In-medium pion weak decay constants

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Abstract

In nuclear matter, the pion weak decay constant is separated into the two components $f^t, f^s$ corresponding to the time and space components of the axial-vector current. Using QCD sum rules, we compute the two decay constants from the pseudoscalar-axial vector correlation function in the matter $\int d^4x \ e^{ip\cdot x} \langle \rho | T[\bar{d}(x)i\gamma_\mu u(x) \bar{u}(0)\gamma_\mu \gamma_5 d(0)] | \rho \rangle$. It is found that the sum rule for $f^t$ satisfies the in-medium Gell-Mann–Oakes–Renner (GOR) relation precisely while the $f^s$ sum rule does not. The $f^s$ sum rule contains the nonnegligible contribution from the dimension 5 condensate $\langle \bar{q}iD_0 iD_0 q \rangle_N + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_N$ in addition to the in-medium quark condensate. Using standard set of QCD parameters and ignoring the in-medium change of the pion mass, we obtain $f^t = 105$ MeV at the nuclear saturation density. The prediction for $f^s$ depends on values of the dimension 5 condensate and slightly on the Borel mass. However, the OPE constrains that $f^s/f^t \geq 1$, which does not agree with the prediction from the in-medium chiral perturbation theory. Depending on the value of the dimension 5 condensate, $f^s$ at the saturation density is found to be in the range $112 \sim 134$ MeV at the Borel mass $M^2 \sim 1$ GeV$^2$.

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The pion weak decay in nuclear matter is expected to be different from its decay in free space. In hadronic models, the in-medium shift of the decay constant occurs through the scalar meson exchange with the background nucleons as well as through isobar-hole (or nucleon-hole) excitations that intermediates a pion and the axial-vector current [1]. The decay constant is directly connected to the renormalization of the induced pseudoscalar coupling, which therefore affects muon capture in nuclei [1–4].

One peculiar feature of the in-medium decay constant is its separation into the time and space components in the matter [1,4]. As we will see below, the separation seems to be a natural consequence when a nuclear matrix element of the axial-vector current involving a pion state is considered in defining pion decay constants. The two decay constants are scaled differently with the nuclear density and the induced pseudoscalar constant is separated correspondingly [1]. Interference between the two decay components may be important for explaining the muon capture data in nuclei. Therefore, it is necessary to determine the decay constants in the matter. Indeed, Kirchbach and Wirzba [3] calculated the in-medium decay constants using the in-medium chiral perturbation theory developed in Ref. [5]. It is found that, while the time component is quenched moderately, the space component is quenched somewhat substantially. At the nuclear saturation density, its value is only 20% of the free decay constant, which seems to be a too much quenching.

Another important motivation for studying the in-medium decay constants is in connection with the so-called Brown-Rho scenario [6]. In this scenario, the in-medium decay constant is a scaling parameter that governs the in-medium reduction of hadron masses in a very simple manner. The quenching of the decay constant can be regarded as a signature of a partial restoration of chiral symmetry. To estimate the degree of the restoration, it is important to calculate the in-medium decay constant model-independently. Such a calculation can be done using QCD sum rules [7,8].

In this work, we perform QCD sum rule calculation of the in-medium decay constant using the pseudoscalar-axial vector (PA) correlation function in nuclear matter. The in-medium chiral perturbation theory [3] will be used to represent the phenomenological side of the correlation function. In this approach, the low-lying intermediate state involves an in-medium quasi-pion whose pole is well separated from higher resonance poles. Problematic instanton contributions [9] can be avoided due to the presence of the axial-vector current in the correlator. Therefore, this correlation function may be relevant for studying pion properties in the matter.

In the QCD side, we will use the finite density QCD sum rule techniques developed in Ref. [10–12]. The finite density QCD sum rule has and used for investigating the self-energies of the nucleon embedded in the matter [10–12]. Similar sum rule has been applied to the calculation of vector meson masses at the finite density [13] and temperature [14]. It can provide a useful constraint for various effective model predictions [15]. A crucial element in the construction of these sum rules is the linear density approximation, which allows to write nuclear matrix elements in terms of the nucleon matrix elements plus their vacuum values. In-medium sum rules therefore are valid as long as the density is low. Following these in-medium frameworks, we construct sum rules for the PA correlation function.

This paper is organized as follow. In Section II, we discuss a subtlety in the definition
of the in-medium decay constants. The isospin symmetry must be assumed in the usual separation of the time and space components of the decay constant. In Section III, we construct the phenomenological side of the PA correlator by following the in-medium chiral perturbation theory. The OPE calculation is given in Section IV and the sum rule analysis for the decay constants is presented in Section V.

