General Relativity, Cosmological Constant and Modular Forms.

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Abstract

Strong field (exact) solutions of the gravitational field equations of General Relativity in the presence of a Cosmological Constant are investigated. In particular, a full exact solution is derived within the inhomogeneous Szekeres-Szafron family of space-time line element with a nonzero Cosmological Constant. The resulting solution connects, in an intrinsic way, General Relativity with the theory of modular forms and elliptic curves and thus to the theory of Taniyama-Shimura. The homogeneous FLRW limit of the above space-time elements is recovered and we solve exactly the resulting Friedmann Robertson field equation with the appropriate matter density for generic values of the Cosmological Constant $\Lambda$ and curvature constant $K$. A formal expression for the Hubble constant is derived. The cosmological implications of the resulting non-linear solutions are systematically investigated. Two particularly interesting solutions i) the case of a flat universe $K = 0, \Lambda \neq 0$ and ii) a case with all three cosmological parameters non-zero, are associated with elliptic curves with the property of complex multiplication and absolute modular invariant $j = 0, 1728$ respectively. The possibility that all non-linear solutions of General Relativity are expressed in terms of theta functions associated with Riemann-surfaces is discussed.

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1 Introduction

1.1 Motivation

The Cosmological Constant $\Lambda$ is presently at the epicentre of contemporary physics [1] and plays a significant role in fields such as: cosmology, astronomy, General Relativity, particle physics, and string theory.

Experimental evidence has been growing suggesting that the Cosmological Constant has a small but non-zero value [8],[9],[4]. In particular, the results from high redshift Type Ia supernovae observations suggest an accelerating Universe (positive Cosmological Constant) where the age of the Universe and Hubble’s Constant $H$ are approximately, $\sim 14.5$ billion years and $70 \text{Kms}^{-1}\text{Mpc}^{-1}$, respectively. Other indirect experimental evidence (Turner [2], Bahcall et al.[5]) has put bounds on the energy densities ratio’s of matter, dark energy and curvature, given by: $\Omega_m \equiv 8\pi G_N \rho_{\text{matter}}/(3H^2) \sim 0.3$, $\Omega_{\Lambda} \equiv c^2\Lambda H^2/3H^2 \sim 0.6 - 0.7$, $\Omega_K \equiv -Kc^2/(R(t)H)^2 \sim 0$, respectively. In these definitions $G_N$ is Newton’s constant, $\rho_{\text{matter}}$ denotes the density of matter, $K$ the curvature coefficient and $R(t)$ the scale factor.

We note here that The Cosmological Constant has been identified with a “dark” exotic form of energy that is smoothly distributed and which contributes $60 -70\%$ to the critical density of the Universe [2] (See Cohn [3] and Carroll and Press [4] for a pedagogical review of the Cosmological Constant and Turner [2] for a review of Dark Matter and Dark Energy).

Recent data from the Boomerang experiment [6] provide a firm evidence for the case of flat Universe, $\Omega_K \sim 0$.

It was suggested that the effects of the Cosmological Constant could be felt on a Galactic scale [1] which would lead to the modification of Newton’s inverse square law. This approach was used in order to try and explain the behaviour of the Galactic velocity rotation curves. The dynamical effects of $\Lambda$ at megaparsec scale were investigated by Axenides et al [7].

Some of the theoretical justifications for a non-zero Cosmological Constant are discussed by Overduin et al [9].

The main objective of this paper is to derive new exact solutions of General Relativity, for a non-zero Cosmological Constant, and to discuss in a systematic manner their cosmological implications. The material of this paper is organized as follows. In section 2 we derive a full exact solution of the gravitational field equations with a non-zero Cosmological Constant in the inhomogeneous class of Szafron-Szekeres models. The solution in a closed analytic form is expressed by the Weierstrass elliptic function. Our motivation for studying this particular class of inhomogeneous models is twofold. First the solution described above is new, and in general the inhomogenous cosmology is only a partially explored territory. Secondly, a homogeneous Friedmann, Robertson, Lemaitre, Walker limit can be derived from this family and therefore a comparison between inhomogeneous and homogeneous cosmology can in principle be made. However, in this paper we do not attempt any such comparison. In section 3 we derive
the FRLW limit and solve the Friedmann equation for the scale factor in terms of Weierstraßsche modular forms using the techniques of section 2. A formal expression for the Hubble’s Constant is also derived. Section 4 provides the necessary mathematical background for the use of the elliptic functions and elliptic curves in the solutions obtained. In section 5 we systematically investigate the cosmological implications of the exact solution of the Friedmann equation. Sections 6 and 7 briefly explore the connections of the physical solutions obtained of General Relativity with some special arithmetic elliptic curves and in particular the connection with complex multiplication. It is also emphasized that the same techniques are also useful for the solution of other important non-linear equations of mathematical physics. Finally, section 8 is used for our conclusions.

1.2 Inhomogeneous versus Homogeneous Cosmologies

As already mentioned in the introduction one of the reasons for studying inhomogeneous cosmologies was to understand the soundness of the “belief” of the overwhelming number of the cosmological community in an isotropic and homogeneous universe. In short the suitability of the Friedmann, Lemaitre, Robertson, Walker (FLRW) model of the Universe. There is a general belief that FLRW models, without substantial evidence, represent a “good” approximation to the actual Universe.

Together the books, "Exact Solutions of Einstein’s Field Equations" [10], and "Inhomogeneous Cosmological Models" [11], including references within, provide a comprehensive review of the presently known families of exact solutions to Einstein’s Field equations.

The Szekeres–Szafron family of solutions were chosen for study as they represented a well-known inhomogeneous cosmological model, whose metric is given by:

\[
    ds^2 = dt^2 - e^{2\beta(x,y,z,t)} (dx^2 + dy^2) - e^{2\alpha(x,y,z,t)} dz^2. \quad (1)
\]

The characteristics, properties and formal solutions for the energy densities, for this family of exact inhomogeneous solutions has been collated and documented by Krasinski [11].

The Einstein equations with a non-zero Cosmological Constant, and a perfect fluid source energy momentum tensor \( T_{\mu\nu} \) have the form, \( (c \equiv 1 \equiv 8\pi G_N) \):

\[
    G_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu}. \quad (2)
\]

3
\[ T_{\mu\nu} = (P + \rho)u_{\mu}u_{\nu} - Pg_{\mu\nu}, \quad u_{\mu} = \delta^0_{\mu}, \quad (3) \]

here \( u_a \), \( P \) and \( \rho \) are the four-velocity, pressure and fluid density respectively. Our exact solution of the non-linear partial differential field equations (2) resulting from the metric Eq.(1) is new. In particular, as we shall see \( \alpha(x, y, z, t) \), \( \beta(x, y, z, t) \) and the energy density are expressed in terms of the Weierstraß's modular forms. As we will explain in detail in section 4 the Weierstraß function and its derivative parametrise the equation of an elliptic curve which can be regarded as a Riemann surface of genus one. Consequently, our exact solutions for the inhomogeneous Szafron-Szekeres space-time element with a non-zero \( \Lambda \) lie on an elliptic curve.

The above metric, equation (1) has two interesting properties: firstly, the Cosmological Constant will play a significant role, and secondly, that for suitable values of the arbitrary constants the Friedmann metric is recovered. The second property is useful in that it allows direct comparison of exact homogeneous and inhomogeneous solutions, where the Friedmann solutions can be thought of as approximations to the more realistic inhomogeneous solutions. The comparison of the results, as applied to cosmology, will give a measure of the validity for an isotropic and homogeneous Universe.

The comparison of homogeneous and inhomogeneous solutions of General Relativity will not be considered in the work presented here but will form part of another paper. However, the Friedmann equations, having been recovered by judicious choice of arbitrary parameters will be analysed in detail.

To the best of our knowledge the exact solution presented in the next section for the Szekeres–Szafron family of non-linear partial differential equations (PDF’s) and a non-zero Cosmological Constant is new and represents the first solution in a closed analytic form. Here we acknowledge the paper by Barrow and Stein-Schabes [12] who were the first to derive a solution which was defined as a formal integral.

\section{Exact Solution of the Gravitational Field Equations for the Szekeres-Szafron Line Element}

\subsection{Approach, Definitions and Assumptions}

This section will outline the approach, definition and assumptions required to derive the general expression for \( \alpha(x, y, z, t) \) and \( \beta(x, y, z, t) \) from the Field Equations. The following sections will describe the determination of the Einstein's Field Equations in the Szafron form, and the derivation of the exact solution for the line element.

We start by assuming that there exists co-ordinates which support the metric line element:
The source is taken to be a perfect fluid and the co-ordinates of the line element are assumed to be comoving, so that,

\[ u^\mu = \delta_0^\mu, \]  

which implies that \( \dot{u}^\mu = 0 \) and \( P = P(t) \). Szafrań [13] considered explicitly the case \( \kappa P = 43q(1-q)t^{-2} \) with \( q \) constant and assumed \( K(z) = 0 \). We explicitly consider the equation of state involving the Cosmological Constant, \( \Lambda = -\kappa P \). This differs from the approach of Szafrań. These class of solutions all under certain conditions reproduce the Friedmann, Lemaitre, Robertson and Walker (FLRW) solutions. The solutions are classified as either type \( \beta' = 0 \) or \( \beta' \neq 0 \), where \( \beta' = \partial \beta / \partial z \), depending on what initial trial solution is chosen. In this section the more general \( \beta' \neq 0 \) type will be studied [11].

### 2.2 Einstein and Szafron Form of the Field Equations

Determining the Einstein field equations from the metric is straightforward but tedious. Expressions for the non-zero components of the Ricci tensor \( R_{\mu\nu} \), and scalar \( R \): \( R_{00}, R_{11}, R_{22}, R_{33} \), \( R \) are given in appendix A for completeness.

