms). It is worth mentioning, however, that the standard BI action (1.1) is actually ghost-free because of its special non-polynomial (or non-perturbative) structure. The quartic terms in the BI action are proportional to the Euler-Heisenberg term, $(F^2)^2 + (F \tilde{F})^2$, whereas the following terms are not ghost-free at any given (finite) order. It would be interesting to check whether any of the actions (1.16) or (1.17) is ghost-free or not.

As will be shown in the next sections, manifest local $N = 1$ supersymmetry leads to certain restrictions on the bosonic BI gravity, while the ghost-freedom is not automatically resolved by introducing supersymmetry alone. We believe that it is very natural to search for a supergravitational BI action by using a reformulation of gravity as a non-abelian gauge theory. This then implies first a construction of the Non-abelian Born-Infeld (NBI) theory. The $N = 1$ supersymmetric NBI action, proposed by one of the authors in ref. [17] (see sect. 2), is going to be used in sect. 3 for a construction of the BI supergravity action.
2 \( N = 1 \) supersymmetric BI and NBI actions

In this section we briefly recall the \( N = 1 \) supersymmetric abelian BI action in superspace \([5,6]\) and its \( N = 1 \) Non-abelian (NBI) generalization in four dimensions \([17]\).

The BI action (1.1) in the form (1.6) is most convenient for supersymmetrization in superspace. One replaces the abelian bosonic field strength \( F_{\mu\nu} \) by the abelian \( N = 1 \) chiral spinor superfield strength \( W_\alpha \) obeying the \( N = 1 \) superspace Bianchi identities
\[
\bar{D}_\dot{\alpha} W_\alpha = 0, \quad D^\alpha W_\alpha - \bar{D}_\dot{\alpha} \bar{W}^\dot{\alpha} = 0 , \quad \alpha = 1,2 ,
\]
while the superextension of \( \omega \) is simply given by \( W^2 \). The \( N = 1 \) manifestly supersymmetric abelian BI action \([5]\) in the NLSM form reads \([6]\)
\[
S_{1BI} = \int d^4x d^2\theta X + \text{h.c.},
\]
where the \( N = 1 \) chiral superfield Lagrangian \( X \) obeys the non-linear constraint
\[
X = \frac{1}{2}X \bar{D}^2 \bar{X} + \frac{1}{2}W^\alpha W_\alpha .
\]
The iterative solution to eq. (2.3) gives rise to the superfield action \([5]\)
\[
S_{1BI} = \frac{1}{2} \left( \int d^4x d^2\theta W^2 + \text{h.c.} \right) + \int d^4x d^4\theta \mathcal{Y}(\frac{1}{2}D^2W^2, \frac{1}{2}\bar{D}^2\bar{W}^2)W^2\bar{W}^2
\]
with \textit{the same} structure function (1.5) of the bosonic BI theory.

It is worth mentioning that the NLSM form (2.2) of the \( N = 1 \) BI action is also most useful in proving its invariance under the second (non-linearly realized and spontaneously broken) supersymmetry with rigid spinor parameter \( \eta^\alpha \) \([5,6]\),
\[
\delta_2 X = \eta^\alpha W_\alpha , \quad \delta_2 W_\alpha = \eta_\alpha \left( 1 - \frac{1}{2} \bar{D}^2 \bar{X} \right) + i\bar{\eta}^\gamma \partial_\gamma \cdot X ,
\]
and its \( N = 1 \) supersymmetric electric-magnetic self-duality as well. The latter amounts to a verification of the non-local constraint \([11]\)
\[
\int d^4x d^2\theta(W^2 + M^2) = \int d^4x d^2\bar{\theta}(\bar{W}^2 + \bar{M}^2) , \quad \text{where} \quad \frac{i}{2}M_\alpha = \frac{\delta S_{1BI}}{\delta W^\alpha} .
\]

It is not difficult to put the \( N = 1 \) supersymmetric BI action (2.2) into the \( N = 1 \) superconformal form by inserting the conformal compensator \( N = 1 \) chiral superfield) \( \Phi \) into the non-linear constraint (2.3) as follows \([11]\):
\[
X = \frac{X}{2\Phi^2} \bar{D}^2 \left( \frac{\bar{X}}{\Phi^2} \right) + \frac{1}{2}W^\alpha W_\alpha .
\]
Equation (2.2) is recovered from eq. (2.7) in the gauge $\Phi = 1$.