II. SEPARATION INTO TWO IN-MEDIUM DECAY CONSTANTS

There is a subtlety in the definition of the pion decay constants in nuclear matter. In this section, we address this issue by defining the in-medium pion decay constant in close analogies with the one in vacuum. To do this, we first recall that in vacuum, the pion decay constant $f_\pi$ is defined by the relation [16],

$$\langle 0 | \bar{d} \gamma^\mu \gamma_5 u | \pi^+(k) \rangle = i f_\pi k^\mu, \ (f_\pi = 131 \text{ MeV}).$$

The well-known Gell-Mann–Oakes–Renner (GOR) relation

$$-4m_q \langle \bar{q}q \rangle = m_\pi^2 f_\pi^2.$$  

can be easily derived from PCAC by taking the divergence and subsequently using the soft-pion theorem.

A straightforward generalization to the nuclear matter case is to write the following nuclear matrix element into the form [1,3]

$$\langle \rho | \bar{d} \gamma^\mu \gamma_5 u | \pi^+(k) \rho \rangle = (i f^t k_0, i f^s k),$$

where $| \rho \rangle$ denotes the ground state of the nuclear matter. This is a usual definition that can be found in literature but one can show that taking this definition for the in-medium pion decay constants is equivalent to assuming an exact isospin symmetry in the isospin symmetric matter. The appearance of the two decay constants, $f^t$ and $f^s$, is also related to the isospin symmetry.

To illustrate this point in detail, we notice that this nuclear matrix element can contain a term proportional to the four-velocity vector $u_\mu$ (with $u^2 = 1$) of the nuclear matter as well as a term proportional to the pion momentum $k_\mu$. The part proportional to $u_\mu$ can be further separated into two parts, one that vanishes in the soft-pion limit and the other that does not. Hence, we may write the nuclear matrix element into the form,

$$\langle \rho | \bar{d} \gamma^\mu \gamma_5 u | \pi^+(k) \rho \rangle = iak^\mu + ibu^\mu (u \cdot k) + icu^\mu,$$

with some in-medium (dimensionful) constants $a$, $b$ and $c$. The third term that does not vanish in the soft-pion limit is proportional to the isospin breaking in the medium. This can be easily seen by applying the soft-pion theorem directly to the nuclear matrix element,

$$\langle \rho | \bar{d} \gamma^\mu \gamma_5 u | \pi^+(k) \rangle \mid_{k_\mu \to 0} \sim \langle \rho | \bar{d} \gamma^\mu d - \bar{u} \gamma^\mu u | \rho \rangle.$$  

Thus, $c$ in Eq. (4) is proportional to the isospin breaking in the medium and it is zero either when the isospin symmetry is exact or when the nuclear density is zero. Also under the
isospin symmetry, there are only two in-medium constants involved for the matrix element
Eq. (4). In the nuclear rest frame $u^\mu = (1, 0)$, we recover Eq. (3) with the identifications

$$a + b = f^t; \quad a = f^s.$$

When the isospin symmetry is broken in the matter, one may quickly notice that the
time component $f^t$ is not well defined. In this case, the time component of Eq. (4) is
$iak_0 + ibk_0 + ic$ and it is not clear how to deal with the $c$ term in defining $f^t$. Thus, $f^t$
defined in Eq. (3) is only for the isospin symmetric case. The ambiguity in the isospin
breaking case is intrinsic in the matter and it may be important when the isospin breaking
of the in-medium decays is studied. Somewhat larger breaking can be expected from the
kaon channel as the strange vector current is involved. This breaking in fact makes $K^+$ and
$K^-$ decays different. In future, it will be interesting to study how the breaking effect is
accommodated systematically in meson decays in the matter. In this work, we will neglect
the isospin breaking and focus on the part that can be represented by Eq. (3). The isospin
breaking terms in the pion sum rules however are suppressed by the small quark masses and
the density.

III. PSEUDOSCALAR AND AXIAL VECTOR CORRELATION FUNCTION

In our QCD sum rule study of the in-medium pion decay constants, we consider the
pseudoscalar-axial vector (PA) correlation function in the nuclear medium,

$$\Pi^\mu \equiv i \int d^4x \ e^{ip\cdot x} \langle \rho | T \left[ \bar{d}(x) i\gamma_5 u(x) \bar{u}(0) \gamma^\mu \gamma_5 d(0) \right] | \rho \rangle,$$

where $| \rho \rangle$ denotes the ground state of the nuclear matter. Because of the separation in
Eq. (3), there are two independent components for this in-medium correlator, time and
space components. We will consider the following limit of the two components

$$\lim_{p^2 \to 0} \frac{\Pi^0}{ip_0}; \quad \lim_{p^2 \to 0} \frac{\Pi^i}{ip^i}$$

so that the resulting functions in the nuclear rest frame depend on one scalar variable $p_0^2$. In
this limit, the OPE calculation can be performed at $p_0^2 \to -large$ unambiguously, which is
then straightforwardly matched with the hadronic representation that we will construct in
this section. We will follow Ref. [3] in making a phenomenological ansatz of this correlation
function.