The original solution for this metric was given by Szafrań [13]. His starting point was to introduce a pair of complex variables given by:

\[ \xi = x + iy, \quad \bar{\xi} = x - iy. \]  

Following his approach we re-derived the field equations which are given below (due to an ordering difference of terms in the metric, our subscript one is equivalent to his three, i.e. \( G_{11} \equiv G_{33} \)). The field equations are listed below:

\[ G_0^0 - 2G_\xi^\xi - G_3^3 = 2 (\ddot{\alpha} + 2\dot{\beta} + \dot{\alpha}^2 + 2\dot{\beta}^2) = \kappa (\rho + 3P), \]  

\[ 12G_3^0 = \ddot{\beta} - \beta'(\dot{\alpha} - \dot{\beta}) = 0, \]  

In section 3 we outline how one can derives the FLRW limit from the inhomogeneous Szafrań-Szekeres space-times.
\[ G_3^3 = 4e^{-2\beta} \beta_\xi + e^{-2\alpha} \beta'^2 - 2\beta - 3\beta^2 = -\kappa P, \]  

(9)

\[ G_\xi^\xi = -e^{-2\alpha} (-\beta'' + \alpha' \beta' - \beta'^2) + 2e^{-2\beta}(\alpha_\xi + \alpha_\xi \alpha_\xi) - (\ddot{\alpha} + \ddot{\beta} + \alpha^2 + \beta^2 + \dot{\alpha} \dot{\beta}) = -\kappa P, \]  

(10)

\[ G_0^\xi = \dot{\alpha}_\xi + \dot{\beta}_\xi - \alpha_\xi (\dot{\beta} - \dot{\alpha}) = 0, \]  

(11)

\[ e^{2\alpha} G_\xi^3 = -\beta'_\xi + \beta' \alpha_\xi = 0, \]  

(12)

\[ 12e^{2\beta} G_\xi^\xi = \alpha_{\xi\xi} + (\alpha_\xi)^2 - 2\beta_\xi \alpha_\xi = 0, \]  

(13)

where \( \dot{\prime} = \partial/\partial z \), and \( \cdot = \partial/\partial t \), and

\[ \alpha_\xi \equiv \partial \alpha/\partial \xi \equiv 12(\partial \alpha/\partial z - \partial \alpha/\partial y), \quad \pi \equiv \partial \alpha/\partial \xi \equiv 12(\partial \alpha/\partial z + \partial \alpha/\partial y). \]  

(14)

### 2.3 Exact Solution of the Szafron-Szekeres Line Element, \( \beta' \neq 0 \)

Extending the work of Szekeres [14], Szafron [13] proved that \( \dot{\beta}_\xi = 0 \), which allowed him to propose a trial solution for the field equations of the form:

\[ \beta = \text{Log} \left[ \Phi(t, z) \right] + \nu(z, \xi, \overline{\xi}), \]  

(15)

\[ \alpha = \text{Log} \left[ h(z, \xi, \overline{\xi}) \Phi'(t, z) + h(z, \xi, \overline{\xi}) \Phi(t, z) \nu'(z, \xi, \overline{\xi}) \right]. \]  

(16)
It is straightforward to show that the trial solution identically satisfies equations (7) - (13) with \( h = h(z) \), leading to two defining equations given by,

\[
e^{-\nu} = A(z)\xi + B(z)\xi + \overline{B(z)}\xi + C(z), \tag{17}
\]

\[
2\dot{\Phi} + \ddot{\Phi}^2 + \kappa P + K(z)\Phi^2 = 0, \tag{18}
\]

where \( P = P(t) \), and \( A(z), B(z), C(z), h(z) \), are arbitrary functions and \( K(z) \) is determined by\(^2\),

\[
A(z)C(z) - B(z)\overline{B(z)} = 14[h^{-2}(z) + K(z)]. \tag{19}
\]

Equation (18) can be integrated to give\(^1\):

\[
\dot{\Phi}^2 = -K(z) + 2M(z)/\Phi - 13\kappa \Phi \int P\Phi^3 \Phi^3 dt. \tag{20}
\]

The solution for \( \Phi \) gives us \( \alpha(x, y, z, t) \), \( \beta(x, y, z, t) \) and the general expression for the metric. Szafron\(^3\) considered explicitly \( \kappa P = 43q(1 - q)t^{-2} \) with \( q \) constant and assumed \( K(z) = 0 \). In this paper by putting \( \Lambda = -\kappa P \) and performing the integration in the above equation, we get the following non-linear partial differential equation:

\[
\dot{\Phi}^2 = -K(z) + 2M(z)/\Phi + 13\Lambda \Phi^2. \tag{21}
\]

Equation (21) can be integrated parametrically by Weierstraßche elliptic functions\(^3\)(see appendix B).

The solution for \( \Phi \) is:

\[
\Phi(t, z) = M(z)/2\varphi(u + \epsilon) - \varphi(v_0), \text{ where } \varphi(v_0) = -K(z)12, \tag{22}
\]

\(^2\)Szafron used \( h(z) = 1 \)

\(^3\)For other examples of elliptic integrals see [15]
with the Weierstraß invariants given by

\[ g_2 = K(z)^2 \, 12, \quad g_3 = 1216K(z)^3 - 112\Lambda M^2 \] (23)

and where the time \( t \) is given by:

\[ t + f(z) = 1 \psi'(v_0) \left[ \log (u + \epsilon - v_0) \sigma(u + \epsilon + v_0) + 2 \left( u + \epsilon \right) \zeta(v_0) \right] (M(z)2), \] (24)

or alternatively

\[ t + f(z) = \sqrt{3\Lambda} \left[ \log (u + \epsilon - v_0) \sigma(u + \epsilon + v_0) + 2 \left( u + \epsilon \right) \zeta(v_0) \right], \] (25)

here \( \psi(z|\omega, \omega') \), \( \zeta(z) = \zeta(z|\omega, \omega') \), and \( \sigma(z) = \sigma(z|\omega, \omega') \) denote the Weierstraß’ family of functions, Weierstrass, and Weierstraß zeta and sigma functions respectively. Also \( \epsilon \) is a constant of integration and \( f(z) \) is an arbitrary function of \( z \). The definitions and properties of these functions will be discussed in detail in section 4.

Substituting our expressions for \( \alpha(x, y, z, t) \) and \( \beta(x, y, z, t) \), given by:

\[ e^\beta = \Phi(t, z)e^{\nu(z,x,y)}, \]
\[ e^\alpha = h(z)e^{-\nu(x,y,z)}(e^\beta)_{,z} \] (26)
\[ = h(z)e^{-\nu(x,y,z)}e^\beta \left[ \partial \log (\Phi(t, z)) \partial z + \partial \nu \partial z \right], \] (27)

into the metric, \( ds^2 = dt^2 - e^{2\beta}(dx^2 + dy^2) - e^{2\alpha}dz^2 \), the final solution becomes:

\[ ds^2 = dt^2 - (M(z)/2)^2(\psi(u + \epsilon) - \psi(v_0))^2e^{2\nu}(dx^2 + dy^2) - h^2(z)(M(z)/2)^2(\psi(u + \epsilon) - \psi(v_0))^2(\partial \log (\Phi(t, z)) \partial z \] (29)

Following extensive research of the literature, and to the best of our knowledge, the solution for the metric presented above is new.

\[ \psi(z) = -\zeta(z), \zeta(z) = \sigma'(z)\sigma(z) \]
2.3.1 Solution for the Matter Density in terms of the Cosmological Constant

In this subsection we derive the expression of the energy density $\rho$ that constitutes a solution of the Einstein field equations within the Szafron-Szekeres family of space-time elements.

The mass density $\rho$, is obtained from the first field equation (see Eq. (7)),

$$G_0^0 - 2G_\xi^\xi - G_3^3 = \kappa(\rho + 3P),$$

Substituting the solution of $\alpha, \beta$ in Eq.(7) (see Appendix C for details) the general expression for the energy density $\rho$ is given by:

$$\kappa\rho = 1\Phi_z + \Phi_{\nu_z} \left\{ 2M_z\Phi^2 - 13\kappa\Phi^{-2} \int P \left( \partial^2\Phi^3 \partial t \partial z \right) dt + 6M_{\nu_z}\Phi^2 - \kappa\Phi^{-2}\nu_z \int P \left( \partial\Phi^3 \partial t \right) dt \right\}.$$

Redefining the energy density as:

$$\kappa\tilde{\rho} = \kappa\rho - \Lambda,$$

the final expression for the matter density is given by:

$$\kappa\tilde{\rho} = 2M_z + 6M_{\nu_z}\Phi^2(\Phi_z + \Phi_{\nu_z}).$$

The Friedmann, Lemaitre, Robertson, Walker (FLRW) limit of the above expression will be discussed in the following section. Here we will only state that in this limit $\kappa\tilde{\rho} \propto 1R(t)^3$, where $R(t)$ is the standard scale factor in the FLRW line-element.

3 Homogeneous Cosmology

3.1 Recovery of the Friedmann Equation

In this section we will show how the Friedmann equation (FLRW limit) can be recovered from the Szekeres-Szafron metric by judicious choice of the various functions, parameters and constants (See Krasinski [11]).

The starting point is the metric:
\[ ds^2 = dt^2 - e^{2\beta(x,y,z,t)}(dx^2 + dy^2) - e^{2\alpha(x,y,z,t)}dz^2, \]  
\[ (34) \]

where \( e^{\beta(x,y,z,t)}, e^{\alpha(x,y,z,t)} \) and \( e^{-\nu(x,y,z)} \) are given by:

\[ e^{\beta} = \Phi(t, z)e^{\nu(z,x,y)}, \]
\[ e^{\alpha} = h(z)e^{-\nu(z,y,z)}(e^{\beta}), \]
\[ e^{-\nu} = A(z)(x^2 + y^2) + 2B_1(z)x + 2B_2(z)y + C(z). \]  
\[ (35) \]

Here \( A(z), B_1(z), B_2(z), C(z) \) and \( h(z) \) are arbitrary functions while \( K(z) \) is determined by:

\[ AC - B_1^2 - B_2^2 = 14 \left[ h^{-2}(z) + K(z) \right]. \]  
\[ (36) \]

Now by choosing:

\[ \Phi(t, z) = f(z)R(t), \quad K(z) = K_0f^2(z), \quad M(z) = M_0z^3 \]  
\[ (37) \]

the FLRW limit [11] in the Goode & Weinwright form [16], can be derived, namely,

\[ ds^2 = dt^2 - R^2(t)[e^{2\nu}(dx^2 + dy^2) + W^2f^2\nu^2dz^2], \]  
\[ (38) \]

where

\[ W^2 = (\epsilon - Kf^2)^{-1}, \]
\[ e^{\nu} = f(z)[a(z)(x^2 + y^2) + 2b(z)x + 2c(z)y + d(z)]^{-1}, \]  
\[ (39) \]

\[ (40) \]

\( \epsilon, K \) are arbitrary constants and \( R(t), f(z), a(z), b(z), c(z), d(z) \) are arbitrary functions subject to the constraint \( ad - b^2 - c^2 = \epsilon/4 \).