The simple structure of the $N = 1$ supersymmetric abelian BI action (2.2) dictated by the Gaussian non-linear constraint (2.3) allows us to easily construct its non-abelian (NBI) generalization [17] that may also be relevant for the effective description of the D3-brane clusters (i.e. the D3-branes on the top of each other).

The $N = 1$ Super-Yang-Mills (SYM) theory in $N = 1$ superspace is defined by the standard off-shell constraints [16]:

$$\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_\alpha, \nabla_\beta^*\} = 0, \quad \{\nabla_\alpha, \nabla_\beta^*\} = -2i \nabla_\alpha^*,$$

$$[\nabla_\alpha, \nabla_\beta^*] = 2i \varepsilon_{\alpha\beta} \hat{W}_\beta^*, \quad [\nabla_\alpha^*, \nabla_\beta^*] = 2i \varepsilon_{\alpha\beta} \hat{W}_\beta,$$  \hspace{1cm} (2.8)

in terms of the $N = 1$ covariantly-chiral (Lie algebra-valued) gauge superfield strength $\hat{W}_\alpha = \hat{W}_\alpha^a t_a$ obeying the Bianchi identities \(^4\)

$$\nabla_\alpha \hat{W}_\alpha = 0, \quad \nabla_\alpha \hat{W}_\alpha^* = \nabla^\alpha \hat{W}_\alpha.$$  \hspace{1cm} (2.9)

The natural $N = 1$ supersymmetric NBI action is [17]

$$S_{NBI} = \int d^4x d^2\theta \text{tr} \hat{\Phi} + \text{h.c.},$$  \hspace{1cm} (2.10)

whose $N = 1$ covariantly chiral Lagrangian $\hat{\Phi}$ is subject to the ‘minimal’ non-abelian generalization of the abelian non-linear constraint (2.3),

$$\hat{\Phi} = \frac{1}{2} \hat{\Phi} \nabla^2 \hat{\Phi} + \frac{1}{2} \hat{W}^2.$$  \hspace{1cm} (2.11)

The leading contribution to the NBI action (2.10) is the standard $N = 1$ SYM action in superspace [16],

$$S_{SYM} = \frac{1}{2} \int d^4x d^2\theta \text{tr} \hat{W}^2 + \text{h.c.}$$  \hspace{1cm} (2.12)

The next (NBI) correction in the Yang-Mills sector (in components) [17],

$$\frac{1}{4} \text{tr} \left[ (F^2)^2 + (F\tilde{F})^2 \right],$$  \hspace{1cm} (2.13)

appears to be the non-abelian version of the Euler-Heisenberg term $\frac{1}{4} \left[ (F^2)^2 + (F\tilde{F})^2 \right]$ which is present in the BI Lagrangian (1.1) as the leading $F^4$ correction to the Maxwell term.

The $N = 2$ super BI and NBI actions [9, 10, 11, 12, 17, 18] have similar features.

\(^4\)The Lie algebra generators $t_a$ obey the relations $[t_a, t_b] = f_{abc} t_c$ and $\text{tr}(t_a t_b) = -2\delta_{ab}$.
3 Born-Infeld supergravity