Following the in-medium PCAC [4], the pseudoscalar current couples to an in-medium
quasi-pion. Hence, the low-lying state that intermediates the PA correlator is $|\pi(k)\rho\rangle$. Of

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1 One can expect that $a$ and $b$ in Eq. (4) also contain isospin breaking parts. As the isospin symmetry is assumed throughout this work, we will neglect such terms as well as the terms belonging to $c$. Thus, $a$ and $b$ in this equation are slightly different from the ones that appear in Eq. (4) by small isospin breaking parts.
course, other states containing \( \pi' \) and \( \pi'' \) can also intermediate the correlator but we will treat them as the continuum. Introducing a coupling strength of the pseudoscalar current to the in-medium quasi-pion \( \langle \rho | i \bar{\gamma}_5 \gamma_\mu \rho \pi | \rangle \equiv G^*_\pi \) and using Eqs. (4) (with \( c = 0 \)), we obtain the phenomenological side of the PA correlator [3],

\[
\Pi^\mu = -iG^*_\pi \alpha p^\mu + \frac{b u^\mu (u \cdot p)}{p^2 - m^*_\pi^2}.
\]

Here we have introduced the in-medium pion mass \( m^*_\pi \). As we will take the limits given in Eq. (8), a possible \( p \) dependence in \( m^*_\pi \) does not affect our sum rule. PCAC tells us that in vacuum, \( G^*_\pi \rightarrow G_\pi = f_\pi m^2_{\pi}/2m_q \). In the matter, we should have \( G^*_\pi = \frac{f t}{m^*_{\pi}^2/2m_q} \) so that it satisfies a set of relations given in Ref. [3]. Putting Eq. (6) into Eq. (9) and evaluating it in the nuclear rest frame \( u^\mu = (1, 0) \), we obtain the phenomenological ansatz for the time and space components of the correlator \( \Pi^0 \) and \( \Pi^j \) \((j = 1, 2, 3)\) in terms of \( f_t \) and \( f_s \) [defined in Eq. (3)],

\[
\Pi^t_{\text{phen}} \equiv \lim_{p \rightarrow 0} \frac{\Pi^0}{ip_0} = -m^*_\pi^2 \frac{f_t^2}{2m_q p_0^2 - m^*_\pi^2}; \quad \Pi^s_{\text{phen}} \equiv \lim_{p \rightarrow 0} \frac{\Pi^j}{ip^j} = -m^*_\pi^2 \frac{f_t f_s}{2m_q p_0^2 - m^*_\pi^2}.
\]

**IV. THE OPERATOR PRODUCT EXPANSION (OPE)**

The OPE calculation for the pseudoscalar-axial vector (PA) correlation function Eq. (7) is performed in this section. For technical details, we mostly follow Refs. [11,17]. Before we proceed to the in-medium OPE calculation, it may worth mentioning on how the OPE looks like in the twist expansion. The leading term in the expansion, the scalar part, must take the same form as the vacuum sum rule. In the vacuum case, we know that the correlator function must reproduce the GOR relation at the order \( O(m_q) \). Note, the phenomenological side Eq. (10) contains quark mass in the denominator. To reproduce the GOR relation, only OPE term that does not vanish in the limit \( m_q \rightarrow 0 \) must be the quark condensate. All others in the OPE must be zero in the limit \( m_q \rightarrow 0 \) because the GOR relation is an exact relation at the order \( O(m_q) \). The modification to this picture in the medium is driven by the higher twist operators, which do not need to be scaled with quark mass but they must be proportional to the nuclear density.

Another thing to mention is that the isospin symmetry in the medium is assumed. It means that \( \pi^+ \) or \( \pi^- \) decays are assumed to be the same. Note, once the isospin breaking is introduced, the time component \( f_t \) is not well-defined. Anyway, under the isospin symmetry, the OPE must be symmetric under \( u \leftrightarrow d \) as it should be.

The OPE of the correlator Eq. (7) can be calculated by using the standard techniques in the fixed-point gauge [18]. To include the finite quark mass effectively, we express the correlator in the momentum space. Introducing an in-medium quark propagator \( S_j(k) \) \((j = u, d)\) in the momentum space, we write Eq. (7) as

\[
\Pi^\mu = -i \int \frac{d^4k}{(2\pi)^4} Tr \left[ i S_d(k) i \gamma_5 i S_u(k - p) \gamma^\mu \gamma_5 \right].
\]
The trace is taken over the Dirac and color indices. The perturbative part can be calculated by using the free propagator

\[ iS_j^{\text{free}}(k) = \frac{i\not{k} + m_j}{k^2 - m_j^2} \quad (j = u, d) \]  

in Eq. (11). Using standard techniques of the dimensional regularization and the Feynman parameterization, we readily compute the perturbative part,

\[ \Pi^\mu_{\text{pert}}(p^2) = -\frac{3}{4\pi^2} ip^\mu \int_0^1 du \frac{m_q\ln[-u(1-u)p^2 + m_q^2]}{\ln [-u(1-u)p^2 + m_q^2]} \] 

with \( m_q = (m_d + m_u)/2 \).

