Fermion masses and mixings in gauge theories

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Abstract

The recent evidence for neutrino oscillations stimulate us to discuss again the problem of fermion masses and mixings in gauge theories. In the standard model, several forms for quark mass matrices are equivalent. They become ansatze within most extensions of the standard model, where also relations between quark and lepton sectors may hold. In a seesaw framework, these relations can constrain the scale of heavy neutrino mass, which is often related to the scale of intermediate or unification gauge symmetry. As a consequence, two main scenarios arise. Hierarchies of masses and mixings may be explained by broken horizontal symmetries.
I. INTRODUCTION

Gauge theories have become a standard framework for the description of fundamental interactions [1]. They rely on the invariance of the Lagrangian with respect to a group of local transformations. If the vacuum state has a restricted invariance, the theory is said to be spontaneously broken. This condition allows the gauge bosons and the fermions to take a mass, without spoiling the renormalizability. Within gauge theories, fermion masses and mixings are important physical parameters. Masses are intrinsic properties of particles, while mixings are related to the possibility of their reciprocal transformations. Currently, there are many experimental and theoretical developments in this field. In particular, the SuperKamiokande Collaboration has recently confirmed the oscillation of atmospheric neutrinos as an explanation of the atmospheric neutrino anomaly [2], that is the deficit of neutrinos produced by cosmic rays in the atmosphere. There are also strong indications of oscillation for solar neutrinos in order to explain the long standing solar neutrino problem [3,4], the deficit of neutrinos produced by nuclear reactions in the sun.

Neutrino oscillations are easily accounted for if neutrinos have nonzero masses. However, in the minimal standard model (MSM), that is the gauge theory for strong, weak and electromagnetic interactions, based on the group $SU(3) \times SU(2) \times U(1)$ and with only one complex Higgs doublet and no right-handed neutrinos, such particles are massless. Several extensions of the MSM allow for nonzero neutrino mass. Some of them do not require the extension of the gauge group, but only of the particle contents, for example adding the right-handed neutrino or the Higgs triplet, although gauge extensions such as the $SO(10)$ model, which does include the right-handed neutrino in its fundamental representation, provide a natural framework for a small neutrino mass. On the other hand, the study of neutrino masses and mixings should be related to the more general problem of elementary fermion masses and mixings in gauge and string theories, and in fact the SuperKamiokande results stimulate us to consider again what we know about this interesting and difficult subject.

In this short review we deal with fermion mass terms and matrices in gauge theories, taking into account the gauge allowed mass terms and the relation between mass matrices and physical parameters (masses and mixings), that is the quantities which are observable in principle. Possible relations between quark and lepton masses are reported. Also neutrino masses and lepton mixings are discussed.

In renormalizable gauge theories, fermion masses arise from bare mass terms like $\bar{\psi} M_0 \psi$, and from Yukawa terms like $\bar{\psi} Y \phi \psi$, which after spontaneous symmetry breaking (SSB) give mass terms $\bar{\psi} M_1 \psi$, with $M_1 = Y v$ and $v$ the vacuum expectation value (VEV) of the Higgs field $\phi$. The structure of mass terms is further determined by the fact that the components of the Higgs representation which can have VEV different from zero must be neutral and colorless, since charge and color are always conserved. The gauge theories that we consider here are the MSM plus three right-handed neutrinos (which we would like to call the standard model (SM)), the left-right model (LRM) $SU(3) \times SU(2) \times SU(2) \times U(1)$, the Pati-Salam partial unification model $SU(4) \times SU(2) \times SU(2)$, the $SU(5)$ and $SO(10)$ grand unification models. Our concern is mainly about the nonsupersymmetric theory. However, some important comments on the supersymmetric version [5] are included in the text. We try to isolate the main features of each model, without entering the detailed realizations existing in the literature.
A clear feature of the quark and charged lepton mass spectrum is the hierarchy of masses belonging to different generations:

$$m_u \ll m_c \ll m_t, \quad m_d \ll m_s \ll m_b, \quad m_e \ll m_\mu \ll m_\tau.$$  \hfill (1)

We do not discuss in detail the origin of such hierarchies, and only mention two mechanisms that have been proposed, namely the radiative mechanism [6], based on loop effects, and the Froggatt-Nielsen mechanism [7]. The latter is based on broken horizontal (flavor or family) symmetries,\(^1\) generating effective mass operators through the coupling of ordinary fermions to hypothetical heavy states (universal seesaw mechanism). Nonsupersymmetric and supersymmetric scenarios which include the effects of this mechanism are outlined in section VII.