If in addition we choose, \( B_1 = B_2 = 0, C = 1, A = 14, \) and \( f(z) = z \), then the FLRW limit is recovered, leading to the standard Friedmann equation, given by:

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\[ R^2 = 8\pi G_N 3 \rho_{\text{matter}} R^2 + c^2 \Lambda 3 R^2 - Kc^2, \quad (41) \]

and the matter energy density eq.(33), which is a solution of the gravitational field equations, in this limit becomes:

\[ k\tilde{\rho} = 2M_{\nu} \Phi^2(\Phi_{\nu} + \Phi_{\nu} \Phi) = 6M_0 R^3(t), \quad (42) \]

where \( M_0 \) is a Friedmann mass integral [17]. In Eq.(41) \( K \) and \( \Lambda \) denote the curvature coefficient, and Cosmological Constant, respectively.

Now by choosing suitable parameters, namely,

\begin{align*}
2M &= 8\pi G_N 3 \times 6M_0, \\
K' &= Kc^2, \\
\Lambda' &= \Lambda c^2. \quad (43)
\end{align*}

equation (41) can be rewritten in the standard Friedmann form (Eq.(21)),

\[ \dot{R}^2 = 2MR + \Lambda' 3 R^2 - K', \quad (44) \]

whose solution is given by:

\begin{align*}
R(t) &= M/2\phi(u + \epsilon, \tau) + K'/12, \\
t &= 1c\sqrt{3\Lambda} \left[ \log \left( \sigma(u + \epsilon - \nu_0)\sigma(u + \epsilon + \nu_0) \right) + 2u \zeta(\nu_0) \right]. \quad (45)
\end{align*}

Alternatively, the solution for the scale factor and the time in terms of Jacobi theta functions are given by:

\begin{align*}
R(t) &= M2 \left\{ e_\alpha + 14\omega^2 [\theta'_1(0) \theta_{\alpha+1}(0) \theta_{\alpha+1} (\gamma) \theta_{1} (\gamma)]^2 + K'12 \right\}^{-1}, \quad (46) \\
t &= \sqrt{3\Lambda} \left[ \log (\theta_{1} (\gamma - \gamma_0) \theta_{1} (\gamma + \gamma_0)) + 2\gamma \theta'_{1} (\gamma_0) \theta_{1} (\gamma_0) \right], \quad (47)
\end{align*}

where \( \gamma = u + \epsilon 2\omega, \gamma_0 = \nu 2\omega \) and \( e_\alpha \) are the three roots of the given cubic equation.
3.1.1 Theoretical Expression for the Hubble Constant

The theoretical expression for Hubble’s Constant is given below in terms of Weierstraß, or alternatively, Jacobi theta functions.

The scale factor and its derivative with respect to time are given by:

\[
R(t) = \frac{M}{2} \wp(u + \epsilon) + K' \sqrt{2}, \quad (48)
\]
\[
\dot{R}(t) = \wp'(u + \epsilon) \wp(u + \epsilon) + K' \frac{1}{2}, \quad (49)
\]

where \( \wp'(z) = \partial \wp(z) / \partial z \).

The theoretical expression for Hubble’s parameter (the expansion rate of the Universe) for a homogeneous cosmology is given by:

\[
H(t) \equiv \frac{\dot{R}(t) R(t)}{R(t)} = -2M \wp'(u + \epsilon), \quad (50)
\]

where the derivative of the Weierstraß function is determined by the equation of the elliptic curve:

\[
\wp'(u + \epsilon) = \sqrt{4 \wp^3(u + \epsilon) - g_2 \wp(u + \epsilon) - g_3}. \quad (51)
\]

We note that Hubble’s Constant is proportional to \( \wp'(u) \). Equation (50) is remarkable since it connects the expansion rate of the Universe, a dynamical parameter, with the cosmological parameters and constants of nature in a highly non-linear way.

Alternatively, Hubble’s Constant can be expressed in terms of Jacobi theta functions, as follows:

\[
H(t) = 2M 14 \omega^3 \left( \theta_2 (\gamma) \theta_3 (\gamma) \theta_4 (\gamma) \theta_3^3 (0) \theta_2 (0) \theta_3 (0) \theta_4 (0) \theta_4^3 (\gamma) \right). \quad (52)
\]

Also the deceleration parameter \( q = -\ddot{R} R \dot{R}^2 \) is given by the following expression:

\[
q = \wp''(u + \epsilon) [\wp(u + \epsilon) + K' \sqrt{2}] \wp^2(u + \epsilon) - 1,
\]
\[
= \left[ 6 \wp^2 - 12g_2 \right] [\wp(u + \epsilon) + K' \sqrt{2}] \wp^2(u + \epsilon) - 1. \quad (53)
\]

The definition and the properties of the Weierstraß \( \wp, \sigma, \zeta \) functions and the primitive half-periods \( \omega, \omega' \) that appear in the formulas above will be given in section 4.
4 Definition and properties of the Weierstrass Functions

In this section we present the definitions and useful properties of the Weierstraßsche elliptic functions. In particular we will discuss the important differential equations that the \( \varphi \) function obeys (one of them is the defining equation of an elliptic curve) and also at which regions of the complex plane the \( \varphi \) function and its derivative are real. This is required since we demand that the scale factor, Hubble’s Constant and time are real physical quantities. It is therefore essential to provide a comprehensive discussion of the properties of the elliptic functions appearing in our solutions. For further formulas the reader is referred to the book by Abramowitz [18].

The Weierstraß \( \wp \) function, for \( \tau \in \Im \) (upper half-complex plane) and \( u \in C \) is defined as follows:

\[
\wp(u, \tau) = 1u^2 + \sum_{w \in \mathbb{Z} + \tau \mathbb{Z}} (1(u + w)^2 - 1w^2), \quad w \neq 0.
\] (54)

Next we note that the Weierstraß elliptic function \( \wp \) obeys the following important differential equations:

\[
(\wp')^2 = 4\wp^3 - g_2(\tau)\wp - g_3(\tau),
\] (55)

\[
\wp'''(z) = 12\wp(z)\wp'(z).
\] (56)

The first equation (55), shown below, is the equation of an elliptic curve, a Riemann surface of genus 1. The second differential equation that the \( \wp \) function obeys (56) is the famous Kordeweg-de Vries (KdV) partial differential equation of soliton physics (in a time independent form).

A cubic equation in the Weierstraß form is given by:

\[
y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in C.
\] (57)

Assuming that the polynomial on the right of Eq.(57) has distinct roots such a curve is called an elliptic curve.
The mapping:

\[ f : u \mapsto (\wp(u), \wp'(u)) \tag{58} \]

represents a homomorphism from the additive group of complex numbers on to the group of complex points on a cubic algebraic curve, an Elliptic Curve, a Riemann Surface of genus 1. The prime denotes differentiation w.r.t. to \( u \), i.e

\[ \wp' = \partial \wp(u) \partial u. \]

The kernel of this homomorphism is the lattice cell (the lattice \( L \) is generated by the periods \( \omega, \omega' \)). The factor group, \( CL = CKer(f) \approx f(u) \), of the complex plane modulo the lattice \( L \) is isomorphic to the group of complex points on the elliptic curve. Therefore the group of complex points on the elliptic curve is that of a torus, the direct product of two cyclic groups. The numbers \( g_2 \) and \( g_3 \) are the invariants of the cubic and are defined as follows:

\[ g_2 = 60 \sum' w^{-4}, \quad g_3 = 140 \sum' w^{-6}, \tag{59} \]

with

\[ w = 2 \ m \ \omega + 2 \ n \ \omega', \quad m, n \in \mathbb{Z}, \ w \in L\setminus\{0\}. \tag{60} \]

Here the Weierstraß invariants \( g_2, g_3 \) are related to the cosmological parameters by:

\[ g_2 = K^2(z)12, \quad g_3 = 1216K^3(z) - \Lambda 12M^2(z), \tag{61} \]

and where the discriminant \( \Delta \) is given by:

[\text{Latter in this section the lattice \( L \) of periods associated to the given cubic equation (that corresponds to a choice of the physical parameters \( \Lambda, M, K(z) \) will be calculated by integrating certain Abelian differentials \( dxy \).}]

[\text{Interesting arithmetic elliptic curves arise when the coefficients \( g_2, g_3 \) are algebraic numbers or more specifically, rational numbers. These curves have interesting number theory properties in connection with the Taniyama-Shimura conjecture, and indirectly to Fermat’s last theorem. These matters are discussed in later sections of the paper.}]

14
\[ \Delta = g_2^3 - 27g_3^2. \]  

(62)

The curve has genus 1 if it is non-singular; the non-singularity is detected by the non-vanishing of the discriminant \( \Delta \).

Finally we note that the Weierstraß function \( \wp \) (as it can be proven from the definition of the function) has non trivial modular transformation properties. It is a meromorphic Jacobi modular form of weight 2 and index 0 and signature 1\[20\], namely:

\[
\wp(u + \lambda \tau + \mu, \tau) = \wp(u, \tau) \quad ((\lambda, \mu) \in \mathbb{Z}^2).
\]

(64)

The unified theory of two apparently different mathematical subjects the elliptic curves and modular forms as first envisioned by Taniyama \[26\] has lead to the proof of Fermat’s Last Theorem by A. Wiles \[22\].

The determination of the cosmological quantities such as the scale factor \( R \), involves finding the roots of the cubic polynomial \( 4z^3 - g_2 z - g_3 \) of the elliptic curve, for various choices of the cosmological parameters \( \Lambda, M, K \).

