Duality Symmetry in Momentum Frame

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Abstract

Siegel’s action is generalized to the $D = 2(p + 1)$ ($p$ even) dimensional space-time. The investigation of self-duality of chiral $p$-forms is extended to the momentum frame, using Siegel’s action of chiral bosons in two space-time dimensions and its generalization in higher dimensions as examples. The whole procedure of the investigation is realized in the momentum space which relates to the configuration space through the Fourier transformation of fields. These actions correspond to non-local Lagrangians in the momentum frame. The self-duality of them with respect to dualization of chiral fields is uncovered. The relationship between two self-dual tensors in momentum space, whose similar form appears in configuration space, plays an important role in the calculation, that is, its application realizes solving algebraically an integral equation.

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1 Introduction

Many models of chiral bosons and/or their generalizations to higher (than two) space-time dimensions, \textit{i.e.}, chiral \( p \)-forms have been proposed \cite{1-12}. Among them, some \cite{1-7} are non-manifestly space-time covariant, while the others \cite{8-12} manifestly space-time covariant. Moreover, these chiral \( p \)-form models have close relationship among one another, especially various dualities that have been demonstrated \cite{12-14} in detail. Incidentally, the self-duality that exists beyond chiral \( p \)-form actions has also been uncovered \cite{15}.

As the investigations of duality symmetries mentioned above are limited only in configuration space, it may be interesting to extend these investigations to the momentum space that relates to the configuration space through the Fourier transformation of fields. It is quite natural to have this idea because the Fourier transformation plays an important role in field theory. Perhaps motivated similarly, the duality in harmonic oscillators obtained by Fourier decomposition was discussed \cite{16} by considering several simple models, such as the free scalar, Maxwell and Kalb-Ramond theories, as examples. However, it is quite unsatisfactory to leave a variety of attractive chiral \( p \)-form models unnoticed. In this note we re-investigate in momentum space the duality symmetries of various chiral \( p \)-forms that exist in the configuration space, and we do find something non-trivial. The non-triviality we mention here means duality investigation of non-local Lagrangians and algebraic solution of integral equations.

We choose Siegel’s action as our example. To this end, we have to generalize the action to the \( D = 2(p+1) \) dimensional space-time\footnote{Note that the space-time is twice odd dimensional, \textit{i.e.}, \( p \) is even throughout this paper.}. The main reason to make this choice is that after one makes the Fourier transformation to fields its formulation is non-trivial because of its cubic Lagrange-multiplier term as shown below. Starting from this formulation, the whole procedure of investigation is realized in the momentum space. As a result, the self-duality of Siegel’s action with respect to dualization of chiral fields is uncovered in the momentum frame. In the next section we deal with the special case in \( D = 2 \) space-time dimensions, and in section 3 we turn to the general case in \( D = 2(p+1) \) dimensions. Finally, section 4 is devoted to a conclusion.
The metric notation we use throughout this note is

\[ g_{00} = -g_{11} = \cdots = -g_{D-1,D-1} = 1, \]
\[ \epsilon^{012\cdots D-1} = 1. \]  \hspace{1cm} (1)

Greek letters stand for indices \((\mu, \nu, \sigma, \cdots = 0, 1, 2, \cdots, D - 1)\) in both the configuration and momentum spaces.

2 Self-duality of the chiral 0-form action in \(D=2\) momentum frame

We begin with Siegel’s action \([8]\) in \(D=2\) space-time dimensions,

\[ S_c = \int d^2x \left\{ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} \lambda_{\mu\nu}(x) [\partial^\mu \phi(x) - \epsilon^{\mu\sigma} \partial_\sigma \phi(x)] [\partial^\nu \phi(x) - \epsilon^{\nu\rho} \partial_\rho \phi(x)] \right\}, \]  \hspace{1cm} (2)

where \(\phi(x)\) is a scalar field, and \(\lambda_{\mu\nu}(x)\) a symmetric auxiliary second-rank tensor field. Substituting the Fourier transformations of \(\phi(x)\) and \(\lambda_{\mu\nu}(x)\),

\[ \phi(x) = \frac{1}{2\pi} \int d^2ke^{-ik\cdot x} \phi(k), \]
\[ \lambda_{\mu\nu}(x) = \frac{1}{2\pi} \int d^2ke^{-ik\cdot x} \lambda_{\mu\nu}(k), \]  \hspace{1cm} (3)

into eq.(2), one arrives at the Siegel action in the momentum frame spanned by \(k_\mu\),

\[ S_m = -\frac{1}{2} \int d^2k(-ik_\mu)(ik^\mu)\phi(k)\phi(-k) \]
\[ + \frac{1}{4\pi} \int d^2kd^2k' \lambda_{\mu\nu}(-k - k')(ik^\mu - \epsilon^{\mu\sigma} ik_\sigma)(ik'^\nu - \epsilon^{\nu\rho} ik'_\rho)\phi(k)\phi(k'). \]  \hspace{1cm} (4)