As is well-known, $N = 1$ supergravity in four dimensions is most naturally described in curved superspace $z^M = (x^m, \theta^\mu, \bar{\theta}^\dot{\mu})$, $m = 0, 1, 2, 3$ and $\mu = 1, 2$, where we now have to distinguish between curved $(M)$ and flat $(A)$ indices related by a supervielbein $E_A^M$ and its inverse $E_M^A$ with $E = \text{Ber}(E_A^M) \neq 0$ [16]. The supervielbein $E_A^M$ and a superconnection $\Omega_A$ are most conveniently described by (super) one-forms,

$$E_A = E_A^M(z) \partial_M \quad \text{and} \quad \Omega = dz^M \Omega_M(z) = E^A \Omega_A,$$

where $\Omega_A$ take their values in the Lorentz algebra,

$$\Omega_A = \frac{1}{2} \Omega_A^{bc}(z) M_{bc} = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \Omega_A^{\beta\dot{\gamma}} \bar{M}^{\dot{\gamma}}_{\beta\dot{\gamma}},$$

and $M_{bc} \sim (M_{\beta\gamma}, M_{\beta\dot{\gamma}})$ are the Lorentz generators. The curved superspace covariant derivatives

$$D_A = (D_a, D_\alpha, D^\dot{\alpha}) = E_A + \Omega_A,$$

obey the algebra

$$[D_A, D_B] = T_{AB}^C D_C + R_{AB},$$

where the supertorsion $T_{AB}^C$ and the (Lorentz algebra-valued) supercurvature $R_{AB} = \frac{1}{2} R_{AB}^{cd} M_{cd} = R_{AB}^{\beta\gamma} M_{\beta\gamma} + R_{AB}^{\beta\dot{\gamma}} \bar{M}^{\dot{\gamma}}_{\beta\dot{\gamma}}$ have been introduced.

The universal (conformal) supergravity constraints can be divided into three sets. The first set of the constraints is needed for the existence of chiral superfields in curved superspace — these are the so-called representation-preserving constraints:

$$T_{\alpha\beta}^{\beta \gamma} = T_{\alpha \beta}^{\beta \dot{\gamma}} = T_{\alpha \beta}^{\gamma} = 0.$$

(3.5a)

The second set of the constraints is needed to solve the vector covariant derivative in terms of the spinor ones — these are the so-called conventional constraints of type-I:

$$T_{\alpha \beta} = T_{\alpha \beta}^{\gamma} = R_{\alpha \beta}^{\sigma^{cd}} = 0, \quad T_{\alpha \beta}^{\sigma} = -2i (\sigma^{\sigma})_{\alpha \beta}.$$

(3.5b)

The third set of the constraints is used to determine the spinor supervielbeins in terms of the spinor supervielbeins — these are the so-called conventional constraints of type-II:

$$T_{\alpha \beta}^{\gamma} = T_{\alpha \beta}^{\dot{\gamma}} = T_{\alpha \beta}^{\beta} = T_{\alpha \beta}^{\beta \dot{\gamma}} = 0.$$

(3.5c)

The so-called minimal $N = 1$ supergravity arises by adding extra constraints,

$$T_{\alpha \beta \gamma \gamma} = \varepsilon_{\alpha \beta \dot{\gamma}} T_{\gamma} = 0 \quad \text{and} \quad T_{\alpha \beta \gamma \gamma} = \varepsilon_{\alpha \beta \dot{\gamma}} T_{\dot{\gamma} \gamma} = 0.$$

(3.5d)
Taken together, equations (3.5) are just the standard (Wess-Zumino) $N = 1$ supergravity constraints [19]. As a result of the constraints and the Bianchi identities, all the superfield components of the supertorsion and the supercurvature appear to be merely dependent upon three (constrained) supertorsion tensors: the complex (covariantly) chiral scalar superfield $R$, the real vector superfield $G_a$ and the complex (covariantly) chiral superfield $W_{\alpha\beta\gamma}$ that is totally symmetric with respect to its spinor indices [16]. The bosonic superfield $R$ has an auxiliary complex scalar $B$ as the leading component, while it also contains the spacetime scalar curvature as another bosonic field component. Similarly, the bosonic vector superfield $G_a$ has the spacetime Ricci curvature amongst its field components. The fermionic superfield $W_{\alpha\beta\gamma}$ has the gravitino field strength as its leading component, while it also contains the spacetime Weyl tensor $C_{\alpha\beta\gamma\delta}$ (totally symmetric on its spinor indices) as the fermionic field component.