**II. STANDARD MODEL**

The standard description of strong, weak and electromagnetic interactions is a gauge theory based on the group \(SU(3)_c \times SU(2)_L \times U(1)_Y\), with coupling constants \(g_3, g_2, g_1\), respectively. The group \(SU(3)_c\) describes the strong interaction of quarks [8], while the product \(SU(2)_L \times U(1)_Y\) describes the electroweak model [9], where a relationship between weak and electromagnetic interactions stands out. The SSB by the Higgs mechanism [10] cuts the symmetry \(SU(2)_L \times U(1)_Y\) down to the abelian symmetry \(U(1)_Q\) of quantum electrodynamics. In such a way the weak gauge bosons take a mass, while gluons and the photon remain massless. For energy scales lower than the electroweak symmetry breaking scale there are two exact symmetry, that is color \(SU(3)\) for the strong interaction and electric charge \(U(1)\) for the electromagnetic interaction. The SSB of the standard group requires the existence of a neutral Higgs boson, not yet discovered. The recent indication to the discovery, by LEP experiments, needs to be confirmed. Let us summarize the particle contents of the MSM.

The classification of left-handed fermions belonging to the first generation within representations of the group \(SU(3)_c \times SU(2)_L \times U(1)_Y\) is the following:

\[
\begin{pmatrix}
u_e \\
e^c
\end{pmatrix} \sim (1, 2, -1/2), \quad e^c \sim (1, 1, 1) \hfill (3)
\]

for leptons and the positron (the antineutrino is absent). The twelve gauge bosons are the gluons \((8, 1, 0)\), the \(W\) bosons \((1, 3, 0)\), and the \(B\) boson \((1, 1, 0)\). In this writing the first number indicates the representation of \(SU(3)_c\), the second one the representation of

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\(^1\)Horizontal symmetries link together particles of different generations. On the contrary, usual gauge symmetries are vertical, because they relate particles of the same generation or family.
and the third number is the hypercharge $Y$, with $Q = T_3 + Y$, where $T_3$ is the quantum number related to the third generator of the group $SU(2)_L$ and $Q$ is the electric charge. Right-handed charge-conjugate states belong to the conjugate representations of $(2),(3)$. Quarks are color triplets. The other two generations have an analogous classification, according to $u \sim c \sim t, d \sim s \sim b, e \sim \mu \sim \tau$. There is also the scalar Higgs doublet $\varphi$,

$$\varphi = \begin{pmatrix} \varphi^0 \\ \varphi^- \end{pmatrix} \sim (1, 2, -1/2),$$

which breaks $SU(2)_L \times U(1)_Y$ down to $U(1)_Q$, due to its nonvanishing VEV. Instead, as said above, $SU(3)_c$ is not broken. This framework, in particular the presence of $SU(2)_L$ in the gauge group, includes the $V−A$ theory of weak interactions [11]. Moreover, the $SU(3)_c$ component allows for asymptotic freedom [12] and confinement [13] of quarks and gluons inside hadrons.