Note that \(S_m\) contains quartic integrals over the momentum space because of the cubic Lagrange-multiplier term in \(S_c\), which shows that \(S_m\) is non-trivial. This non-triviality can also be understood as the non-locality of the corresponding Lagrangian. Therefore, we have to envisage the duality investigation of non-local Lagrangians.

We investigate the duality property of \(S_m\) with respect to dualization of the field \(\phi(k)\). By introducing two independent vector fields in the momentum space, \(F_\mu(k)\) and \(G_\mu(k)\), we construct a new action to replace \(S_m\),

\[ S'_m = -\frac{1}{2} \int d^2kF_\mu(k)F^\mu(-k) \]
\begin{align*}
+ \frac{1}{4\pi} \int d^2k d^2k' \lambda_{\mu\nu}(-k - k') [F^\mu(k) - e^{\mu\sigma} F_\sigma(k)] [F^{\nu}(k') - e^{\nu\rho} F_\rho(k')] \\
+ \int d^2k G^\mu(-k) [F_\mu(k) + ik_\mu \phi(k)],
\end{align*}

where the third term is nothing but the \( k \)-space formulation of \( \int d^2x G^\mu(x)[F_\mu(x) - \partial_\mu \phi(x)] \).

Variation of \( S'_m \) with respect to \( G^\mu(-k) \) gives

\[
F_\mu(k) = -ik_\mu \phi(k),
\]

which, when substituted into \( S'_m \), yields the classical equivalence between the two actions, \( S_m \) and \( S'_m \). Furthermore, variation of \( S'_m \) with respect to \( F^\mu(k) \) leads to the expression of \( G^\mu(k) \) in terms of \( F^\mu(k) \),

\[
G^\mu(k) = F^\mu(k) - \frac{1}{2\pi} \int d^2k' (g^{\mu\nu} + e^{\mu\nu}) \lambda_{\nu\sigma}(k - k') [F^\sigma(k') - e^{\sigma\rho} F_\rho(k')] .
\]

Note that eq.(7) is an integral equation other than an algebraic one that happens in configuration space, which is induced by the non-locality of the corresponding Lagrangian of eq.(4). At first sight it seems difficult to solve \( F^\mu(k) \) from eq.(7). In fact, one can deal with this problem in terms of an algebraic method shown in the following. In order to avoid solving this integral equation, one defines two self-dual tensors as

\[
\mathcal{F}^\mu(k) \equiv F^\mu(k) - \epsilon^{\mu\nu} F_\nu(k),
\]

\[
\mathcal{G}^\mu(k) \equiv G^\mu(k) - \epsilon^{\mu\nu} G_\nu(k),
\]

and establishes their relationship by using eq.(7),

\[
\mathcal{F}^\mu(k) = \mathcal{G}^\mu(k).
\]

We should emphasize that a similar relationship exists in configuration space for various chiral \( p \)-form actions as pointed out in Ref.[14]. With the aid of eq.(9), one can easily obtain algebraically from eq.(7) \( F^\mu(k) \) expressed in terms of \( G^\mu(k) \),

\[
F^\mu(k) = G^\mu(k) + \frac{1}{2\pi} \int d^2k' (g^{\mu\nu} + e^{\mu\nu}) \lambda_{\nu\sigma}(k - k') [G^\sigma(k') - e^{\sigma\rho} G_\rho(k')] .
\]

We can check from eq.(7) that when the self-duality condition is satisfied, i.e., \( \mathcal{F}^\mu(k) = 0 \), which is also called an “on mass shell” condition, \( F^\mu(k) \) and \( G^\mu(k) \) relate with a duality,
\[ G^\mu(k) = \epsilon^{\mu\nu} F_{\nu}(k). \]

Substituting eq.(10) into eq.(5), we obtain the dual action of \( S_m \),

\[
S_{m,\text{dual}} = \frac{1}{2} \int d^2 k G_\mu(k) G^\mu(-k) \\
+ \frac{1}{4\pi} \int d^2 k d^2 k' \lambda_{\mu\nu}(-k - k') \left[ G^\mu(k) - \epsilon^{\mu\sigma} G_\sigma(k) \right] \left[ G^\nu(k') - \epsilon^{\nu\rho} G_\rho(k') \right] \\
+ \int d^2 k \phi(k) \left[ ik_\mu G^\mu(-k) \right].
\] (11)