The three roots of the cubic \( e_i \), \( i = 1, 2, 3 \) can be determined by using the algorithm developed by Tartaglia and Gardano \[23\] and their expressions in terms of \( \Lambda, M, K \) are given by:

\[
e_1 = 112 \left\{ K^3 - 18M^2\Lambda + 6\sqrt{-K^3\Lambda M^2 + 9M^4\Lambda^2} \right\}^{1/3} + 112 \left\{ K^3 - 18M^2\Lambda - 6\sqrt{-K^3\Lambda M^2 + 9M^4\Lambda^2} \right\}^{1/3},
\]

\[
e_2 = \rho 12 \left\{ K^3 - 18M^2\Lambda + 6\sqrt{-K^3\Lambda M^2 + 9M^4\Lambda^2} \right\}^{1/3} + \rho^2 12 \left\{ K^3 - 18M^2\Lambda - 6\sqrt{-K^3\Lambda M^2 + 9M^4\Lambda^2} \right\}^{1/3},
\]

\[
e_3 = \rho^2 12 \left\{ K^3 - 18M^2\Lambda + 6\sqrt{-K^3\Lambda M^2 + 9M^4\Lambda^2} \right\}^{1/3} + \rho 12 \left\{ K^3 - 18M^2\Lambda - 6\sqrt{-K^3\Lambda M^2 + 9M^4\Lambda^2} \right\}^{1/3},
\]

(65)
where $\rho = e^{2\pi i/3}$.

In general there are three types of solution depending on whether the discriminant is either zero, positive or negative. The solutions are outlined below:

1. $\Delta = 0$, the roots are all real and at least two of them coincide. The curve is no longer an elliptic curve, since it now has a singular point. The solutions are not given by elliptic functions and do not have modular properties.

2. $\Delta < 0$, the roots $e_1$ and $e_3$ form a complex conjugative pair with $e_2$ real. These solutions are given by elliptic functions and have modular properties.

3. $\Delta > 0$, all the roots are real and the solution is expressed in terms of elliptic functions which have modular properties.

The lattice $L$ of periods $2\omega, 2\omega'$ associated to the given cubic equation (that corresponds to a choice of the parameters $\Lambda, M, K$) is determined by integrating certain Abelian differentials $dz$ described below.

Consider the case $\Delta < 0$. In this case two of the roots $e_1, e_3$ are conjugate complex and one $e_2$ is real.

There is a complex conjugate pair of semi fundamental periods, $\omega$ and $\omega'$, which are given by:

$$\omega = \int_{e_1}^{\infty} dz \sqrt{4z^3 - g_2 z - g_3}, \quad \omega' = \int_{e_3}^{\infty} dz \sqrt{-4z^3 + g_2 z + g_3}. \quad (66)$$

For $\Delta > 0$, all the three roots $e_1, e_2, e_3$ of $4z^3 - g_2 z - g_3$, are real and if we order the $e_i$ so that $e_1 > e_2 > e_3$ we can choose the periods as:

$$\omega = \int_{e_1}^{\infty} dt \sqrt{4t^3 - g_2 t - g_3}, \quad \omega' = i \int_{-\infty}^{e_3} dt \sqrt{-4t^3 + g_2 t + g_3}. \quad (67)$$

In this case $\omega$ and $\omega'$ are real and totally imaginary respectively. The period ratio $\tau$ is defined by $\tau = \omega'/\omega$.

The Weierstraß zeta function is a meromorphic function with simple poles, given by:

$$\zeta(u, L) = \zeta(u|\omega, \omega') = 1u + \sum_{w \in L \setminus \{0\}} [(u - w)^{-1} + w^{-1} + uw^{-2}]. \quad (68)$$
Given a lattice $L$ the Weierstraß sigma function is an entire function whose logarithmic derivative is the zeta function:

$$
\sigma(u, L) = \sigma(u|\omega, \omega') = u \prod_{w \in L \setminus \{0\}} \left\{ (1 - uw) e^{uw^2 + u^2w^2} \right\}, \quad u \in \mathbb{C},
$$

(69)

$$
\zeta = \sigma'/\sigma, \quad \varphi = -\zeta',
$$

(70)

where prime in the previous equation indicates $d/du$.

A complex valued function $f$ of two complex variables $\omega, \omega'$ with $\Im(\omega'/\omega) > 0$ is called a homogeneous modular form with weight $k \in \mathbb{Z}$ if

$$
f(\lambda \omega, \lambda \omega') = \lambda^{-k} f(\omega, \omega'), \quad \text{for all non-zero } \lambda \in \mathbb{C},
$$

(71)

$$
\in \Gamma, \quad (72)
$$

$$
f(a \omega + b \omega', c \omega + d \omega') = f(\omega, \omega') \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,
$$

(73)
4.0.2 The Fundamental Period Parallelogram for $\Delta < 0$

The Fundamental Period Parallelogram (FPP) for a typical solution, $\Delta < 0$, is shown in figure 1. The figure also shows the position of the real root $e_2$, and the two complex conjugates $e_1$ and $e_3$. In order to obtain physically meaningful (i.e. real) values for the scale factor $R$, we restrict ourselves to a set of solutions for the Weierstrass function and its derivative that lie along both of the diagonals. Mathematically this can be described by:

The Weierstrass function $\wp(z) = \wp(u + \epsilon)$, $z = u + \epsilon$, where $\epsilon$ is a constant of integration, is real when the argument $z = \text{Re}(z) + \text{Im}(z) = m(\omega + \omega') + n(\omega - \omega')$, with $m, n \in \mathbb{R}$.

Note that the Weierstrass function and its derivative are real for the diagonal that goes left to right, and real and totally imaginary, respectively, for the right to left diagonal. As $z$ varies along diagonals of period parallelograms from 0 to $2(\omega + \omega')$ $\wp(z)$ decreases from $+\infty$ to $e_2$ to $\infty$.

We also note that the zeros of the derivative of the Weierstrass function

$\wp'(x) = 2\sqrt{(\wp - e_1)(\wp - e_2)(\wp - e_3)}$ are at the point $\omega, \omega', \omega + \omega'$ of the FPP in the complex plane. From the Euclidean uniformization (58) of the Weierstrass elliptic curve these points correspond to the three points of the elliptic curve of order two $^8$.

4.0.3 The Fundamental Period Parallelogram for $\Delta > 0$

The Fundamental Period Parallelogram (FPP) for, $\Delta > 0$, is shown in figure (2). There are three real roots $e_1, e_2, e_3 \in \mathbb{R}$, and $\omega, \omega'$ are real and totally imaginary, respectively. The real values of the Weierstrass function lie on the vertical and horizontal lines defined by:

\[ z = u + \epsilon, \wp(z = a + ib) \in \mathbb{R} \]

for $a = 0$ or $\omega$ and $b = 0$ or $\omega'$.

The Weierstrass derivative is real on the horizontal lines and totally imaginary on the vertical ones. More specifically, in the interesting case of $g_2 = K^2 12 \neq$ (m summands) $^8$A point $P$ is said to have order $m$ if $mP = P + \cdots + P = \mathcal{O}$ but $m'P \neq \mathcal{O}$ for all integers $1 \leq m' < m$, and $\mathcal{O}$ is the identity element of the elliptic curve group. If such an $m$ exists, then $P$ has finite order; otherwise it has infinite order. In order to find points of order dividing $m$, we look for points in the FPP such that $mP \in L$. 

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0, \ g_3 = 0, \text{ the resulting lattice is a square lattice with } \omega' = i\omega, \text{ i.e., the lattice } L \text{ is the Gaussian integer lattice expanded by a factor of } \omega'. \text{ In this case we will see in section (4.1.2) that as } z \text{ travels along the straight line from } \omega' \text{ to } \omega' + 2\omega, \text{ the point } (x, y) = (\wp(z), \wp(z')) \text{ moves around the real points of the elliptic curve } y^2 = 4x^3 - g_2x \text{ between } -K'/(4\sqrt{3}) \text{ and } 0, \text{ and as } z \text{ travels along the straight line from } 0 \text{ to } 2\omega, \text{ the point } (x, y) = (\wp(z), \wp(z')) \text{ travels through all the real points of this elliptic curve which are to the right of } (K'4\sqrt{3}, 0).

We also note that in the case } \Delta \neq 0, \text{ the points of order 3 correspond to the inflection points of the elliptic curve. We found that at these points and for a positive Cosmological Constant the deceleration parameter tends to } -1, \text{ leading to the asymptotic value for the Hubble’s Constant given by } \Lambda c^2 = 3H^2. \text{ Geometrically, points of order three are the points where the tangent line to the cubic has a triple order contact.}
5 Analysis and General Results

5.1 Introduction

Before presenting the results for the scale factor $R(t)$ in the general case, in which both the discriminant $\Delta$ and the cubic invariants $g_2$ and $g_3$ are non-zero, we shall present three interesting cases in which one of the above mentioned parameters vanishes. Each of these cases has a particular physical interpretation and special mathematical properties which, with the exception of $\Delta = 0$, have not been discussed in the literature before. The three cases under consideration

\[
\Delta = 0, \quad g_2 \neq 0, \quad g_3 \neq 0 \quad (75)
\]

\[
\Delta > 0, \quad g_2 \neq 0, \quad g_3 = 0 \quad (76)
\]

\[
\Delta < 0, \quad g_2 = 0, \quad g_3 \neq 0 \quad (77)
\]

The presentation and ordering of the cosmological results has been dictated by the mathematical properties of the elliptic curve: the mathematics determines the nature of the allowed physics.

5.2 Analysis

5.2.1 The Case of Vanishing Discriminant, $\Delta = 0$

There are three limiting cases in which the discriminant $\Delta$ of the cubic equation vanishes. This occurs when at least two of the roots $\epsilon_i$ coincide. As one can see from the form of the cubic invariants, $g_2, g_3$ Eq.(61), the discriminant $\Delta$ vanishes when either (or both) of $\Lambda, M$ is zero.

In the third case, $\Delta$ is identically zero for the choice of parameters given by:

\[
\Lambda' = 19R^3M^2 \equiv \Lambda_{\text{Critical}}. \quad (78)
\]

In all three cases the solutions for the cubic equation are no longer described by elliptic functions and they lose their modular properties (the Weierstraß functions degenerate). The resulting solutions are expressed in terms of elementary functions and represent the standard textbook solutions of Cosmology [19].

5.2.2 Class of Solutions with a non-zero Discriminant and One of the two Cubic Invariants zero.

This section examines the two special cases where one of the cubic invariants $g_2, g_3$ is zero for a non-zero discriminant. It will be shown that the resulting associated elliptic curves have important mathematical and physical properties.
The first case is when $g_2 = 0$, i.e. $\dot{K} = 0$ (case of a flat universe), $g_3 = -112\Lambda' M^2 \neq 0$. Here the discriminant $\Delta$ is negative and the resulting two types of solutions are characterised by the sign of $\Lambda$, which in mathematical terms correspond to the fixed points in the upper half complex plane ($\tau = \pm 12 + \sqrt{32i}$).