The $N = 1$ (Einstein) supergravity action [16]

$$S_{SG} = -\frac{3}{\kappa^2} \int d^8z E^{-1}$$

is just the supervolume of curved $N = 1$ superspace. Here $\kappa$ is the gravitational coupling constant of dimension of length. A chiral local density also exists in curved $N = 1$ superspace [16],

$$\mathcal{E} = -\frac{1}{4} \mathcal{R}^{-1} (\bar{D}^2 - 4 \mathcal{R}) E^{-1} .$$

The simple ‘covariantizing’ rules in the chiral $N = 1$ superspace are given by

$$d^4x d^2\theta \rightarrow d^4x d^2\theta \mathcal{E} \quad \text{and} \quad \bar{D}^2 \rightarrow (\bar{D}^2 - 4 \mathcal{R}) .$$

The BI-type non-linear superfield constraint (2.11) can be considered as the powerful tool converting any fundamental (input) chiral superfield Lagrangian $(\hat{W}^2)$ into the corresponding BI-type chiral Lagrangian $(\hat{\Phi})$ in superspace. A natural candidate for the BI supergravity action just arises along these lines. Indeed, the supergravitational analogue of the (covariantly chiral) $N = 1$ SYM spinor superfield strength $W^\alpha t_a$ is given by the super-Weyl curvature tensor $W_{\alpha\beta\gamma} M^{\beta\gamma}$ that is covariantly chiral in $N = 1$ curved superspace of the $N = 1$ minimal supergravity. This essentially amounts to replacing the Yang-Mills gauge group in the $N = 1$ NBI action (sect. 2) by the Lorentz group. The Weyl supergravity action [16]

$$S_W = \int d^4x d^2\theta \mathcal{E} \text{tr} W^2 + \text{h.c.}$$

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can then be extended to the corresponding Born-Infeld-Weyl (BIW) supergravity action

\[ S_{BIW} = \int d^4x d^2\theta \text{tr} \mathcal{F} + \text{h.c.} \quad (3.10) \]

whose covariantly chiral (Lorentz algebra-valued) Lagrangian \( \mathcal{F} \) is a solution to the non-linear superfield constraint

\[ \mathcal{F} = \frac{1}{2} \mathcal{F}(\mathcal{D}^2 - 4\mathcal{R}) + W^2 \quad (3.11) \]

It is worth noticing that the superfield \( \mathcal{R} \) also enters the action (3.11).

The subleading correction to the Weyl supergravity action in the BIW theory (3.10) is given by

\[ S_{BR} = \frac{1}{2} \int d^4x d^4\theta E^{-1} W^2 \mathcal{W}_{\alpha\beta\gamma} \mathcal{W}^{\alpha\beta\gamma} \quad (3.12) \]

whose purely bosonic (gravitational) part is proportional to the square of the Bel-Robinson (BR) tensor [20]

\[ T_{mnpq} = R_{mspt} R_n^s t + R_{msqt} R_n^s t - \frac{1}{2} g_{mn} R_{prst} R_{q}^{rst} \quad (3.13) \]

In four dimensions, the BR tensor (3.13) can be identically rewritten to

\[ T_{mnpq} = R_{mspt} R_n^s t + \tilde{R}_{mspt} \tilde{R}_n^s t \quad (3.14) \]

where \( \tilde{R}_{mspt} \) is the dual curvature. Moreover, in four dimensions the BR tensor is known to be symmetric in all four indices and pairwise traceless [20]. The BR tensor squared, \( T_{mnpq}^2 \), should therefore be considered as the gravitational analogue of the subleading (Euler-Heisenberg) term, \( (F^2)^2 + (\tilde{F}\tilde{F})^2 \), in the abelian BI action (1.1).

Of course, unlike the Weyl supergravity action (3.9), the BIW action (3.10) is no longer invariant under the super-Weyl transformations (this is similar to the BI action vs. the Maxwell action). However, the conformal invariance can be easily restored by introducing a conformal compensator (the covariantly chiral \( N = 1 \) scalar superfield) of Weyl weight \((-1)\), as in eq. (2.7).