Variation of eq.(11) with respect to \( \phi(k) \) gives

\[
G^\mu(k) = \epsilon^{\mu\nu} (-ik_\nu) \psi(k) \equiv \epsilon^{\mu\nu} F_{\nu}[\psi(k)],
\] (12)

where \( \psi(k) \) is an arbitrary scalar field in momentum space. Substituting eq.(12) into eq.(11), one finally obtains the dual action in terms of \( \psi(k) \),

\[
S_{m,\text{dual}} = - \frac{1}{2} \int d^2 k (-ik_\mu)(ik^\mu) \psi(k) \psi(-k) \\
+ \frac{1}{4\pi} \int d^2 k d^2 k' \lambda_{\mu\nu}(-k - k') (ik^\mu - \epsilon^{\mu\sigma} ik_\sigma)(ik^{\nu'} - \epsilon^{\nu'\rho} ik'_\rho) \psi(k) \psi(k').
\] (13)

This action has the same form as the original one, eq.(4), only with the replacement of \( \phi(k) \) by \( \psi(k) \). As analysed above, \( \phi(k) \) and \( \psi(k) \) coincide with each other up to a constant when the self-duality condition is imposed. Therefore, the \( k \)-space formulation of Siegel’s action is self-dual with respect to \( \phi(k) - \psi(k) \) dualization expressed by eq.(6) and eq.(12).

3 Self-duality of the chiral p-form action in \( D=2(p+1) \) momentum frame

By introducing a real \( p \)-form field, \( A_{\mu_1...\mu_p}(x) \), we generalize the Siegel action to the \( D = 2(p+1) \) dimensional space-time,

\[
S_c = \int d^D x \left\{ -\frac{1}{2(p+1)!} \partial_{[\mu_1} A_{\mu_2...\mu_p+1]}(x) \partial^{[\mu_1} A_{\mu_2...\mu_p+1]}(x) \\
+ \frac{1}{2} \lambda_{\nu}(x) \left[ \partial_{[\mu_1 A_{\mu_2...\mu_p]}(x) - \frac{1}{(p+1)!} \epsilon_{\mu_1...\mu_p\nu_1...\nu_{p+1}} \partial^{[\nu_1} A_{\nu_2...\nu_{p+1]}(x) \right] \\
\times \left[ \partial^{[\nu} A_{\mu_1...\mu_p]}(x) - \frac{1}{(p+1)!} \epsilon^{\nu\mu_1...\mu_p\sigma_1...\sigma_{p+1} \partial_{\sigma_1} A_{\sigma_2...\sigma_{p+1}}(x) \right] \right\}. \] (14)

It can be verified that \( A_{\mu_1...\mu_p}(x) \) indeed describes a chiral \( p \)-form by following Ref.[8] in which only the \( D = 2 \) and \( D = 6 \) cases are dealt with.
As done in the above section, substituting the Fourier transformations,

\[ A_{\mu_1\cdots\mu_p}(x) = \frac{1}{(2\pi)^{D/2}} \int d^D ke^{-ik\cdot x} A_{\mu_1\cdots\mu_p}(k), \]

\[ \lambda_{\mu\nu}(x) = \frac{1}{(2\pi)^{D/2}} \int d^D ke^{-ik\cdot x} \lambda_{\mu\nu}(k), \] (15)

into eq.(14), one arrives at Siegel’s action in the $D = 2(p+1)$ momentum space spanned by $k_\mu$,

\[
S_m = -\frac{1}{2(p+1)!} \int d^D k \left[ -ik_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}(k) \right] \left[ i k^{[\mu_1 A^{\mu_2\cdots\mu_{p+1}]}(-k) \right] 
+ \frac{1}{2(2\pi)^{D/2}} \int d^D k d^D k' \lambda^\nu(-k - k') 
\times \left[ i k_{[\mu_1 A_{\mu_2\cdots\mu_p]}(k) - \frac{1}{(p+1)!} \epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{p+1}} i k^{[\nu_1 A^{\nu_2\cdots\nu_{p+1}]}(k) \right] 
\times \left[ i k'^{[\nu_1 A^{\nu_2\cdots\mu_{p+1}]}(k') - \frac{1}{(p+1)!} \epsilon^{\nu_1\cdots\nu_p\sigma_1\cdots\sigma_{p+1}} i k'_{[\sigma_1 A_{\sigma_2\cdots\sigma_{p+1}]}(k') \right] . \] (16)

Here we note that eq.(16) includes $2D$ momentum integrals. This shows the non-locality of the corresponding Lagrangian as was seen in the $D = 2$ case.