Nonperturbative contributions involving the quark condensate can be obtained by replacing one propagator by the disconnected nonlocal quark condensate, \( iS_{ab}^{(2)}(k) \rightarrow \int d^4x \ e^{ik\cdot x} \langle q^a(x)q^b(0)\rangle_p \) while the other propagator remains to be the free quark propagator. The subscript \( \rho \) indicates the in-medium expectation value. A Taylor expansion around \( x_\mu = 0 \) leads to [11,19],

\[ \int d^4x \ e^{ik\cdot x} \langle q^a(x)q^b(0)\rangle_p = -\frac{\delta_{ab}}{12(2\pi)^4} \left\{ \frac{\delta^{\alpha\beta}A(k) + \gamma^{\alpha\beta}B^\lambda(k)}{m_q} \right\} \] 

where

\[ A(k) = \langle \bar{q}q \rangle_\rho \tilde{\delta}^{(4)}(k) + \langle \bar{q}D_\mu q \rangle_\rho \frac{\partial}{\partial \not{k} \not{\nu}} \delta^{(4)}(k) + \frac{1}{2} \langle \bar{q}D_\mu D_\nu q \rangle_\rho \frac{\partial}{\partial \not{k} \not{\nu}} \frac{\partial}{\partial \not{k} \not{\rho}} \delta^{(4)}(k) + \cdots, \]

\[ B^\lambda(k) = \langle \bar{q}\gamma^\lambda q \rangle_\rho \delta^{(4)}(k) + \langle \bar{q}\gamma^\lambda D_\mu q \rangle_\rho \frac{\partial}{\partial \not{k} \not{\nu}} \delta^{(4)}(k) + \frac{1}{2} \langle \bar{q}\gamma^\lambda D_\mu D_\nu q \rangle_\rho \frac{\partial}{\partial \not{k} \not{\nu}} \frac{\partial}{\partial \not{k} \not{\rho}} \delta^{(4)}(k) + \cdots. \]  

A few technical remarks are in order. Note that the dimension of the correlator is even. In the phenomenological side Eq. (9), one dimension is taken up by \( p^\mu \) so the rest has an odd dimension. It means, the constants \( a \) and \( b \) in Eq. (9) pick up nonzero contributions only from odd dimensional operators in the OPE. Even dimensional operators are not symmetric under \( u \leftrightarrow d \) and constitute the isospin breaking parts of the OPE which we ignore in this work. Then, this observation yields an interesting consequence. When the part \( A(k) \) in Eq. (14) is taken for the disconnected quark propagator, then only the \( k \) part of the free quark propagator Eq. (12) survives through the trace in Eq. (11). In this case, the term like \( \langle \bar{q}D_\mu q \rangle_\rho \) constitutes even dimensional contributions, thus cannot contribute to our isospin symmetric decomposition of Eq. (9). Similarly when the part \( B^\lambda \) in Eq.(14) is taken for the disconnected quark propagator, then the \( m_j \) part of the free quark propagator gives nonzero contributions. Odd dimensional operators in \( B^\lambda \) (for example \( \langle \bar{q}\gamma^\lambda q \rangle_\rho \)) combine with \( m_j \) to form even dimensional operators and they can not contribute to Eq. (9). In-medium condensates that contribute to our sum rules can be decomposed into [11]

\[ \langle \bar{q}D_\mu D_\nu q \rangle_\rho = \frac{4}{3} \langle \bar{q}(u \cdot D)^2 q \rangle_\rho \left[ u_\mu u_\nu - \frac{1}{4} g_{\mu\nu} \right] - \frac{1}{3} \langle \bar{q}D^2 q \rangle_\rho (u_\mu u_\nu - g_{\mu\nu}) , \] 

\[ \langle \bar{q}\gamma_\mu D_\nu q \rangle_\rho = \frac{4}{3} \langle \bar{q}u \cdot D q \rangle_\rho \left[ u_\mu u_\nu - \frac{1}{4} g_{\mu\nu} \right] - \frac{1}{3} \langle \bar{q}q \rangle_\rho (u_\mu u_\nu - g_{\mu\nu}) . \]
Another nonperturbative contributions come from the quark-gluon mixing operator. To form such an operator, we need to take the quark propagator with one gluon line attached [8]

$$iS_{cd}(k) \rightarrow \frac{g_s}{2} G^A_{\alpha\beta} t^A_{cd} \frac{1}{(k^2 - m_q^2)^3} \langle \bar{q}(k + m_q) \gamma^\beta (\bar{q} + m_q) \gamma^\alpha (\bar{q} + m_q) \rangle ,$$

where $G^A_{\alpha\beta}$ is the gluon field tensor, $t_A$ the SU(3) generators in color space with the normalization $Tr(t^A t^B) = \delta^{AB}/2$, $c$ and $d$ color indices. Taking the gluon tensor and moving it into the disconnected quark propagator, one can form a quark-gluon mixed condensate in the medium [11]