The second case is when $g_3 = 0$, i.e. $\dot{\Lambda} = 118K^3 M^2$ and $g_2 \neq 0$. Here $\Delta = g_2^2 > 0$ and the resulting solutions are characterised by the sign of the curvature constant $K$. The solutions correspond to the other fixed point in the upper half complex plane ($\tau = i$). These special points for $\tau$ in the upper half complex plane have significant symmetry properties which will be discussed in later sections of this paper.

**Euclidean Universe** ($\Delta < 0$, $g_2 = 0$, $g_3 \neq 0$) For $K = 0$, $\dot{\Lambda} \neq 0$, $M \neq 0$, $g_2$ vanishes while $g_3$ and the discriminant $\Delta$ are both non-zero. This is the case of the Euclidean Universe where the solutions are described by elliptic functions.

In this case:

$$g_2 = 0, \quad g_3 = -112\Lambda' M^2, \quad \Delta = -27g_3^2 < 0,$$

the roots are given by:

\[
\begin{align*}
  e_1 &= 112\sqrt{-36\Lambda' M^2}, \\
  e_2 &= \rho 12\sqrt{-36\Lambda' M^2}, \\
  e_3 &= \rho^2 12\sqrt{-36\Lambda' M^2},
\end{align*}
\]

and the cubic equation becomes:

$$\wp^2(z) = 4\wp^3(z) - g_3.$$  \hspace{1cm} (81)

For this case the discriminant is always negative and that there are two types of solutions depending upon the sign of the Cosmological Constant. Interesting physics and mathematics arise when equation (81) is solved at the zeros of the Weierstraß function, $\wp(u_0) = 0$, it reduces to:

\[
\wp^2(z) = 4\wp^3(z) - g_3.
\]

A beautiful formula for the zeros of the Weierstraß function $\wp$ obtained by Eichler and Zagier [20] is:

$$\wp(\tau, z) = 0 \iff z = \lambda \tau + \mu \pm \left(12 + \log(5 + 2\sqrt{6})2\pi i + 144\pi i \sqrt{6} \int_0^\infty (t - \tau) \Delta(t) E_6(t)^{3/2} dt\right)^{1/3}.$$ 

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\[ \psi'(u_0) = i\sqrt{g_3}. \]  
\[ (82) \]

For \( \Lambda' > 0 \) the defining equation becomes:

\[ \psi'(u_0) = i\sqrt{g_3}, \]
\[ \psi'(u_0) = i\sqrt{-\Lambda'M^212} = -M2\sqrt{\Lambda3}, \]  
\[ (83) \]

leading to the expression for Hubble’s Constant given by:

\[ H = -2M\psi'(u_0), \]
\[ = \sqrt{\Lambda^3}, \]
\[ = c\sqrt{\Lambda3}. \]  
\[ (84) \]

This is the expression of the Hubble parameter in de-Sitter phase (inflationary scenario). At the point of the complex plane that corresponds to a zero of the Jacobi modular form \( \psi \), a point of order 3, the flat universe scenario\(^\text{10}\) predicts the ultimate de-Sitter phase, given by:

\[ \Lambda c^2 = 3H^2_\infty, \]  
\[ (85) \]

where \( H_\infty \) is the limiting value of the Hubble’s Constant as \( t^- \rightarrow \infty \), and \( R^- \rightarrow \infty \). Equation (85) connects the Cosmological Constant \( \Lambda \), with the asymptotic expansion rate, itself a dynamical parameter.

As emphasised earlier, the limiting value (85) is achieved at the 3-division points of the elliptic curve that correspond to zeros of the \( \psi \) function, while the deceleration parameter \( q \) given by Eq.(53) in this limit tends to \(-1\). The deceleration parameter is shown in fig.(3). There the parameter takes values of 0.5 when the matter density is maximum, is zero at the inflection points and \(-1\) at the points where the Cosmological Constant dominates the dynamics of the universe.

For \( \Lambda' < 0 \) the defining equation becomes:

\[ \psi'(u_0) = \sqrt{g_3}, \]  
\[ (86) \]
\[ \psi'(u_0) = i\sqrt{\Lambda'M^212} = M2\sqrt{\Lambda3}, \]  
\[ (87) \]

\(^\text{10}\)In fact, as we shall see shortly, the asymptotic value \( H_\infty \) of the Hubble parameter for \( \Lambda > 0 \) is reached also in models with non-zero \( K \).
leading to a Hubble’s Constant given by:

\[ H = ic\sqrt{\Lambda}. \]  

(88)

In the Euclidean Universe scenario the lattice \( L \) is highly symmetrical in that it admits the property of complex multiplication (by \( \tau = e^{\frac{2\pi i}{3}} \)) and the Weierstraß function \( \wp \) obeys the functional relationship:

\[ \wp(\tau u) = \tau^{-2} \wp(u). \]  

(89)

The above relationship has been verified, numerically. The associated elliptic curve has absolute modular invariant \( j = 0^{11} \).

The scale factor evolution and other cosmological parameters in the special case of Euclidean Universe will be discussed in more detail below.

An interesting scenario, which is also favoured by recent observational data (Type Ia Supernovae, Cosmic Microwave background radiation experiments) is the case with \( \Lambda > 0 \). In the second graph of fig.(3) we have plotted the scale factor versus time for the choice of cosmological parameters: \( \Lambda = 10^{-56}\text{cm}^{-2}, M = 8.5 \times 10^{47}\text{cm}^{3}\text{s}^{-2}, K = 0 \).

Here we have integrated (44) along the diagonal \( m(\omega + \omega'), m \in \mathbb{R} \) of the FPP in the complex plane. Since \( \Delta < 0 \) there is one real (negative) root \( e_2 \) and two complex conjugate roots. At the 2-division point \( (\omega + \omega') \), \( \wp(\omega + \omega') = e_2 \). Fig.(4) shows the behaviour of the Weierstraß \( \wp \) function along the diagonal of the FPP. As we can see the \( \wp(u) \) function starts from \( \infty \), passes through a zero, and reaches the negative real root \( e_2 \) as \( u \) varies from 0, through \( v_0 \) and onto \( \omega + \omega' \). When \( u \to v_0 \), \( R(t) \to \infty \), \( t \to \infty \). Hubble’s Constant goes to the limiting value given by, \( H^2_\infty \to c^2\Lambda \), this is the inflationary scenario.

Finally we note that when the Cosmological Constant is negative we obtain an oscillatory solution see fig.(7). The scale factor has a maximum value at the 2-division point \( \omega + \omega' \) in the FPP. At this point the \( \wp \) function is equal to the real root, i.e. \( \wp(\omega + \omega') = e_2 \), and the first derivative \( \wp' \) vanishes leading to a zero value for the Hubble Constant as expected.

\(^{11}\)The absolute modular invariant is defined as: \( j \equiv 1728 \times g_2^3 \Delta = 1728 \times g_2^3(g_2^3 - 27g_3^3) \).

We discuss in more detail its properties in section 7.
Non-Euclidean Universe \( (\Delta > 0, g_2 \neq 0, g_3 = 0) \) The second special case is given by the following conditions:

\[ g_2 = K^2 12, \quad g_3 = 0, \quad \Delta = g_2^3. \]  

(90)

In this case \( \Delta > 0 \) and there are three real roots, one of which is zero and the other two are equal in magnitude and opposite in sign. The elliptic curve is given by the expression:

\[ y^2 = 4x^3 - g_2 x, \]  

(91)

or in the Weierstraß parametrisation:

\[ \wp^2(z) = 4\wp^3(z) - g_2 \wp(z). \]  

(92)

The vanishing of \( g_3 \) leads to the expression for the cosmological parameters, given by:

\[ \Lambda' = 118K^6 M^2 = 12\Lambda'_{crit} \]  

(93)

The sign of the Cosmological Constant depends on the sign of the curvature coefficient. Also, the elliptic curve has the property of complex multiplication (CM), in that the square lattice \( L \) admits complex multiplication by \( i \). The Weierstraß function \( \wp(x) \) satisfies the functional relationship:

\[ \wp(ix) = i^{-2} = -\wp(x). \]  

(94)

The absolute modular invariant function for the elliptic curve (92) has the value \( j = 1728^{12} \), and since the elliptic curve has the property of CM the absolute invariant is an algebraic integer.

There is a very interesting physical feature: at the point \( v_0 \) that inverts the equation \( \wp(v_0) = -K^2 12 \), the derivative of the Weierstraß function satisfies the elliptic equation, namely:

12As discussed in section 7, the absolute modular invariant function \( j \) characterises the lattice \( L \) up to homothety. In other words if \( L \) and \( L' \) are lattices in \( C \), then \( j(L) = j(L') \) if and only if \( L \) and \( L' \) are homothetic, i.e if \( \exists \lambda \in C \mid L' = \lambda L \).
\[ \varphi'^2(v_0) = 4 \times (-K'12)^3 - K'^212 (-K'12) = 8K'^312^3, \quad (95) \]

or

\[ \varphi'^2(v_0) = \Lambda'M^212, \quad (96) \]

at this point the Hubble’s Constant becomes:

\[ H = - (2M) \varphi'(v_0) = -\sqrt{\Lambda'3} = -c\sqrt{\Lambda3}. \quad (97) \]

Therefore we arrive at the de-Sitter functional relationship for the Hubble’s constant for \( \Lambda > 0 \), and again the limiting value, \( H_\infty \), for the Hubble’s Constant is recovered Eq.(85).

In fig.(5) we have plotted the scale factor versus time for the choice of parameters: \( K = 1 \), \( M = 3.5 \times 10^{48} \text{cm}^3s^{-2} \), \( \Lambda = K'^218M^2 \). Here we have integrated the Friedmann equation in the region, \( (\omega + \omega' - \omega' + 2 \omega) \) of the FPP in the complex plane, where, \( \varphi(u) \) lies between \( e_3 = -K'4\sqrt{3} \) and \( e_2 = 0 \). In this “bouncing” universe scenario, the scale factor starts at a minimum value, \( R_{min}(t) \), and then expands exponentially forever. Interestingly, such models do not suffer from a \( t = 0 \) singularity.

The deceleration parameter \( q \), versus time, is also plotted in fig (5). In this case the deceleration parameter is always negative see second graph.