The electroweak interactions of quarks and leptons are mediated by the weak gauge bosons $W^+, W^-$ and $Z^0$, and by the photon $A^0$, which are linear combinations of the bosons $W$ and $B$, that is $Z^0 = \cos \theta_W W^0 - \sin \theta_W B^0, A^0 = \sin \theta_W W^0 + \cos \theta_W B^0$, where $\theta_W$ is the Weinberg angle, $\sin^2 \theta_W \approx 0.23$. Such interactions are summarized in the Lagrangian

$$L_{int} = L_{CC} + L_{NC},$$

where

$$L_{CC} = \frac{g_2}{\sqrt{2}} (J^+_{\mu} W^{-\mu} + J^-_{\mu} W^{+\mu})$$

describes charged current weak interaction and

$$L_{NC} = \frac{g_2}{2 \cos \theta_W} J^0_{\mu} Z^\mu + e J^{em}_{\mu} A^\mu$$

describes neutral weak and electromagnetic interactions. The parameter $g_2$ is the coupling constant related to the group $SU(2)_L$, the parameter $e$ is the coupling constant of electrodynamics, and the electroweak relations $g_1 \cos \theta_W = g_2 \sin \theta_W = e$ hold. Currents are defined as $J^- = (J^+)^\dagger$,

$$J^+_{\mu} = \bar{u} \gamma^\mu d' + \bar{c} \gamma^\mu s' + \bar{t} \gamma^\mu b' + \bar{\nu}_e \gamma^\mu e + \bar{\nu}_\mu \gamma^\mu \mu + \bar{\nu}_\tau \gamma^\mu \tau,$$

$$J^{em}_{\mu} = \sum_f Q_f \bar{f} \gamma^\mu f,$$

$$J^0_{\mu} = \sum_f \bar{f} \gamma^\mu (v_f - a_f \gamma_5) f,$$

with $v_f = T_{3f} - 2 Q_f \sin^2 \theta_W, a_f = T_{3f}$, where $Q_f$ and $T_{3f}$ refer to the left-handed fermion field $f_L$. In Eqns.(8),(9) both the left-handed and the right-handed components are allowed, while in Eqn.(7) only left-handed components appear, according to the $V−A$ theory.

$^2$We denote with a bar the conjugate representation. The product of a representation with its conjugate gives the singlet. The adjoint representation, which corresponds to gauge bosons, is autoconjugate or real.
Primed fields in Eqn.(7) mean that the weak eigenstates \((d', s', b')\) are not equal to the corresponding mass eigenstates \((d, s, b)\) but are linear combinations of them

\[
\begin{pmatrix}
d' \\
s' \\
b'
\end{pmatrix} =
\begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\begin{pmatrix}
d \\
s \\
b
\end{pmatrix}.
\]

The unitary matrix \(V\) connecting mass to weak (flavor) eigenstates is the Cabibbo-Kobayashi-Maskawa (CKM) matrix [14,15]. The standard parametrization of this matrix is given by

\[
V =
\begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\]

where \(c_{ij} = \cos \theta_{ij}\), \(s_{ij} = \sin \theta_{ij}\) and \(\delta\) is the CP violating phase. A 3 \times 3 unitary matrix contains three angles and six phases. Five phases in the CKM matrix can be absorbed due to the freedom of five relative phases for the left-handed quarks. The explicit form (11) is obtained by the product of three rotations in sectors 2-3, 1-3, 1-2. Thus, three angles and one phase appear as independent parameters. It is well known that \(s_{23}\) and \(s_{13}\) are small, \(O(10^{-2})\) and \(O(10^{-3})\) respectively, hence \(c_{23} \simeq c_{13} \simeq 1\) and we get \(s_{12} \simeq |V_{us}|\), \(s_{23} \simeq |V_{cb}|\), \(s_{13} = |V_{ub}|\). Setting \(\lambda = s_{12} \simeq 0.22\), \(s_{23} = A\lambda^2\) and \(s_{13}e^{-i\delta} = A\lambda^3(\rho - i\eta)\), with \(A, \rho, \eta\) of order 1, we obtain the Wolfenstein parametrization [16]

\[
V \simeq
\begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix}
\]

where the four independent parameters are \(\lambda, A, \rho, \eta\). In the case of only two generations of quarks, the mixing matrix becomes a rotation, which depends on the unique parameter \(\theta_C\), the Cabibbo angle [14], \(\sin \theta_C = \lambda \simeq 0.22\), without the CP violating phase. It was a great achievement by Kobayashi and Maskawa to show, before the third generation was discovered, that if there were three generations of quarks then the MSM allowed CP violation.