Though the BIW action naively seems to be the most obvious gravitational analogue of the NBI action, even the leading terms of the BIW action contain terms with higher derivatives. Unlike the gauge theory that is quadratic in the field strength, the Einstein action is linear in the curvature, in components. The \( N = 1 \) supergravity action does not contain the supercurvature at all (i.e. of the zeroth order). Hence, it is worthwhile to investigate how the Einstein supergravity may appear in our approach.
A Born-Infeld-Einstein (BIE) supergravity can be generated in several ways, either from the quadratic action in AdS five dimensions (sect. 4) by using the abelian BI machinery in the form (2.2) and (2.3), or simply by inserting supertensors into the action (3.6). In fact, any full superspace action containing a supercurvature (even linearly) gives rise to the terms that are non-linear in the component curvature. As an example, let’s consider the most ‘economical’ superfield Lagrangian whose superspace structure resembles the component Einstein action with a cosmological term,

\[ S_{\text{BIE}} = \int d^8z E^{-1}(\Lambda + \mathcal{R}) + \text{h.c.} \quad , \]  \hspace{1cm} (3.15)

where \( \Lambda \) is a non-vanishing constant. In components, this very simple local \( N = 1 \) superinvariant gives rise to the following bosonic terms:

\[ S_{\text{bos.}} = -\frac{1}{9} \int d^4x \sqrt{-g}(R + \frac{1}{3}B\bar{B})(2\Lambda + B + \bar{B}) \quad , \]  \hspace{1cm} (3.16)

where the auxiliary complex scalar field \( B \) is the leading component of \( \mathcal{R} \). The algebraic \( B \)-equation of motion has an obvious solution

\[ B = \bar{B} = -\frac{1}{3}\Lambda \pm \sqrt{\frac{1}{9}\Lambda^2 - R} \quad . \]  \hspace{1cm} (3.17)

Being inserted back into the action (3.16), this yields

\[ S_{\text{bos.}} = -\frac{4}{27} \int d^4x \sqrt{-g} \left\{ \frac{4}{3}\Lambda R + (\frac{1}{3}\Lambda^2 - R) \left( \frac{1}{3}\Lambda \mp \sqrt{\frac{1}{9}\Lambda^2 - R} \right) \right\} \quad . \]  \hspace{1cm} (3.18)

This action is already of the BI type, while it also implies taming of the scalar curvature from above,

\[ R \leq (\frac{1}{4}\Lambda)^2 \quad . \]  \hspace{1cm} (3.19)

After choosing the upper sign (minus) choice in eq. (3.18) and adjusting the free parameter \( \Lambda \) as

\[ \Lambda = \left( \frac{3}{2\kappa} \right)^2 \quad , \]  \hspace{1cm} (3.20)

where \( \kappa \) is the gravitational constant of dimension of length as usual, the leading term (in the curvature) in the action (3.18) takes the standard (Einstein-Hilbert) form,

\[ -\frac{1}{2\kappa^2}R \quad . \]

As is well known, one of the most beautiful features of the original BI action is its famous taming of the Coulomb (electro-magnetic field) self-energy of a point-like electric charge [1]. Related to this feature is the existence of the maximal value for the electro-magnetic field strength. Similarly, one may expect from a BIE action that it should remove the spacetime singularity of the Schwarzschild (or Schwarzschild-AdS) solution (black hole) in the Einstein or AdS theory. This can only happen if Ricci-flat
solutions are excluded, which is not the case for the BIE action (3.15). We may thus need a better BIE action that would be dependent upon the Weyl supertensor $W_{\alpha \beta \gamma}$ too, e.g., by combining eqs. (3.10) and (3.15). A more natural solution within our approach apparently implies a generation of the Einstein term from some action that is quadratic in the curvatures (sect. 4).

4 AdS-BI supergravity

In this section we propose yet another approach for a construction of gravitational and supergravitational avatars of the BI action. This component approach is based on treating gravity and supergravity as the geometrical theories by gauging the AdS (covering) symmetry group $Sp(4)$ or the AdS supergroup $OSp(1, 4)$, respectively [21]. Here we briefly outline this construction in the supergravity case.

