Multi Hamilton-Jacobi quantization of $O(3)$ nonlinear sigma model

Dumitru Baleanu$^1$ and Yurdahan Güler$^2$

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University-06530, Ankara, Turkey

Abstract

The $O(3)$ non-linear sigma model is investigated using multi Hamilton-Jacobi formalism. The integrability conditions are investigated and the results are in agreement with those obtained by Dirac’s method. By choosing an adequate extension of phase space we describe the transformed system by a set of three Hamilton-Jacobi equations and calculate the corresponding action.

---

$^1$On leave of absence from Institute of Space Sciences, P.O BOX, MG-23, R 76900 Magurele-Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr

$^2$E-Mail: yurdahan@cankaya.edu.tr
1 Introduction

The importance of $O(3)$ nonlinear sigma model lies in theoretical and phenomenological basis [1, 2, 3]. It is a fact that this theory describes classical (anti) ferromagnetic spin systems at their critical points in the Euclidean space, while in the Minkowski one it delineates the long wavelength limit of quantum antiferromagnets. On the other hand the model exhibits solitons, Hopf instantons and novel spin and statistics in 2+1 space-time dimensions with inclusion of the Chern-Simon term [3, 4].

Recently, the quantization of this model was investigated [5] by improved Batalin-Fradkin-Tyutin method [6]. An alternative method of quantization of the systems with constraints was initiated in [7, 8] and it is based on the Carathéodory’s equivalent Lagrangians method [9].

Recently, the singular systems with higher order Lagrangians, the systems which have elements of the Berezin algebra [10, 11, 12], the quantization of Proca’s model [13], the non-relativistic particle on a curved space [14] as well as the supersymmetric quantum mechanics [15] in Witten’s version [16] were investigated using this method.

One of the most interesting application of the formalism is the case when the systems admit second class constraints [17, 18] simply because the corresponding system of equations is not integrable [18]. Is it possible to quantize the system using multi Hamilton-Jacobi formalism in this case? The answer is positive if we transform the system such that it becomes completely integrable and in addition the form of the Hamiltonians are suitable for application of multi Hamilton-Jacobi equations.

On the other hand the multi Hamilton-Jacobi formalism for fields requires a special attention when all the Hamiltonians are densities and, as a consequence, the surface terms [19] play an important role in the process of quantization.

For these reasons the quantization of the $O(3)$ nonlinear sigma model using multi Hamilton-Jacobi formulation is interesting to investigate.

The plan of the paper is the following:

In sec. 2 the multi Hamilton-Jacobi formalism for fields is presented. In sect. 3 the path integral quantization of $O(3)$ model is analyzed. Sec. 4 contains our conclusions.
Multi Hamilton-Jacobi formalism for fields

Let us assume that we start with a singular Lagrangian density having the Hessian matrix of rank \( n-r \). Using the Carathéodory’s equivalent Lagrangian method we find the following Hamiltonian densities

\[
H'_\alpha = H_\alpha(t, q_\alpha, p_\alpha) + p_\alpha, \tag{1}
\]

where \( \alpha, \beta = n-r+1, \ldots, n, a = 1, \ldots, n-r \). The canonical Hamiltonian \( H_0 \) is defined as

\[
H_0 = -L(t, q_\alpha, \dot{q}_\alpha = w_\alpha) + p_\alpha w_\alpha + \dot{q}_\mu p_\mu \mid_{p_\nu = -H_\nu}, \nu = 0, n-r+1, \ldots, n. \tag{2}
\]

and by construction is independent of \( \dot{q}_\mu \). Here \( \dot{q}_\alpha = \frac{dq_\alpha}{d\tau} \), where \( \tau \) is a parameter. The equations of motion are obtained as total differential equations in many variables as follows

\[
dq_\alpha = \frac{\delta H'_\alpha}{\delta p_\alpha} dt_\alpha, dp_\alpha = -\frac{\delta H'_\alpha}{\delta q_\alpha} dt_\alpha, \\
dp_\mu = -\frac{\delta H'_\alpha}{\delta t_\mu} dt_\alpha, \mu = 1, \ldots, r, \tag{3}
\]