Fig.(6) (first graph) presents results obtained by integrating eq.(44), for the same values of the physical parameters \( \Lambda, M, K \), in the region of the complex plane from 0 to \( 2\omega \). In this case a periodic solution is obtained, and in this region of the FPP, the Weierstraß function is positive, \( \varphi(u) > e_3 = K'4\sqrt{3} \). For \( \Lambda < 0 \) an oscillatory solution is obtained (second plot) of Fig.(6).

In both cases \( \Lambda = 12\Lambda_{crit} \), where \( \Lambda_{crit} \) is given by equation (78).
5.3 General Results \((\Delta \neq 0, g_2 \neq 0, g_3 \neq 0)\)

5.3.1 Approach

The results will be presented in three different classes determined by the sign and magnitude of the Cosmological Constant. For each of the classes arbitrary, but realistic, sets of cosmological parameters have been chosen \((\Lambda, M, K)\). Here the pseudomass and Cosmological Constant parameters always have a non-zero value such that \(g_2, g_3\) and \(\Delta\) are all non-zero. The types of solutions can be classified as:

- **Periodic or Oscillating Universe**,

- **Asymptotic Inflationary Universe**,

- **Periodic and Bouncing Universe** (for \(0 < \Lambda < \Lambda_{\text{crit}}\) and \(K = 1\))

Solving the Friedmann equation exactly leads to a set of interdependent cosmological physical quantities:

- Scale Factor
- Age of the Universe
- Hubble’s Constant
- Deceleration Parameter.

For each class of solution we have picked (fixed) values of the cosmological parameters that give ”good” results for the cosmological values. Qualitative similar results to our own for the scale factor have been found in Ref.[21].

In the following section we present our results for the scale factor, Hubble’s Constant and deceleration parameter.

5.3.2 Periodic or Oscillating Universe

For a negative Cosmological Constant the solutions are always periodic. We have chosen a representative set of cosmological parameters given by:

\[
K = 0, \pm 1, \quad M = 2.0 \times 10^{48} \text{ cm}^3 \text{s}^{-2}, \quad \Lambda = -10^{-56} \text{ cm}^{-2}.
\]  

(98)

The resulting solutions are shown in fig(7).

The solutions show a general trend of increasing scale factor and age of the universe as \(K\) goes through 1, 0 and -1. As expected, shown in fig,(7), the deceleration parameter is always positive. For this choice of the cosmological parameters, a ”good” set of results, for \(K = 0\), is given by:
5.3.3 Asymptotic Inflationary Universe

This scenario used the following set of Cosmological Parameters:

\[ K = 0, \pm 1, \quad M = 8.5 \times 10^{47} - 4 \times 10^{49} cm^2 s^{-2}, \quad \Lambda = 10^{-56} cm^{-2}. \quad (99) \]

For a positive Cosmological Constant the solutions are asymptotic inflationary for \( K = 0, -1 \). For \( \Lambda > \Lambda_{crit} \) and \( K = +1 \), the scale factor time evolution is similar to the cases that correspond to \( K = 0, -1 \), where eventually the asymptotic de-Sitter Universe is reached.

Interestingly for the case, \( K = +1 \) and \( 0 < \Lambda < \Lambda_{crit} \), a periodic or a bouncing solution is obtained. This class of solutions includes the special case \( g_3 = 0, g_2 \neq 0 \) discussed earlier in the paper, where \( \Lambda = 12 \Lambda_{crit} \) (see also next section). The resulting solutions are shown in fig.(8).

Again, the solutions show a general trend. Initially matter dominates the evolution of the scale factor, it then goes through the point of inflection and finally tends asymptotically to infinity. The Hubble’s Constant in this last phase is determined entirely by the Cosmological Constant Eq.(85) whose asymptotic value is \( H_\infty^2 \approx (c^2 \Lambda^3) \approx (54 Kms Mpc)^2 \), for the value of \( \Lambda \) chosen in Eq.(99).

Similarly, we observe that the deceleration parameter starts with a positive value (\( q = 0.5 \)), goes through zero at the inflection points and tends asymptotically to minus one. For this choice of the cosmological parameters, a "good" set of results, for \( K = 0 \), is given by:

\[
\begin{align*}
\text{Scale Factor} & \quad 3300 \text{ Mpc} \\
\text{Age of Universe} & \quad 13.6 \text{ Billion Years} \\
\text{Hubbles Constant} & \quad 66 \text{ Kms}^{-1} \text{Mpc}^{-1} \\
\text{Deceleration Parameter} & \quad -0.477
\end{align*}
\]

The above cosmological values are in agreement with recently published experimental results: scale factor [19], deceleration parameter \( q \) [8] and Hubble’s Constant [8][2].

5.3.4 Periodic (special case) and Bouncing Universe

This section examines the class of models that lead to the periodic and bouncing Universe. These solutions are produced when, \( K = 1 \) and \( 0 < \Lambda < \Lambda_{Crit} \). For this case we have chosen a parameter set given by,
\[ K = +1, \quad M = 9 \times 10^{47} \text{cm}^2 \text{s}^{-2}, \quad \Lambda = 10^{-56} \text{cm}^{-2}. \]  

(100)

The resulting solutions are shown in the first two graphs of fig.(9). For comparison the cases with \( K = -1, 0 \) and same value for the \( \Lambda, M \) parameters are shown in the last two graphs of fig.(9).

The bouncing solution is of particular interest in that there is no \( R = 0 \) singularity, and the periodic solution, for this special case, is similar to solutions with negative Cosmological Constant and positive curvature coefficient.

### 5.4 Cosmological Redshift

In the preceding sections of this paper we have developed a predictive, non-linear theory of cosmology. The hope is that the parameters \( K, M \) and \( \Lambda \) can be experimentally determined. Knowledge of these parameters will allow the geometry of the Universe to be calculated, which in turn will allow, for example, the measurement of galactic redshifts to determine absolute distances. This section of the paper discusses just such a problem.

The cosmological redshift is defined by:

\[ 1 + z \equiv R_0/R(t), \]  

(101)

where the subscript “0” denotes quantities measured at the present time; i.e. at redshift \( z = 0 \).

As an example we have calculated the Redshift versus distance curve, from the second graph of fig(3), by looking back in time from our present position.

This graph shows that the Redshift versus distance calibration curve Fig.(10) is linear up to approximately \( z = 0.16 \) (500 Mpc), after which the non-linear effects of general relativity become apparent.

In summary, measuring the redshift and calculating the geometry directly gives the associated distance, negating the need for such things as ”standard candles” in order to work around difficulties in determining cosmological distance.
6 Mathematical Connections

6.1 Connections between General Relativity (Geometry) and Topology

This part of the paper explores the connections between the physical world of General Relativity and its associated geometry and the profound world of pure mathematics in the form of topological entities, specially a torus or an elliptic curve, a Riemann surface of genus 1.

Figure (11) show the connections between General Relativity, non-linear partial differential equations (PDE’s), elliptic Weierstraß functions, Elliptic Curves, Modular Forms and the topology of a torus.

The figure is completed by Wiles’ [22] famous solution of Fermat’s Last Theorem, where he proved the Taniyama-Shimura conjecture for the family of semi-stable elliptic curves establishing the one to one connection between elliptic curves and modular forms 13. The Taniyama-Shimura conjecture for complex multiplication elliptic curves has been proven by Shimura in 1971[37].

6.2 Fermat’s Last Theorem and the Taniyama-Shimura Conjecture

Fermat’s Last Theorem remained an outstanding problem for pure mathematicians for approximately three hundred and fifty years, the proof of which was relatively recently given by Wiles[22] to universal acclaim.

Fermat stated that he could prove the equation,

\[ X^n + Y^n \neq Z^n, \quad (102) \]

admitted no integer solutions for \( n \) greater than 2, i.e. for \( n > 2, \not\exists \quad n, X, Y, Z \in \mathbb{Z} | X^n + Y^n = Z^n. \]

From a historical point of view, the starting point for the modern proof was the Taniyama-Shimura conjecture. At that time, they made, the startling conjecture that there was an exact one to one correspondence between elliptic curves and modular forms [26].

Frey [27] made the connection between Fermat’s Last Theorem and the Taniyama-Shimura conjecture, in that, if there existed a solution to the problem, the associated elliptic curve would correspond to a particular modular form of weight 2, that in principle, should not exist (this idea was formulated in a more...

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13 According to a recent preprint by Breuil et al, [34] the Taniyama-Shimura conjecture for rational elliptic curves is a theorem—all elliptic curves over \( \mathbb{Q} \) are modular elliptic curves.
precise manner by Serre). In a very beautiful work Ribet [29] proved that if
the Taniyama-Shimura conjecture holds than Frey’s elliptic curve would also be
associated with the particular modular form of weight 2 in the way predicted
by Serre’s conjecture [28]. This deep result reduced Fermat’s Last Theorem to
proving Taniyama-Shimura conjecture for semi-stable elliptic curves.

Finally Wiles, using the associated Galois [22] representations, duly obliged
with the proof.

6.3 Connection with non-linear physics

6.3.1 General Relativity

Fig. (11) shows the connections between General Relativity, the resulting non-
linear PDE’s whose solutions are given in terms of Weierstraß functions, elliptic
curves, modular forms and a Riemann surface of genus 1. We have noted that
several non-linear equations of General Relativity (Lemaitre [31], Omer [32], Zecca
[33]) have solutions described by Weierstraß functions. These connections have
led us to propose a conjecture, namely:

Conjecture 1 That all non-linear exact solutions of General Relativity with a
non-zero \( \Lambda \) can be given in terms of the Weierstraß Jacobi modular form. In this
case the exact solutions of the resulting non-linear partial differential equations
of General Relativity are described by theta functions \( \vartheta(z|\tau) \) associated with
Riemann surfaces.

6.3.2 Non-Linear Equations of Physics

The techniques developed in this paper for solving the gravitational field equa-
tions have applications to other non-linear differential equations of mathemat-
ical physics. In particular non-linear PDEs of solid state physics such as the
Kordeweg-de Vries (KdV), its two dimensional generalization, the Kadomtsev-
Petviashvili (KP), the Sine-Gordon as well as the non-linear Schrödinger equa-
tion of Quantum mechanics can readily be solved by elliptic theta functions
associated with Riemann surfaces. The key property is that the theta func-
tions associated with a period \( \tau \) (or the matrix \( \Omega \) for a higher genus Riemann
surface) that comes from a Riemann surface posses special function theoretic
properties. One of the most striking properties is that these special \( \vartheta \)’s satisfy
simple non-linear partial differential equations of fairly low degree, similar to
the differential equation that the Weierstrass, \( \wp \), function obeys.