From Eqn.(12) we see that \(V\) is near the identity, that is quark mixings are small. Moreover, we have the mixing hierarchy \(V_{ub} \ll V_{cb} \ll V_{us}\). These features probably have the same origin as the mass hierarchies (1), as we will see. Values of quark masses and mixings at several scales in the SM and in the supersymmetric SM are collected in Ref. [17]. Mass hierarchy of charged leptons is similar to that of down quarks, while for up quarks it is enhanced:

\[
\frac{m_d}{m_s} \sim \frac{m_s}{m_b} \sim \lambda^2, \quad \frac{m_u}{m_c} \sim \frac{m_c}{m_t} \sim \lambda^4.
\]

Any theory of fermion masses and mixings should explain these mass ratios and the structure of Eqn.(12). As a matter of fact, we have only hints on how such a pattern may arise.
A. Fermion mass terms

As we said in the introduction, fermion masses in gauge theories can be generated by bare mass terms or Yukawa terms. These things must be gauge invariant, they must transform as the gauge group singlet. Moreover, a mass term is the Lorentz invariant part in the product of two left-handed spinor fields of a particle type, but can also be written using the bispinor notation (see the Appendix). We will use spinors in the product of representations and bispinors in mass terms. In the MSM it is not possible to build gauge invariant bare bispinor notation (see the Appendix). We will use spinors in the product of representations as the gauge group singlet. Moreover, a mass term is the Lorentz invariant part in the mass terms or Yukawa terms. These things must be gauge invariant, they must transform as the gauge group singlet. After the SSB they give the mass terms for charged leptons and quarks, respectively

\[ \mathcal{L}_m = \bar{e}_L^c M_e e_R^c + \bar{d}_L^c M_d d_R^c + \bar{u}_L^c M_u u_R^c, \]  

where \( M_\ell = Y_\ell v \) are matrices in generation space, \( Y_\ell \) are Yukawa coupling matrices. Sum with the hermitian conjugate is understood. The physical parameter \( v = 174 \text{ GeV} \) is the VEV of the Higgs field \( \varphi^0 \) (the weak scale). At the tree level the masses of the weak gauge bosons are \( m_W = g_2 v/\sqrt{2} \) and \( m_Z = m_W/\cos \theta_W \). When we diagonalize the mass matrices \( M_\ell \) and \( M_d \), by means of biunitary transformations, \( V_\ell^\dagger M_\ell V_\ell, V_d^\dagger M_d V_d \), the CKM matrix \( V_{CKM} = V_{uL}^\dagger V_{dL} \) appears in the charged current interaction. Unitary transformations with \( V_\ell, V_d \) diagonalize \( M_\ell M_\ell^\dagger \) and \( M_d M_d^\dagger \), respectively.

Some remarks are soon in order. First, the value of \( v \) is nearly equal to the top quark mass \( m_t \). Moreover, in the supersymmetric case two distinct Higgs doublets \( H_1 \) and \( H_2 \), with VEVs \( v_1 \) and \( v_2 \), are necessary. One has \( v_1^2 + v_2^2 = v^2 \) and \( v_2/v_1 = \tan \beta \), so that \( v_2 = v \sin \beta, v_1 = v \cos \beta \). The doublet \( H_2 \) generates \( M_\ell \) while the doublet \( H_1 \) generates both \( M_d \) and \( M_e \), a possible hint towards the hierarchy \( m_t \gg m_b \sim m_\tau \). Considering only the third generation, in the MSM we have \( m_t = y_t v, m_b = y_b v, \) so that \( y_t \approx 1, y_b \approx 10^{-2} \). In the supersymmetric case \( m_t = y_t v \sin \beta, m_b = y_b v \cos \beta \), and \( m_t \gg m_b \) may follow from high \( \tan \beta \), keeping \( y_t \approx y_b \approx 1 \) valid. The last relation will be justified in the \( SO(10) \) model.