\[
z = \int (-H_\alpha + p_\alpha \frac{\delta H'_\alpha}{\delta p_\alpha}) dt_\alpha, \tag{4}
\]

where \( z = S(t_\alpha, q_\alpha) \) and \( \frac{\delta H}{\delta x} \) represents the variation of \( H \) with respect to \( x \). Since the equations of motion are total differential equations the integrability conditions play an important role. More exactly eqs. (3,4) are integrable iff \( dH'_\alpha = 0 \). If the variations of \( H'_\alpha \) are not zero then additional constraints may arise. Thus, we may have Hamiltonian densities other than (1). The essence of the formalism is to express all Hamiltonian densities in the form (1) and (2).
3 Multi Hamilton-Jacobi treatment of O(3) nonlinear sigma model

The model is described by the Lagrangian

\[ L = \int d^2 x \left[ \frac{1}{2f}(\partial_\mu n^a \partial^\mu n^a) - \dot{\lambda}(n^a n^a - 1) \right], \tag{5} \]

where the metric has signature (+, −, −), \( n^a(a = 1, 2, 3) \) is a multiplet of three real scalar field with a constraint

\[ n^a n^a - 1 = 0 \tag{6} \]

and \( \dot{\lambda} = \frac{d\lambda}{dt} \) is a Lagrange multiplier.

In multi Hamilton-Jacobi formalism we have two Hamiltonian densities which corresponds to (5)

\[ H'_0 = \int d^2 x [p_0 + \frac{f}{2} \pi^a n^a + \frac{1}{2f} (\partial_i n^a)(\partial_i n^a)], \quad i = 1, 2 \]
\[ H'_1 = \int (\pi_\lambda + n^a n^a - 1)d^2 x. \tag{7} \]

Here \( \pi_a = \frac{1}{f} \dot{n}^a \) and \( \pi_\lambda \) represents the momentum conjugate to \( \lambda \).

Integrability condition \( dH'_2 = 0 \) gives

\[ n^a \pi^a = 0. \tag{8} \]

The variation of (8) determines \( \lambda \) as the solution of the differential equation

\[ \dot{\lambda} = -\frac{1}{n^a n^a} (\frac{f}{2} \pi^a n^a + \frac{1}{2f} n^a \partial_i^2 n^a). \tag{9} \]

In conclusion we have three Hamiltonian densities for our model

\[ H'_0 = \int d^2 x [p_0 + \frac{f}{2} \pi^a n^a + \frac{1}{2f} (\partial_i n^a)(\partial_i n^a)], \]
\[ H'_1 = \int (\pi_\lambda + n^a n^a - 1)d^2 x, H'_2 = \int n^a \pi^a d^2 x. \tag{10} \]

We would like to stress on the fact that \( \pi_\lambda \) is a constant and the results (10) are in agreement with Dirac’s analysis [17]. The next step is to quantize the
model by transforming the Hamiltonian densities (10) to be in involution and calculate the corresponding characteristic.

The essence of Batalin-Fradkin-Tyutin formalism is to enlarge the phase space with some extra variables such that the modified canonical Hamiltonian and modified second class constraints to be in involution (for more details see Refs. [20]). Recently, a formalism, which is called improved Batalin-Fradkin-Tyutin has been proposed (see Refs. [6] and the references therein). Let us consider a constrained system having only second class constraints $\Phi_\alpha$. This formalism contains two steps, the first step consists in finding a set of constraints in involution and the second step deals with transformation of all fields and corresponding momenta in such a way that they are in involution with the transformed constraints.

In our specific problem, since we have only two second class constraints, we need only two extra fields $\theta, \pi_\theta$ to start with. On the other hand we have to find a set of Hamiltonians in involution and in the form (1). To reach this objective we will exploit the fact that the commutation relations of the extra variables in Batalin-Fradkin-Tyutin formalism are not uniquely defined, so we will try to find a proper choice such that the transformed Hamiltonians to be in form (1). We assume that $\pi_\theta$ is the canonical conjugate momenta of $\theta$. Next step is to make the Hamiltonians $H'_1$ and $H'_2$ in involution.