It is interesting to speculate that all exact solutions of non-linear equations
of physics could be described in terms of Weierstraß functions, whose solutions
lie on a Riemann surface.
7 Elliptic Curves, Modular Forms and the Moduli Space associated with Cosmological Parameters \((\Lambda, M, K)\).

7.1 Introduction

Section 4, defined the Weierstraß function \(\wp\) whose fundamental differential equation parametrizes the equation of the elliptic curve \((E = CL)\), in Weierstraß form, \(y^2 = 4x^3 - g_2x - g_3\). As previously noted, the \(\wp\) function is a Jacobi modular form of weight 2 and index 0.

The solutions, previously described in this paper, lie on a genus 1 Riemann surface or equivalently the elliptic curve. In this section the physical solutions will be mapped to the upper half complex plane with the use of the period ratio parameter, \(\tau\), which is defined as the ratio of the semi fundamental periods of the Weierstrass function, \(\tau = \omega'/\omega\).

An important function which will be required later in this section is the absolute modular invariant function \(j(\tau)\). This can be defined as a function only of \(\tau\) and characterizes the isomorphism class of \(E\) over \(C^{14}\):

\[
j \equiv 1728 \times g_3^3 \Delta = 1728 \times g_2^3(g_2^3 - 27g_3^3).
\]

(103)

It satisfies the functional equation:

\[
j(a\tau + bc\sigma + d) = j(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, Z).
\]

(104)

7.2 Physical Solutions in the Upper Half Complex Plane

Fig 12 shows a representative set of solutions for various positive and negative values of the cosmological parameters. All the solutions lie on the unit circle for \((\Delta < 0^{15})\) and on the imaginary axis for \((\Delta > 0)\). These types of mappings are to be expected, as the cosmological parameters are all real, which is consistent with the absolute modular invariant being real on the boundary of the fundamental domain.

\[14\text{Let } E = CL \text{ and } E' = CL' \text{ be two elliptic curves. Then, } E \simeq E' \text{ if and only if } j(E) = j(E').\]

\[15\text{If instead of the two complex conjugate periods we choose a FPP with } \omega \text{ the half-period that corresponds to the real root and } \omega' \text{ the half-period that corresponds to one of the two complex roots then } \tau = \pm 12 + bi, \text{ with } b \geq \sqrt{32}.\]
When the torus degenerates, the elliptic curve becomes singular ($\Delta = 0$), which corresponds to one of the cases: $\Lambda = 0$, $M = 0$, or $\Lambda = 19K'^3M^2$, here one of the two half-periods becomes infinite.

Two particularly interesting elliptic curves are:

\[ y^2 = x^3 - 1, \text{ with } g_2 = 0 \text{ and } j(\tau = -12 + \sqrt{32}i) = 0, \quad (105) \]
\[ y^2 = x^3 - x, \text{ with } g_3 = 0 \text{ and } j(\tau = i) = 1728. \quad (106) \]

These elliptic curves correspond to the two fixed points of the $PSL(2, Z)$ modular group. Both they have the property of complex multiplication. The first one corresponds to the case of flat Universe ($K' = 0, \Lambda > 0$ or $\Lambda < 0, M \neq 0, \Delta < 0$) and the second to $\Lambda = 118K'^3M^2, K = \pm 1, \Delta > 0$. These cases correspond to solutions with maximal symmetry, described by the triangular and the square lattice respectively. The Euclidean Universe with $\Lambda > 0$ is presently regarded as a favourable model by the astrophysics community [2],[8].

### 7.3 Connections of General Relativity with Number Theory through the Taniyama-Shimura Conjecture

From a number theory point of view, elliptic curves with $g_2$ and $g_3$ rational or algebraic numbers are of some interest and have been studied extensively. In particular the Taniyama-Shimura theory asserts that such special arithmetic elliptic curves admit a hyperbolic uniformization by modular functions (automorphic functions) of subgroups of the full modular group $PSL(2, Z)$, generalizing the Euclidean Uniformization (58) provided by the Weierstraß functions and its derivative. In this case the corresponding elliptic curves are called modular elliptic curves. This conjecture has been proven for the case of CM curves by G. Shimura in 1971 [37]. It also has been proven recently by A. Wiles for the case of semistable elliptic curves in connection with Fermat’s Last Theorem.

As mentioned earlier, according to a recent preprint by Breuil et al [34], the Taniyama-Shimura conjecture for rational elliptic curves is a theorem—all elliptic curves over $Q$ are modular elliptic curves. The main point of interest here is the connection to CM curves since as we saw CM curves describe a highly motivated subclass of physical solutions of General Relativity, the currently favourable model for the Universe, the case with $g_2 = 0, g_3 \neq 0$ and a positive $\Lambda$ being one example. From the mathematical point of view it will be interesting to explore the possibility that the Taniyama-Shimura conjecture might

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16Let $E$ be an elliptic curve. We say that $E$ has complex multiplication if there is an endomorphism $\phi : E \rightarrow E$ which is not multiplication by $n$ map. As seen in section 4.1.2, for such special curves the lattice $L$ has CM because there is a complex number $c$ such that $cL \subset L$. 

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32
be generalised to elliptic curves over general algebraic number fields \( k \neq Q \) as it was envisioned by Taniyama using the connection of complex multiplication and modular forms [35].

8 Discussion

An exact, closed form, solution for the Szekeres-Szafron family of inhomogeneous space time line element with a non-zero Cosmological Constant was derived. By careful choice of parameters and constants the homogeneous FLRW limit of the space time line element was recovered. Within the homogeneous cosmology paradigm a useful, predictive theory of cosmology has been developed. Here knowledge of the cosmological parameters \( K, \Lambda \) and \( M \) leads to the derivation of the geometry of the Universe where actual values for the scale factor, Hubble’s Constant, deceleration parameter and Redshift calibration curves are obtained.

For the cosmological parameters, \( K = 0 \), \( \Lambda = +10^{-56}cm^{-2} \), \( M = 8.5 \times 10^{47}cm^3s^{-2} \), a set of values for the cosmological quantities are derived listed in Table 1,

<table>
<thead>
<tr>
<th>Scale factor ( R(t) )</th>
<th>3300 Mpc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age of the Universe</td>
<td>13.6 Billion years</td>
</tr>
<tr>
<td>Deceleration parameter</td>
<td>-0.477</td>
</tr>
<tr>
<td>Hubble’s Constant</td>
<td>66Kms^{-1}Mpc^{-1}</td>
</tr>
</tbody>
</table>

Table 1: Values for the scale factor, age of the Universe, \( q \) and Hubble’s Constant.

which are in reasonable agreement with the published experimental values [2],[8].

However a set of cosmological parameters with a negative Cosmological Constant also predict a ”reasonable” set of values, the essential difference between the two is the sign and magnitude of the deceleration parameter \( (q = -0.48 \) for \( \Lambda > 0 \), \( q = 2.2 \) for \( \Lambda < 0 \)\). It will be interesting to see how the value for the deceleration parameter changes over time and consequently the geometry of the Universe, as more experimental data become available.

The body of the paper shows how the use of Weierstrass functions, and the associated theory of elliptic curves and modular forms, provides the mathematical framework for solving several non-linear equations arising from General Relativity. Further we have suggested that many other non-linear equations of mathematical physics could be solved from just such an approach.
Weierstraß functions are important in that they provide the link between pure mathematics analysis techniques and the physical problems of theoretical cosmology. In particular, for a positive Cosmological Constant $\Lambda$ and $K = 0$ (Euclidean Universe), the discriminant $\Delta$ is negative resulting in one negative real root. In this case the integration path includes a zero point, $v_0$, of the Weierstraß function ($\wp(v_0) = 0$) which invariably leads to an asymptotically de Sitter phase in which the Cosmological Constant dominates the dynamics of the Universe, i.e. $c^2\Lambda^3 = H^2_{\infty}$. The zeros of the $\wp$ function are points of order 3 (i.e. they are points of the FPP that satisfy $3P = O$) and belong to the torsion subgroup of the elliptic curve group. In the general case of positive Cosmological Constant, and for $K = -1$ and $K = +1$ ($\Lambda > \Lambda_{\text{crit}}$), we also reach the dynamical de Sitter phase asymptotically at the points $v_0$ of the fundamental domain which solve the equation $\wp(v_0) = -K'12$.

For a negative Cosmological Constant and $\Delta < 0$, the Weierstraß function is always positive in the physical region of integration in the FPP, and its values lie between the positive real root and $+\infty$. In this case the integration path in the FPP never includes the points that solve the equation $\wp(v_0) = -K'12$ and a periodic solution is always obtained. The maximum scale factor occurs at the 2--division points, $e_2 = \wp(\omega + \omega')$ of the Weierstraß function. These points are points of order 2 which also belong to the torsion subgroup of the elliptic curve group. The value of the positive real root determines the maximum value for the scale factor. Similar arguments hold for the positive discriminant case.

Oscillating universes have a certain poetic almost transcendent appeal and many ancient cultures fostered the notion of a universe that periodically died, only to rise phoenix-like from the ashes to seed a new creation. As described by Herodotus [38]: “the sacred bird was indeed a great rarity, even in Egypt, only coming there according to the accounts of the people of Heliopolis once in five hundred years when the old phoenix dies.” For a recent account of the oscillating case see the book by Barrow [39].

The theory of Taniyama-Shimura has interesting ramifications, for the special cases of the invariants $g_2 = 0, g_3 \neq 0$ and $g_2 \neq 0, g_3 = 0$, which correspond to the Euclidean and $\Lambda = 12\Lambda_{\text{crit}}$ solutions. These solutions correspond to elliptic curves with the property of complex multiplication and are connected by the Taniyama-Shimura theory, to modular elliptic curves. Consequently, the solutions are intimately connected with the hyperbolic upper half-complex plane.

We believe that the machinery developed in this paper based on exact solutions of General Relativity should be a very useful tool for investigating in a precise manner the implications of various models of theoretical cosmology.