The MacDowell-Mansouri procedure consists of the following steps [21]:

- take the spacetime supersymmetry $OSp(1, 4)$ as the gauge group, and introduce the connection one-forms $h^A = h^A_\mu dx^\mu$ and the corresponding curvature two-forms $W^A$ à la Yang-Mills, $W^A_{\mu \nu} = \partial_\mu h^A_\nu - \partial_\nu h^A_\mu + h^B_\mu h^C_\nu f_{BC}^A$, where $f_{BC}^A$ are the structure constants of $OSp(1, 4)$,
- define the invariant action

\[ S_{AdS} = \int W^A \wedge W^B Q_{AB} , \] (4.1)

where $Q_{AB} = \{\varepsilon_{abcd}, const.(C^5)_{\alpha \beta}\}$, $\varepsilon_{abcd}$ is the Levi-Civita symbol, $C_{\alpha \gamma}$ is a charge conjugation matrix, and $const. \neq 0$,
- varying the action (4.1) with respect to the connections associated with the Lorentz generators allows one to express the Lorentz connections $h^{[ab]}_\mu$ in terms of the remaining gauge fields (the vierbein $h^a_\mu$ and a gravitino $h^a_\mu$). This also converts the first-order action (4.1) into the second-order action and ensures its invariance under supersymmetry transformations,
- a decomposition of the action $S_{AdS}$ with respect to the irreducible representations of $OSp(1, 4)$ results in a sum of the topological Euler-Poincaré characteristic of the four-dimensional Lorentz base manifold (spacetime), the Einstein gravity term, and a cosmological term, together with their fully supersymmetric completion.
The MacDowell-Mansouri approach [21] for a construction of the Einstein supergravity puts both gravity and supergravity theories on equal footing, by generating both of them from the most basic gauge field theory action quadratic in the curvature. Therefore, their approach is perfectly suitable for our purpose of generating a BIE action via the BI equations (1.6) and (1.7), with the MacDowell-Mansouri density \( \varepsilon^{\mu
u\lambda\rho} R_{\mu\nu}^A R_{\lambda\rho}^B Q_{AB} \) as the input (\( \omega \)).

5 Conclusion

In this paper we proposed the new BIW, BIE and AdS-BI actions with manifest local \( N = 1 \) supersymmetry. Our construction is entirely based on the non-abelian and locally supersymmetric generalization of the non-linear constraint (1.7) governing the structure of the BI action (1.1). This mechanism is related to spontaneous partial supersymmetry breaking, while it seems to be similar to the renormalization procedure converting the bare (input) coupling constant into the ‘running’ (effective) coupling constant in a renormalizable quantum field theory. Of course, it would be interesting to know whether some of our actions survive tests of causal propagation and/or positivity of energy. We are also going to investigate a possible connection between our actions and the effective actions for D-branes.

We conclude with two comments.

The bosonic non-abelian BI action is well-known to suffer from the non-abelian ambiguities [4]. Our NBI prescription does not have these ambiguities since the iterative solution to the NBI constraint (2.11) also implies definitive ordering of the non-abelian quantities. This equally applies to our BI supergravity actions.

It is not difficult to generalize our \( N = 1 \) BI supergravity actions to \( N = 2 \) and \( N = 4 \) extended BI supergravity too. The off-shell \( N = 2 \) supercurvature, supertorsion and chiral density superfields are well known in the standard curved \( N = 2 \) superspace [22]. For example, the Weyl tensor is hiding in the \( N = 2 \) bosonic (covariantly chiral) superfield \( W_{\alpha\beta} \) that is symmetric on its spinor indices, being the \( N = 2 \) analogue to the \( N = 1 \) Weyl superfield \( W_{\alpha\beta\gamma} \) (see, e.g., ref. [23]). Similarly, there exists a complex chiral scalar \( N = 4 \) superfield \( W \) containing the Weyl tensor in the \( N = 4 \) superfield supergravity [24].
Acknowledgements

We are grateful to Stanley Deser and Gary Gibbons for discussions.

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