$$\langle g_s q^\alpha_d \bar{q}^\beta_c G^A_{\mu\nu} \rangle = - \frac{t^A}{96} \left\{ \langle \bar{q}q \sigma \cdot G \rangle \rho [\gamma_{\mu\nu} + i(u_{\mu} \gamma_{\nu} - u_{\nu} \gamma_{\mu})]^{\alpha\beta} + \langle \bar{q}q \bar{q} \sigma \cdot G \rangle \rho [\gamma_{\mu\nu} + i(u_{\mu} \gamma_{\nu} - u_{\nu} \gamma_{\mu})]^{\alpha\beta} - 4 \left[ \langle \bar{q}(u \cdot D)^2 q \rangle + i m_q \langle \bar{q} q u \cdot D q \rangle \right] [\gamma_{\mu\nu} + i(u_{\mu} \gamma_{\nu} - u_{\nu} \gamma_{\mu})]^{\alpha\beta} \right\} .$$

Here note, $G_{\mu\nu} \equiv G^A_{\mu\nu} t^A$.

Combining all these ingredients, we readily compute nonperturbative parts that can contribute up to dimension 5. After a lengthy but straightforward calculation, we obtain

$$\Pi^\mu_{\text{nonpert}} = \frac{ip^2}{p^2 - m_q^2} \left[ \frac{2}{3} \langle \bar{q}q \rangle \rho \right] + \frac{4}{3} m_q \langle \bar{q}q u \cdot D q \rangle \rho \left[ \frac{p^\mu - 4p \cdot uu^\mu}{(p^2 - m_q^2)^2} \right] - \frac{2i}{3} \left[ \langle \bar{q}q \rangle \rho \right] \left[ \frac{p^\mu - 2p \cdot uu^\mu}{(p^2 - m_q^2)^2} + \frac{4(p \cdot u)^2 p^\mu}{(p^2 - m_q^2)^3} \right] - \frac{16}{3} m_q \langle \bar{q}q \rangle \rho \left[ \frac{p \cdot uu^\mu - p^\mu/4}{(p^2 - m_q^2)^2} + \frac{4}{3} m_q \langle \bar{q}q \rangle \rho \frac{p \cdot uu^\mu - p^\mu}{(p^2 - m_q^2)^2} \right].$$

Here, we have not written down even dimensional operators belonging to the isospin breaking. Though the OPE is messy, we can make a simple check. If we collect only the scalar part of each condensate, all the dimension five contributions become proportional to $m_q^2 \langle \bar{q}q \rangle \rho$, whose Wilson coefficient becomes zero. Through this delicate cancellation, we precisely obtain the pion sum rule given in Ref. [17], which seems to support that our OPE calculation is correct.

At the dimension 5, one may expect another nonperturbative contributions (with odd dimensional condensate) containing $m_q \langle \bar{q}q \rangle \rho$. To compute this, we use quark propagators with one and two gluon attached [8] in the calculation of Eq. (11). Somewhat lengthy calculations show that the Wilson coefficient of this term is infrared divergent when the quark mass goes to zero. It means the integral is dominated by the soft region of the internal momentum $k^2 \sim m_q^2$ that has to be subtracted from the standard OPE [20].

Since we are constructing sum rules for the two scalar functions given in Eq. (10), we can write down from Eqs. (13) (20) the time and space components of $\Pi^\mu = \Pi^\mu_{\text{pert}} + \Pi^\mu_{\text{nonpert}}$ in the nuclear rest frame $u^\mu = (1, 0)$,
\[ \Pi_{\text{ope}}^t \equiv \lim_{p_\rho \to 0} \frac{\Pi^0}{ip_\rho} = -\frac{3}{4\pi^2} \int_0^1 du \ m_q \ln[-u(1-u)p_0^2 + m_q^2] + \frac{2\langle \bar{q}q \rangle_\rho}{p_0^2 - m_q^2} + \frac{8m_q \langle q^i i_D q \rangle_\rho - 2m_q^2 \langle \bar{q}q \rangle_\rho}{(p_0^2 - m_q^2)^2} , \tag{21} \]

\[ \Pi_{\text{ope}}^s \equiv \lim_{p_\rho \to 0} \frac{\Pi^i}{ip_\rho} = -\frac{3}{4\pi^2} \int_0^1 du \ m_q \ln[-u(1-u)p_0^2 + m_q^2] + \frac{2\langle \bar{q}q \rangle_\rho}{p_0^2 - m_q^2} - \frac{8m_q \langle q^i i_D q \rangle_\rho}{3 (p_0^2 - m_q^2)^2} - \frac{2m_q^2 \langle \bar{q}q \rangle_\rho}{(p_0^2 - m_q^2)^2} + \frac{32}{3} \left[ \langle \bar{q}iD_0 iD_0 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_\rho \right] \frac{1}{(p_0^2 - m_q^2)^2} . \tag{22} \]

In obtaining this, we have neglected the terms whose dimension in the numerator is 7.