Finally, the presently accepted model of the Universe corresponds to a solution that admits complex multiplication and therefore has maximal symmetry. We believe that this is not coincidental but points to a deep connection between General Relativity and topology. This raises the question of Einstein: “Did God had any choice in creating the Universe?”.
A Ricci Tensor and Scalar for the Szafron-Szekeres Metric

Below are the equations for the non-zero components of the Ricci tensor, \( R_{\mu\nu} \) and \( R \) scalar.

\[
R_{00} = 2\partial^2 \beta \partial t^2 + \partial^2 \alpha \partial t^2 + 2(\partial \beta \partial t)^2 + (\partial \alpha \partial t)^2 . \tag{107}
\]

\[
R_{11} = -e^{2\beta} \left[ \partial^2 \beta \partial t^2 + 2(\partial \beta \partial t)^2 + \partial \beta \partial t \partial \alpha \partial t \right] + \partial^2 \beta \partial x^2 + \partial^2 \alpha \partial x^2 + (\partial \alpha \partial x)^2 - \partial \beta \partial x \partial \alpha \partial x + \partial^2 \beta \partial y^2 + \partial \beta \partial y \partial \alpha \partial y \\
+ e^{2(\beta - \alpha)} \left[ \partial^2 \beta \partial z^2 + 2(\partial \beta \partial z)^2 - \partial \alpha \partial z \partial \beta \partial z \right] . \tag{108}
\]

\[
R_{22} = -e^{2\beta} \left( \partial^2 \beta \partial t^2 + 2(\partial \beta \partial t)^2 + \partial \beta \partial t \partial \alpha \partial t \right) + \partial^2 \beta \partial x^2 + \partial \alpha \partial x \partial \beta \partial x \\
+ \partial^2 \beta \partial y^2 + \partial^2 \alpha \partial y^2 - \partial \beta \partial y \partial \alpha \partial y + (\partial \alpha \partial y)^2 \\
+ e^{2(\beta - \alpha)} \left( \partial^2 \beta \partial z^2 + 2(\partial \beta \partial z)^2 - \partial \alpha \partial z \partial \beta \partial z \right) . \tag{109}
\]

\[
R_{33} = -e^{2\alpha} \left( \partial^2 \alpha \partial t^2 + (\partial \alpha \partial t)^2 + 2\partial \alpha \partial t \partial \beta \partial t \right) + e^{2(\alpha - \beta)} \left( \partial^2 \alpha \partial x^2 + (\partial \alpha \partial x)^2 \right) \\
+ e^{2(\alpha - \beta)} \left( \partial^2 \alpha \partial y^2 + (\partial \alpha \partial y)^2 \right) \\
+ 2 \left( \partial^2 \beta \partial z^2 + (\partial \beta \partial z)^2 - \partial \alpha \partial z \partial \beta \partial z \right) . \tag{110}
\]

\[
R = 4\partial^2 \beta \partial t^2 + 6(\partial \beta \partial t)^2 + 4\partial \beta \partial t \partial \alpha \partial t + 2(\partial \alpha \partial t)^2 + 2\partial^2 \alpha \partial t^2 \\
- 2e^{-2\beta} \left( \partial^2 \beta \partial x^2 + \partial^2 \alpha \partial x^2 + (\partial \alpha \partial x)^2 \right) \\
- 2e^{-2\beta} \left( \partial^2 \beta \partial y^2 + \partial^2 \alpha \partial y^2 + (\partial \alpha \partial y)^2 \right) \\
- e^{-2\alpha} \left( 4\partial^2 \beta \partial z^2 + 6(\partial \beta \partial z)^2 - 4\partial \alpha \partial z \partial \beta \partial z \right) . \tag{111}
\]

B Integration of the Non-Linear PDE

The elliptic integral for time can be defined as:
let us first consider the integral,

\[
\Phi = \frac{1}{X} = \frac{M(z)}{2} \wp(u + \epsilon) + K(z)/12.
\]

(117)

Finally, substituting back into our previous equations, gives:

\[
\Phi = -1/X = M(z)/2\wp(u + \epsilon) + K(z)/12
\]

(117)

\[
t = \int \Phi du = \int M(z)/2\wp(u + \epsilon) + K(z)12du.
\]

(118)
C Derivation for the Energy Density expression

In this appendix we derive the expression for the energy density using the first field equation 7, given by,

\[ G_0^0 - 2G_ξ^ξ - G_3^3 = 2 (\ddot{α} + 2\ddot{β} + \dot{α}^2 + 2\dot{β}^2) = \kappa (\rho + 3P) , \quad (119) \]

\[ G_0^0 - 2G_ξ^ξ - G_3^3 = -2 \left[ \phi (\phi'' + \nu' \phi_{,tt}) \phi + \phi'' + \phi_{,tt} \phi (\phi' + \phi'') \right] \]

\[ = -2 (\phi' + \phi'') \left[ \phi'' + \nu' \phi_{,tt} + 2 \phi_{,tt} \phi (\phi' + \phi'') \right] \]

\[ = -2e' \left[ e^\beta \right]^{-1} \left\{ \phi'' + \nu' \phi_{,tt} + 2 \phi_{,tt} \phi (\phi' + \phi'') \right\} \quad (120) \]

Expression (120) can be written as follows:

\[ \phi'' + \nu' \phi_{,tt} + 2 \phi_{,tt} \phi (\phi' + \phi'') = \]

\[ = 2M(z)\phi^3\phi\phi_{,z} - 1\phi^2M_{,z}(z) - 13\kappa\phi_{,z}\phi^3 \int P (\partial \phi^3 \partial t) dt - \kappa P^2 \phi_{,z} \]

\[ + 16\kappa \phi^2 \int P (\partial^2 \phi^3 \partial t \partial z) dt - \nu z M(z) \phi^2 + \nu z \kappa \phi^3 \int P (\partial \phi^3 \partial t) dt \]

\[ - \kappa P^2 \phi_{,z} + \left[ -2M(z)\phi^3 + 13\kappa \phi^3 \int P (\partial \phi^3 \partial t) dt - \kappa P \right] \phi_{,z} + \phi'_{,z} \]

\[ = -M_{,z}(z)\phi^2 + 16\kappa \phi^2 \int P (\partial^2 \phi^3 \partial t \partial z) dt - 3M(z)\nu z \phi^2 \]

\[ + 12\kappa \nu z \phi^2 \int P (\partial \phi^3 \partial t) dt \]

\[ - 3\kappa P^2 (\phi_{,z} + \phi'_{,z}) \quad (121) \]

Eq.(120) can be written using (121),

\[ G_0^0 - 2G_ξ^ξ - G_3^3 = \]

\[ \left[ 2M_{,z}(z)\phi^2 - 13\kappa \phi^2 \int P (\partial^2 \phi^3 \partial t \partial z) dt + 6M(z)\nu z \phi^2 - \kappa \phi^2 \nu z \int P (\partial \phi^3 \partial t) dt + \right. \]

\[ \left. 3\kappa P (\phi_{,z} + \phi'_{,z}) \right] \times \left\{ e' \left[ e^\beta \right]^{-1} \right\} \]

\[ = \kappa (\rho + 3P) \quad (122) \]

Finally we get,
\[ \kappa \rho = e^{\nu|e|v} \left\{ 2M_z(z)\Phi^2 - 13\kappa \Phi^2 \int P \left( \partial^2 \Phi^3 \partial t \partial z \right) dt + 6M(z)\nu_z \Phi^2 - \kappa \Phi^2 \nu_z \int P \left( \partial \Phi^3 \partial t \right) dt \right\}^{-1} \]

(123)

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Figure 1: Fundamental domain for $\Delta < 0$
Figure 3: Deceleration parameter and scale factor in the Euclidean Universe, with parameters $\Lambda = 10^{-56} \text{cm}^{-2}, M = 8.5 \times 10^{47} \text{cm}^3 \text{s}^{-2}, K = 0$. 
Figure 4: Values of the Weierstrass $\wp$ function along the diagonal $\omega + \omega'$ of the FPP.
Figure 5: Scale factor and deceleration parameter versus time in the bouncing universe for $K = +1, \Lambda = (1/18)K^3M^{-2}, M = 10^{48}\text{cm}^3\text{s}^{-2}, \Lambda = 118K^3M^2$, in the special case $g_3 = 0$. 
Figure 6: Periodic solution for $\Lambda > 0$ (first graph) and $\Lambda < 0$ (second graph) in the special case $g_3 = 0$. 
Figure 7: Periodic solutions for $\Lambda < 0$. 

Negative Cosmological Constant, $\Lambda = -10^{-56}\text{cm}^{-2}, M = 2 \times 10^{48}\text{cm}^{3}\text{s}^{-2}$
Figure 8: Asymptotic inflationary cosmological scenarios for positive Cosmological Constant.

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\( \Lambda = 10^{-55} \text{cm}^{-2} \)

- **\( K = +1 \)**
  - \( M = 4 \times 10^{49} \text{cm}^{3} \text{s}^{-2} \)

- **\( K = -1 \)**
  - \( M = 9 \times 10^{47} \text{cm}^{3} \text{s}^{-2} \)

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\( \Lambda = 10^{-56} \text{cm}^{-2} \)

- **\( K = +1 \)**
  - \( M = 4 \times 10^{49} \text{cm}^{3} \text{s}^{-2} \)

- **\( K = 0 \)**
  - \( M = 8.5 \times 10^{47} \text{cm}^{3} \text{s}^{-2} \)
Figure 9: Scale factor vs time for $0 < \Lambda < \Lambda_{\text{crit}}$. 

Positive Cosmological Constant $\Lambda = 10^{-56}\text{cm}^{-2}$ 

$K = +1$, bouncing Universe 

$M = 9 \times 10^{47}\text{cm}^3\text{s}^{-2}$ 

$K = -1$ 

$M = 9 \times 10^{47}\text{cm}^3\text{s}^{-2}$ 

$K = 0$ 

$M = 8.5 \times 10^{47}\text{cm}^3\text{s}^{-2}$
Figure 10: Redshift versus Distance calibration curve.

\[ M = 8.5 \times 10^{47} \text{ cm}^2 \text{ s}^{-2}, \ K = 0, \ \Lambda = 10^{-56} \text{ cm}^{-2} \]
Figure 11: Connections between non-linear differential equations in General Relativity, elliptic curves and modular forms.
Figure 12: Mapping of the physical solutions in the upper half-complex plane.