We are looking to find $H''_i$ as

$$H''_i = \sum_{n=0}^{\infty} \Phi^n_i, \; i = 1, 2$$

(11)

fulfilling the boundary conditions $\Phi^0_i = H'_i$ such that

$$\{H''_i(x), H''_j(y)\} = 0.$$  

(12)

Here $\Phi^n_i$ are polynomials in $\theta$ and $\pi_\theta$.

Due to linearity in the auxiliary fields we can make the following ansatz

$$\Phi^1_A = \int d^2y X_{AB}(x, y) \Gamma^B(y),$$

(13)

where $\Gamma^B$ represents the set $(\theta, \pi_\theta)$[20] and $X_{AB}(x, y)$ is a matrix to be determined.

From (12) and (13) we find the equations corresponding to $X_{AB}$ as

$$\Delta_{AB} + X_{AC} \omega^{CD} X_{BD} = 0,$$

(14)
where $\Delta_{AB} = 2\varepsilon^{AB}\eta^{\alpha}\delta(x - y)$ and $\varepsilon^{12} = -\varepsilon^{21} = 1$. The solution of (13) is

$$X_{AB} = \begin{pmatrix} 2 & 0 \\ 0 & \eta^{\alpha}\eta^{\beta} \end{pmatrix} \delta(x - y).$$  \hspace{1cm} (15)

Using (13) and (15) we find the following strongly involutive Hamiltonians

$$H_{2}'' = \int d^{2}x(\pi_{\lambda} + \eta^{\alpha}\eta^{\beta} - 1 + 2\pi_{\theta}), H_{3}''' = \int d^{2}x(\eta^{\alpha}\pi^{\beta} + \eta^{\alpha}\eta^{\beta}\theta).$$  \hspace{1cm} (16)

The next step is to transform $H_{0}'$ such that it is in involution with $H_{2}''$ and $H_{3}'''$. Denoting $\Sigma = (\eta^{\alpha}, \pi^{\alpha})$ we would like to make a transformation such that \{\eta^{\alpha}, H_{2}'\} = \{\pi^{\alpha}, H_{2}'\} = 0 and as a consequence \{\eta^{\alpha}, H_{3}'\} = \{\pi^{\alpha}, H_{3}'\} = 0 \hspace{1cm} [6]. Expressing $\Sigma$ to be the set $(\hat{\eta}^{\alpha}, \hat{\pi}^{\alpha})$ we can reach this objective making the assumption

$$\hat{\Sigma} = \Sigma + \sum_{n=1}^{\infty} \hat{\Sigma}^{(n)}, \hat{\Sigma}^{(n)} \sim \Gamma^{(n)},$$  \hspace{1cm} (17)

where the $(n + 1)$-th order of iteration has the form \hspace{1cm} [6]

$$\hat{\Sigma}^{(n+1)} = -\frac{1}{n + 1} \int d^{2}x d^{2}y d^{2}z \Gamma^{A}(x) \omega_{AB} X^{BC}(y, z) G_{C}^{(n)}(z) \hspace{1cm} (18)$$

with

$$G_{A}^{(n)}(x) = \sum_{m=0}^{n} \{H_{A}''^{(n-m)}, \hat{\Sigma}^{(m)}\}_{(\Sigma)} + \sum_{m=0}^{n-2} \{\{H_{A}''^{(n-m)}, \hat{\Sigma}^{(m+2)}\}\}_{(\Gamma)} + \{H_{A}''^{(n+1)}, \hat{\Sigma}^{(1)}\}_{(\Gamma)]}. \hspace{1cm} (19)$$

The calculations gives the following expressions for the set $\hat{\Sigma}$

$$\hat{\eta}^{\alpha} = \eta^{\alpha} \left( 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n} \pi^{\beta}_{\theta}(2n - 3)!!}{(n^{\alpha}n^{\beta})^{n}n!} \right),$$

$$\hat{\pi}^{\alpha} = (\pi^{\alpha} + \eta^{\beta}\theta) \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n} \pi^{\beta}_{\theta}(2n - 1)!!}{(n^{\alpha}n^{\beta})^{n}n!} \right),$$  \hspace{1cm} (20)