**V. THE IN-MEDIUM PION DECAY CONSTANTS**

Having calculated the OPE for the PA correlation function Eq. (7), we can easily construct QCD sum rules for the in-medium pion decay constants \( f_\pi^t \) and \( f_\pi^s \). Taking imaginary part of Eqs. (10), (21), (22) and matching them within the duality region with a Borel weight \( e^{-s/M^2} \),

\[ \int_0^{S_0} ds \ e^{-s/M^2} \frac{1}{\pi} \text{Im} [\Pi_{\text{phen}}^t(s) - \Pi_{\text{ope}}^t(s)] = 0 \quad (l = t, s) , \tag{23} \]

we obtain sum rules for the time and space components separately,

\[ \frac{m_q^2}{2m_q} f_\pi^t \ e^{-m_q^2/M^2} = \frac{3m_q}{4\pi^2} \int_{3m_q^2}^{S_0} ds e^{-s/M^2} \sqrt{1 - \frac{4m_q^2}{s}} - 2\langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2} + \frac{8m_q \langle q^i i_D q \rangle_\rho e^{-m_q^2/M^2}}{M^2} - \frac{2m_q^2 \langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2}}{M^2} , \tag{24} \]

\[ \frac{m_q^2}{2m_q} f_\pi^s \ e^{-m_q^2/M^2} = \frac{3m_q}{4\pi^2} \int_{3m_q^2}^{S_0} ds e^{-s/M^2} \sqrt{1 - \frac{4m_q^2}{s}} - 2\langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2} - \frac{8m_q \langle q^i i_D q \rangle_\rho e^{-m_q^2/M^2}}{3M^2} - \frac{2m_q^2 \langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2}}{M^2} + \frac{32}{3M^2} \left[ \langle \bar{q}iD_q iD_0 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_\rho \right] e^{-m_q^2/M^2} . \tag{25} \]

The continuum threshold is denoted by \( S_0 \). The dependence on this parameter is expected to be small as the perturbative part is proportional to the quark mass.

Various nuclear matrix elements appearing in the OPE can be written in terms of corresponding nucleon matrix elements (denoted by the subscript “N” below) in the linear density approximation. In this approximation, the dimension 4 and 5 condensates in the OPE can be written \[11\]

\[ \langle q^i i_D q \rangle_\rho = \rho \langle q^i i_D q \rangle_N + \frac{m_q}{4} \langle \bar{q}q \rangle_\rho , \tag{26} \]

\[ \langle \bar{q}iD_q iD_0 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_\rho = \rho \left[ \langle \bar{q}iD_q iD_0 q \rangle_N + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_N \right] + \frac{m_q^2}{4} \langle \bar{q}q \rangle_\rho . \tag{27} \]
These expressions are slightly different from the ones in Ref. [11] by the quark-mass dependent terms but whose appearance can be easily understood from the equation of motion. Ref. [11] neglected these small quark mass terms. It is interesting to see that, with these quark mass terms, the Wilson coefficient of the condensate \( m_q^2 \langle \bar{q}q \rangle \rho \) vanishes in the sum rules. Anyway, in this separation, the nucleon matrix elements should be understood as the ones in the chiral limit \( m_q \rightarrow 0 \). According to Ref. [11]

\[
\langle q \, iD_0q \rangle_N \sim 0.18 \text{ GeV} ,
\]

\[
\langle \bar{q}iD_0iD_0q \rangle_N + \frac{1}{8} \langle \bar{q}g_s \sigma \cdot Gq \rangle_N \sim 0.08 - 0.3 \text{ GeV}^2 .
\]

The value of the first equation comes from the twist-2 quark distribution function in a nucleon. The value of the second line, the dimension 5 term, is not well-known. This is related to the twist-3 nucleon distribution function whose value however can not be reliably calculated. The bag model estimate gives 0.08 GeV\(^2\) [11] while the other standard estimate yields 0.3 GeV\(^2\) [11]. In our later analysis, we will take these two values as a window of this dimension 5 condensate. But it is important to note that this dimension 5 term can not be negative. The operator \( iD_0iD_0 \) is positive definite as it is a square of the hermitian operator. The operator \( \sigma \cdot G \) in \( \langle \bar{q}g_s \sigma \cdot Gq \rangle_N \) represents the average virtuality [11] of the quarks in a nucleon, which also has to be positive definite. Therefore, \( \langle \bar{q}iD_0iD_0q \rangle_N + \frac{1}{8} \langle \bar{q}g_s \sigma \cdot Gq \rangle_N \) must have the same sign as the positive quantity \( \langle \bar{q}q \rangle_N \).