In order to discuss the duality of $S_m$, we introduce two $(p+1)$-form fields in the momentum space, $F_{\mu_1\cdots\mu_{p+1}}(k)$ and $G_{\mu_1\cdots\mu_{p+1}}(k)$, and replace eq.(16) by the following action,

\[
S'_m = -\frac{1}{2(p+1)!} \int d^D k F_{\mu_1\cdots\mu_{p+1}}(k) F^{\mu_1\cdots\mu_{p+1}}(-k) 
+ \frac{1}{2(2\pi)^{D/2}} \int d^D k d^D k' \lambda^\nu(-k - k') 
\times \left[ F_{\mu_1\cdots\mu_p}(k) - \frac{1}{(p+1)!} \epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{p+1}} F^{\nu_1\cdots\nu_{p+1}}(k) \right] 
\times \left[ F^{\nu_1\cdots\nu_p}(k') - \frac{1}{(p+1)!} \epsilon^{\nu_1\cdots\nu_p\sigma_1\cdots\sigma_{p+1}} F_{\sigma_1\cdots\sigma_{p+1}}(k') \right] 
+ \frac{1}{(p+1)!} \int d^D k G^{\mu_1\cdots\mu_{p+1}}(-k) \left[ F_{\mu_1\cdots\mu_{p+1}}(k) + i k_{[\mu_1 A_{\mu_2\cdots\mu_{p+1}]}(k) \right], \quad (17)
\]

where $F_{\mu_1\cdots\mu_{p+1}}(k)$ and $G_{\mu_1\cdots\mu_{p+1}}(k)$ act, at present, as independent auxiliary fields. Similar to the $D = 2$ case, the third term can be obtained by substituting the Fourier transformations of the fields, i.e., eq.(15), into the following term,

\[
\frac{1}{(p+1)!} \int d^D x G^{\mu_1\cdots\mu_{p+1}}(x) \left[ F_{\mu_1\cdots\mu_{p+1}}(x) - \partial_{\mu_1} A_{\mu_2\cdots\mu_{p+1}}(x) \right].
\]
Variation of eq.(17) with respect to \( G^{\mu_1 \cdots \mu_{p+1}}( -k ) \) gives

\[
F_{\mu_1 \cdots \mu_{p+1}}(k) = -ik_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}(k),
\]

which yields the equivalence between the actions, eq.(16) and eq.(17). On the other hand, variation of eq.(17) with respect to \( F_{\mu_1 \cdots \mu_{p+1}}(k) \) leads to the expression of \( G^{\mu_1 \cdots \mu_{p+1}}(k) \) in terms of \( F^{\mu_1 \cdots \mu_{p+1}}(k) \),

\[
G^{\mu_1 \cdots \mu_{p+1}}(k) = F^{\mu_1 \cdots \mu_{p+1}}(k) - \frac{1}{(2\pi)^{D/2}} \int d^D k' \lambda_{\mu\nu}(k-k') \times \left[ g^{[\mu_1} F^{\mu_2 \cdots \mu_{p+1}]\nu}(k') + \epsilon^{\mu_1 \cdots \mu_{p+1} \nu \mu_1 \cdots \nu_p} F^{\nu \mu_1 \cdots \nu_p}(k') \right],
\]

where \( F^{\mu_1 \cdots \mu_{p+1}}(k) \) is defined as the difference of the field strength \( F^{\mu_1 \cdots \mu_{p+1}}(k) \) and its Hodge dual,

\[
F^{\mu_1 \cdots \mu_{p+1}}(k) \equiv F^{\mu_1 \cdots \mu_{p+1}}(k) - \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}(k),
\]

which is also called the self-dual tensor. Note that eq.(19) is an integral equation. In order to obtain algebraically \( F^{\mu_1 \cdots \mu_{p+1}}(k) \) in terms of \( G^{\mu_1 \cdots \mu_{p+1}}(k) \) without solving the integral equation, as was done in the last section, one defines another field strength difference/self-dual tensor which is relevant to \( G^{\mu_1 \cdots \mu_{p+1}}(k) \),

\[
G^{\mu_1 \cdots \mu_{p+1}}(k) \equiv G^{\mu_1 \cdots \mu_{p+1}}(k) - \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} G_{\nu_1 \cdots \nu_{p+1}}(k),
\]