In the MSM there is no mass term for neutrinos. However, it is natural to complete the fermion spectrum by adding the analogue of \( w^c \) in the lepton sector (see the classification (2),(3)), that is the left-handed antineutrino \( \nu^c \sim (1,1,0) \), along with the right-handed neutrino. Then, a bare Majorana mass term for the right-handed neutrino,

\[ (1, 1, 0) f \times (1, 1, 0) f = (1, 1, 0), \]  

and a Yukawa term,

\[ (1, 2, -1/2) f \times (1, 1, 0) f \times (1, 2, 1/2) \mathcal{P} = (1, 1, 0) + ..., \]
providing a Dirac mass term, are allowed. While the Dirac mass $m_D$ is expected to be of the same order of magnitude as quark or charged lepton masses, because it is generated by the term (19) which is very similar to terms (16) and (14),(15), the Majorana mass $M_R$ is not constrained, because it is not related to the SSB, and can be very large. In such a case the full neutrino mass matrix

$$M = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix}$$  \hspace{1cm} (20)$$
gives a small eigenvalue $m_L \simeq m_D^2/M_R$ for the mass of a left-handed Majorana neutrino and a large eigenvalue $m_R \simeq M_R$ for the mass of a right-handed Majorana neutrino. This is the so-called seesaw mechanism [18], which can explain the smallness of the neutrino mass with respect to the masses of charged fermions.\(^3\) A small mass for the left-handed neutrino is produced through the mixing with a very heavy right-handed neutrino. In the MSM the light neutrino mass term would appear as a nonrenormalizable (dimension-5) effective operator which breaks lepton number conservation by two units [22]. For a numerical estimate on the value of $M_R$, let us assume $m_D \sim m_t \sim 10^2$ GeV and $m_L \simeq 5 \cdot 10^{-2}$ eV, the mass scale obtained by SuperKamiokande [2]. Then we get $M_R \simeq 2 \cdot 10^{14}$ GeV. If one assumes $m_D \simeq m_\tau \simeq 1.8$ GeV, then $M_R \simeq 6 \cdot 10^{10}$ GeV. In both cases a new (and very high) scale appears in the theory. For three generations one has a $6 \times 6$ matrix

$$M = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix},$$  \hspace{1cm} (21)$$
where $M_D$ and $M_R$ are $3 \times 3$ matrices, and the eigenvalues of $M_R$ are much bigger than the elements of $M_D$. The effective Majorana mass matrix for light left-handed neutrinos is given by the seesaw formula

$$M_L \simeq -M_D^T M_R^{-1} M_D,$$  \hspace{1cm} (22)$$
while for the heavy right-handed Majorana neutrinos it is $M_H \simeq M_R$. The Majorana matrices $M_L$ and $M_R$ are symmetric, but the Dirac matrix $M_D$ may be also asymmetric. After diagonalization of $M_e$ and $M_L$, a lepton mixing matrix is induced in the charged weak interaction, allowing for neutrino oscillations [23]. In subsection IIC we discuss the scale and structure of $M_R$ in the case of large lepton mixing and small Dirac mixing (that is $M_D$ and $M_e$ almost diagonal).

\(^3\)If only term (19) is introduced, we have the problem of why neutrino mass is so small with respect to the electron mass. The seesaw mechanism is the most natural and elegant way to account for small neutrino mass, although other mechanisms have been proposed. For example, a small neutrino mass can be generated radiatively at one loop order [19] or at two loop order [20], introducing several new particles in the theory. The possibility of a Yukawa coupling between the lepton doublet and a Higgs triplet [21], with small VEV, is now excluded by the LEP data on the $Z^0$ width.
B. Quark mass matrices

We are now interested in the Lagrangian formed by the quark mass terms and charged weak interaction, simplified as

\[ \mathcal{L}_M = \bar{u}_L M_u u_R + \bar{d}_L M_d d_R + g \bar{u}_L d_L W. \]  

(23)

Diagonalizing \( M_u \) and \( M_d \) we get

\[ \mathcal{L}_M = \bar{u}_L D_u u_R + \bar{d}_L D_d d_R + g \bar{u}_L V_{CKM} d_L W. \]  

(24)