Using Eqs. (26), (27), we rewrite Eqs. (24), (25) as (transferring the quark mass in the LHS to the RHS)

\[
m^*_\pi^2 f^2 t e^{-m^*_\pi^2 / M^2} = \frac{3m^2_q}{2\pi^2} \int_{4m^2_q}^{S_0} ds e^{-s / M^2} \sqrt{1 - \frac{4m^2_q}{s}} - 4m_q \langle \bar{q}q \rangle \rho e^{-m^2_q / M^2} + \frac{16m^2_q}{M^2} \rho \langle q \, iD_0q \rangle_N e^{-m^2_q / M^2} ,
\]

\[
m^*_\pi^2 f^2 t e^{-m^*_\pi^2 / M^2} = \frac{3m^2_q}{2\pi^2} \int_{4m^2_q}^{S_0} ds e^{-s / M^2} \sqrt{1 - \frac{4m^2_q}{s}} - 4m_q \langle \bar{q}q \rangle \rho e^{-m^2_q / M^2} - \frac{16m^2_q}{3M^2} \rho \langle q \, iD_0q \rangle_N e^{-m^2_q / M^2} + \frac{64m^2_q}{3M^2} \rho \left[ \langle \bar{q}iD_0iD_0q \rangle_N + \frac{1}{8} \langle \bar{q}g_s \sigma \cdot Gq \rangle_N \right] e^{-m^2_q / M^2} .
\]

An indirect check for these expressions can be made by taking the limit \( \rho \rightarrow 0 \). In this limit, we have \( f^2 = f^* = f_\pi , \ m^*_\pi = m_\pi \), and the two sum rules become equivalent to the vacuum pion sum rule that appears in Ref. [17],

\[
f^2 m^2_\pi e^{-m^2_\pi / M^2} = \frac{3m^2_q}{2\pi^2} \int_{4m^2_q}^{S_0} ds e^{-s / M^2} \sqrt{1 - \frac{4m^2_q}{s}} - 4m_q \langle \bar{q}q \rangle e^{-m^2_q / M^2} .
\]

The GOR relation is satisfied at the order \( O(m_q) \).

Our two sum rules with nonzero density mainly differ by the dimension 5 condensate \( \langle \bar{q}iD_0iD_0q \rangle_N + \frac{1}{8} \langle \bar{q}g_s \sigma \cdot Gq \rangle_N \). The other terms that make the two sum rules different are suppressed by the small quark mass and the density. We outline a few qualitative results.
In the sum rule for $f^t$ Eq. (30), dominant piece of the OPE comes from $m_q\langle \bar{q}q \rangle_\rho$. The other OPE are down by higher order in quark mass. Thus, at the order $\mathcal{O}(m_q)$ [or at the order in $\mathcal{O}(m_q^2)$], this sum rule precisely reproduces the in-medium GOR relation [3,5,21],

$$m_{\pi}^* f^t \sim -4m_q\langle \bar{q}q \rangle_\rho .$$  

(33)

Such a simple relation can not be obtained from the space component sum rule Eq.(31) due to the nonnegligible dimension 5 term $\langle \bar{q}D_0 D_0 q \rangle_N + \frac{1}{8}\langle \bar{q}g_s \sigma \cdot G q \rangle_N$.

The ratio of Eqs.(30) (31) gives us an interesting relation for $f^t$ and $f^s$. Namely neglecting the terms of order $\mathcal{O}(m_q^2)$ and higher, we find the approximate formula,

$$\frac{f^s}{f^t} \simeq 1 - \frac{16\rho}{3M^2} \frac{\langle \bar{q}D_0 D_0 q \rangle_N + \frac{1}{8}\langle \bar{q}g_s \sigma \cdot G q \rangle_N}{\langle \bar{q}q \rangle_\rho} .$$  

(34)

When $\rho \to 0$, the ratio becomes the unity as it should be. Note, $\langle \bar{q}q \rangle_\rho$ is negative and the dimension 5 condensate in the numerator is positive. Therefore, we should have

$$f^s \geq f^t .$$  

(35)

One may suspect that the OPE does not converge fast enough up to dimension 5 and large contributions from higher dimensional operators can alter this inequality. But nonnegligible higher dimensional operators (even if they exist) must be related to twist operators that scale with the density. The scalar (nontwist) operators with higher dimensions must be proportional to $\mathcal{O}(m_q^2)$ or higher orders because the scalar part of the sum rule at the order $\mathcal{O}(m_q)$ must satisfy the GOR relation. Higher twist operators are suppressed either in the twist expansion and by the high correlator momentum. For operators having the same twist but with higher spin indices, we do not have a systematic way to analyze them but their contributions are usually smaller than the contribution from lower spin operators [13,14] in in-medium sum rules. Therefore, it is unlikely that such higher dimensional terms reverse the inequality.

On the other hand, the in-medium chiral perturbation theory gives the ratio in terms of the chiral perturbation parameters (Note, $f_\pi = 131$ MeV in our notation.) [3],

$$\frac{f^s}{f^t} = \frac{1 + \frac{4c_2\rho}{f_\pi^2}}{1 + \frac{4(c_2+c_3)\rho}{f_\pi^2}} = 1 - \frac{4c_3\rho}{f_\pi^2} .$$  

(36)

With the parameter values, $c_2 = 0.28$ fm and $c_3 = -0.55$ fm given in Ref. [3], the ratio is 0.28 at the nuclear saturation density $\rho = 0.17$ fm$^{-3}$, substantially smaller than the unity. Even taking into account an error in $f^s$ coming from a large uncertainty in the parameter $c_3$ [3], our finding is still inconsistent with this result.