$$H_{0}'' = \int d^{2}x \left( p_{0} + \frac{1}{2} \hat{\pi}^{\alpha} \hat{\eta}^{\alpha} + \frac{1}{2} \partial_{i} \hat{\eta}^{\alpha} \partial_{i} \hat{\eta}^{\alpha} \right). \hspace{1cm} (21)$$
Taking into account (20) and the form of series expansion of \( f(x) = \frac{1}{1 + 2x} \) around \( x = 0 \) and using the conformal map condition \( n^a \partial_i n^a = 0 \) we reexpress (21) in a simpler form

\[
H_0''' = \int d^2 x [p_0 + \frac{f}{2}(\pi^a + n^a \theta)(\pi^a + n^a \theta) \frac{n^b n^b}{n^b n^b + 2 \pi \theta} + \frac{1}{2 f} \partial_i n^a \partial_i n^a n^b n^b + 2 \pi \theta].
\]

Even if the Hamiltonian densities \( H_0''', H_1''', H_2''' \) are in involution \( H_1''' \) and \( H_2''' \) aren’t in the form given by (1) and then we can not describe, yet, the transformed system with three Hamilton-Jacobi equations. For these reasons we must modify the form of \( H_1''' \) and \( H_2''' \). Dividing \( H_1''' \) and \( H_2''' \) by 2 and respectively by \( n^1 \) we get

\[
H_1''' = \int d^2 x \left( \pi_\theta + \frac{\pi_\lambda + n^a n^a - 1}{2} \right),
\]

\[
H_2''' = \int d^2 x \left( \pi^1 + \frac{n^2 \pi^2 + n^3 \pi^3 + n^a n^a \theta}{n^1} \right)
\]

and by inspection we conclude that \( H_0''', H_1''', H_2''' \) are weakly involutive. From (22) and (23) and taking into account (4) we find the form of the action as

\[
\begin{align*}
z &= \int d^2 x d\tau \left[ \frac{1}{2} ((\pi^2)^2 + (\pi^3)^2 - (n^2 \theta)^2 + \frac{-n^a n^a + 1}{2} \theta - (n^3 \theta)^2 \right]
- (\pi^1 + n^1 \theta)^2 \frac{n^b n^b}{n^b n^b + 2 \pi \theta} - \frac{1}{2 f} \partial_i n^a \partial_i n^a n^b n^b + 2 \pi \theta - \frac{n^a n^a \theta}{n^1 \pi}].
\end{align*}
\]

4 Conclusion

The multi Hamilton- Jacobi formalism has the unique attribute of exhibiting the classical analog of the quantum state.

The treatment of the second class constrained systems is problematic for multi Hamilton-Jacobi procedure. In fact two types of problems arises, the first and the most important one is that the initial system is not integrable and the second one is related to the form of the constraints.

To solve these problems we have to transform the system by enlarging or reducing the initial phase space such that the transformed Hamiltonians
become in involution. On the other hand we have to preserve the form of the Hamiltonians such that to be suitable for application of Carathéodory’s equivalent Lagrangian method.

In this paper we investigate both from classical and quantum point of view, the $O(3)$ nonlinear sigma model using multi Hamilton-Jacobi formulation. From the consistency conditions we find the parameter $\lambda$ which is in agreement with Dirac’s procedure.

To quantize the system we extend the phase-space using improved Batalin-Fradkin-Tyutin procedure choosing a specific form of commutation relation between extra fields. After finding the Hamiltonians densities in involution and in the form (1) we calculate the action.

5 Acknowledgments

DB would like to thank M. Henneaux for the valuable communication. This paper is partially supported by the Scientific and Technical Research Council of Turkey.

References


   Haldane F D M 1983 Phy. Rev. Lett. 50 1153

   Polyakov A M 1999 Mod. Phys. Lett. A 3 417


Güler Y 1992 Nuovo Cimento B 107 1389


[17] Dirac P A M Lectures on Quantum Mechanics 1964 (Yeshiva University, New York, N.Y.)
Hanson A, Regge T and Teitelboim C 1976 Constrained Hamiltonian systems (Academia Nationale dei Lincei, Rome)
Sündermeyer K 1982 Constrained Dynamics, Lecture Notes in Physics vol. 169 (Springer -Verlag,New-York)