and then establishes the relationship between these two self-dual tensors by using eq.(19),

\[
F^{\mu_1 \cdots \mu_{p+1}}(k) = G^{\mu_1 \cdots \mu_{p+1}}(k),
\]

whose similar form, as mentioned above, exists in various chiral \( p \)-forms in configuration space [14]. With eq.(22), one can algebraically invert eq.(19) and obtain \( F^{\mu_1 \cdots \mu_{p+1}}(k) \) expressed in terms of \( G^{\mu_1 \cdots \mu_{p+1}}(k) \),

\[
F^{\mu_1 \cdots \mu_{p+1}}(k) = G^{\mu_1 \cdots \mu_{p+1}}(k) + \frac{1}{(2\pi)^{D/2}} \int d^D k' \lambda_{\mu\nu}(k-k') \times \left[ g^{[\mu_1} G^{\mu_2 \cdots \mu_{p+1}]\nu}(k') + \epsilon^{\mu_1 \cdots \mu_{p+1} \nu \mu_1 \cdots \nu_p} G^{\nu \mu_1 \cdots \nu_p}(k') \right].
\]

We can verify from eq.(19) that when the self-duality condition is satisfied, i.e.,

\[
F^{\mu_1 \cdots \mu_{p+1}}(k) = 0,
\]
$F^{\mu_1 \cdots \mu_{p+1}}(k)$ and $G^{\mu_1 \cdots \mu_{p+1}}(k)$ relate with a duality,

$$G^{\mu_1 \cdots \mu_{p+1}}(k) = \frac{1}{(p+1)!} e^{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}(k).$$

Now substituting eq.(23) into the action, eq.(17), and making tedious calculations, we obtain the dual Siegel action in the $D = 2(p+1)$ momentum space,

$$S_{m}^{\text{dual}} = \frac{1}{2(p+1)!} \int d^D k G_{\mu_1 \cdots \mu_{p+1}}(k) G^{\mu_1 \cdots \mu_{p+1}}(-k)$$

$$+ \frac{1}{2(2\pi)^{D/2}} \int d^D k d^D k' \lambda_{\mu \nu} (-k - k') G^{\mu \mu_{1} \cdots \mu_{p}}(k) G_{\nu \mu_{1} \cdots \mu_{p}}(k')$$

$$+ \int d^D k A_{\mu_1 \cdots \mu_{p}}(k) [ik_{\mu} G^{\mu_{1} \cdots \mu_{p}}(-k)].$$

Variation of eq.(24) with respect to $A_{\mu_1 \cdots \mu_{p}}(k)$ gives $ik_{\mu} G^{\mu_{1} \cdots \mu_{p}}(-k) = 0$, whose solution has to be

$$G^{\mu_{1} \cdots \mu_{p+1}}(k) = \frac{1}{(p+1)!} e^{\mu_{1} \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} \left[ -ik_{[\nu_1} B_{\nu_2 \cdots \nu_{p+1}]}(k) \right]$$

$$\equiv \frac{1}{(p+1)!} e^{\mu_{1} \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}} [B(k)],$$

where $B_{\nu_1 \cdots \nu_{p}}(k)$ is an arbitrary $p$-form field. When eq.(25) is substituted into the dual action, eq.(24), one finally obtains the result that the dual action is the same as the original action, eq.(16), only with the replacement of $A_{\nu_1 \cdots \nu_{p}}(k)$ by $B_{\nu_1 \cdots \nu_{p}}(k)$. Consequently, the $k$-space formulation of Siegel’s action in $D = 2(p+1)$ dimensions is self-dual with respect to $A_{\nu_1 \cdots \nu_{p}}(k) - B_{\nu_1 \cdots \nu_{p}}(k)$ dualization given by eq.(18) and eq.(25).

4 Conclusion

In this note we have generalized Siegel’s model to the $D = 2(p+1)$ dimensional space-time, and have extended for this model duality investigations from the configuration to momentum frames and hence uncovered its self-duality with respect to dualization of chiral fields in the momentum space. The characteristic that does not exist in configuration space is duality investigation of non-local Lagrangians and algebraic solution of integral equations. Here we emphasize that the introduction of two self-dual tensors and the establishment of their relationship are crucial in realizing the whole procedure of investigation.

For the PST action [11], the investigation of duality symmetries in the momentum frame
is the same as that of Siegel’s action in the sense that the former is a special case of the latter as explained in detail in Ref.[12]. As to other chiral $p$-forms [1,4,5,7,9,10] whose actions are quadratic, the duality investigations are the same in both the configuration and momentum spaces.

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