To confirm these approximate results, we now numerically analyze the two sum rules, Eqs.(30) and (31). In the phenomenological side, we have three parameters to be determined $f^t f^s$ and $m_{\pi}^*$ but we have only two sum rules. The best fitting method with a trial function of the form $ae^{-b/M^2}$ may not reliably determine the exponent $b$. Further input may be needed. For simplicity, we take the in-medium pion mass to be its free value $m_{\pi}^* = m_{\pi}$. This assumption is supported by Ref. [3,22,23] where the mass change in the medium is
calculated to be about 10 MeV. It should be noted that this assumption does not affect one of our results Eq. (35) as the pion mass is canceled in the ratio.

The in-medium quark condensate in the linear density approximation is given by [11,24]

$$\langle \bar{q}q \rangle_\rho = \langle \bar{q}q \rangle_0 + \rho \frac{\sigma_N}{2m_q}.$$  \hspace{1cm} (37)

The vacuum expectation value is denoted by the subscript “0”. In our analysis, we will use $\langle \bar{q}q \rangle_0 = -(225 \text{ MeV})^3$ that corresponds to the quark mass $m_q \sim 7 \text{ MeV}$ [25,26]. The nucleon sigma term is $\sigma_N \sim 45 \text{ MeV}$ [27]. The values for the other nucleon matrix elements are given in Eq.(29). The continuum threshold $S_0$ will be fixed to the $\pi'$ mass but the sensitivity of our result on this choice is very small.

In Figure 1, the solid lines show the time component of the decay constant $f^t$ with respect to the Borel mass. The upper one is the vacuum case and the lower one is from Eq. (30) at the nuclear saturation density $\rho = 0.17 \text{ fm}^{-3}$. The dashed lines are obtained from the GOR relation. $f^t$ from the OPE is only 2 % larger than the one from the GOR relation. At the saturation density, we obtain

$$f^t = 105 \text{ MeV},$$  \hspace{1cm} (38)

20 % lower than its free space value. It is not so different from the value of the in-medium chiral perturbation theory $f^t = 101 \text{ MeV}$ [3].

Figure 2 shows the Borel curve for the ratio $f^s/f^t$. The solid curves are from Eqs.(30) (31) and the dashed curves are from our approximate formula Eq.(34), which almost indistinguishable from the full curves. In getting the upper two curves, we use the dimension 5 condensate $\langle \bar{q}iD_0iD_0q \rangle_N + \frac{1}{8}\langle \bar{q}g_\sigma \sigma \cdot Gq \rangle_N = 0.3 \text{ GeV}^2$ while the lower two curves use 0.08 GeV$^2$. The Borel curves are not so stable with respect to the Borel mass. Nevertheless, within the region 0.8 GeV$^2 \leq M^2 \leq 1.2 \text{ GeV}^2$, the variation of the ratio is 10 % level for the upper solid curve and 3% for the lower solid curve. At $M^2 = 1 \text{ GeV}^2$, the ratio gives $f^s/f^t = 1.28$ from the upper curve and 1.07 from the lower curve. Using our result of $f^t = 105 \text{ MeV}$, the two extremes give the range

$$112 \text{ MeV} \leq f^s \leq 134 \text{ MeV},$$  \hspace{1cm} (39)

Thus, the space component of the pion decay constant either slightly increases or decreases from its vacuum value depending upon the value of the dimension 5 condensate.

VI. SUMMARY

In this work, we have constructed QCD sum rules for the pion decay constant in nuclear matter. The pseudoscalar-axial vector correlation function in the matter has been used for this purpose. The time component of the decay constant $f^t$ is found to satisfy the in-medium GOR relation. The OPE for the space component contains the non-negligible dimension 5 contribution as well as the quark condensate that yields the in-medium GOR relation. We have established a firm constraint $f^s \geq f^t$ and it does not agree with the result from the in-medium chiral perturbation theory. Neglecting the in-medium shift of the pion mass, we have obtained $f^t = 105 \text{ MeV}$ and $112 \text{ MeV} \leq f^s \leq 134 \text{ MeV}$. 

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FIG. 1. The Borel curves for $f_t$, the solid lines from its sum rule and the dashed lines from the GOR relation. The upper two curves are the vacuum case and the lower two curves are obtained from the in-medium sum rule at the nuclear saturation density $\rho = 0.17$ fm$^{-3}$.
FIG. 2. The Borel curves for the ratio $f^s/f^t$ at the nuclear saturation density $\rho = 0.17$ fm$^{-3}$. The upper curves are obtained with the dimension 5 condensate $0.3$ GeV$^2$ and the lower curves are obtained with $0.08$ GeV$^2$. The solid curves are from our sum rules and the almost indistinguishable dashed curves are from the approximate formula Eq. (34).