In this form of the Lagrangian we see a general property of masses and mixings. Masses link a particle to itself, while mixings link a particle to another. Apart from the weak coupling constant \( g = g_2/\sqrt{2} \), Eqn.(24) contains ten observables quantities, namely six quark masses, three angles and one phase, while Eqn.(23) can have 36 real parameters in \( M_u \) and \( M_d \). This count means that within the SM there is much freedom in the choice of quark mass matrices. In fact, without changing the ten observables in Eqn.(24), it is possible to perform the following unitary transformations on quark fields in Eqn.(23):

\[ u_L \rightarrow U u_L, \quad d_L \rightarrow U d_L, \]  

(25)

\[ u_R \rightarrow V_u u_R, \quad d_R \rightarrow V_d d_R. \]  

(26)

Left-handed states transform together, while right-handed states \( u_R \) and \( d_R \) are independent from each other. Of course, this fact is related to the classification (2). By using transformations (25),(26) one can reduce the number of independent parameters in quark matrices till ten (minimal parameter basis). For example, one can get two hermitian mass matrices [24] (18 parameters) and in particular one matrix diagonal and the other hermitian [25] (12 parameters). In fact, \( M_1 \) can be made hermitian or diagonal by \( U^\dagger M_1 V_1 \), and \( M_2 \) hermitian by \( U^\dagger M_2 V_2 \), since a polar decomposition theorem allows one to write a matrix as product \( HX \), where \( H \) is hermitian and \( X \) unitary. For \( M_u = D_u \) it is \( M_d = V D_d V^\dagger \), and for \( M_d = D_d \) it is \( M_u = V^\dagger D_u V \). Two phases in the hermitian mass matrix are fixed by the CKM representation and we are left with ten observable parameters. The numerical expression for \( M_d \) when \( M_u \) is diagonal looks like

\[ M_d = \begin{pmatrix}
0.009 & 0.019 & 0.010 \\
0.019 & 0.093 & 0.113 \\
0.010 & 0.113 & 2.995
\end{pmatrix}, \]  

(27)

where mass values are in GeV at the \( M_Z \) scale. We note that \( M_{d33} \approx m_b \), that is \( Y_{d33} \approx 1 \) in the supersymmetric case with large \( \tan \beta \).

In addition to the basis with \( M_d \) hermitian, there are also several bases with three zeros in \( M_d \) [26,27], so that the relation \( M_d M_d^\dagger = V D_d^2 V^\dagger \) allows one to get \( M_d \). We report here just two forms for it [28,27], with symmetric zeros,

\[ M_d = \begin{pmatrix}
0 & 0.024 & 0 \\
0.021 & 0.105 & 0.106 \\
0 & 1.333 & 2.685
\end{pmatrix}, \]  

(28)

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\[ M_d = \begin{pmatrix} 0 & 0 & 0.023 \\ 0 & 0.106 & 0.104 \\ 0.541 & 2.687 & 1.213 \end{pmatrix}. \] (29)

There are three parameters (masses) in the diagonal \( M_u \) and six real mass parameters plus one phase, not written, in \( M_d \). In both asymmetric forms there are simple relations between elements. In Eqn.(28) we have \( M_{d21} \approx M_{d21}, M_{d22} \approx M_{d23} \) and \( M_{d33} \approx 2M_{d32} \). In Eqn.(29) \( M_{d22} \approx M_{d23} \) and \( M_{d32} \approx 2M_{d33} \). Moreover, Eqn.(28) can be written in the approximate form

\[
M_d = \begin{pmatrix} 0 & \sqrt{m_d m_s} & 0 \\ \sqrt{m_d m_s} & m_s & m_s \\ 0 & m_b \sqrt{5} & 2m_b \sqrt{5} \end{pmatrix}, \] (30)

which is symmetric in the 1-2 sector and leads to the relations

\[
V_{us} \approx \sqrt{\frac{m_d}{m_s}}, \quad V_{cb} \approx \frac{3}{\sqrt{5}} \frac{m_s}{m_b}, \quad V_{ub} \approx \frac{1}{\sqrt{5}} \sqrt{\frac{m_d m_s}{m_b}}. \] (31)

between quark masses and weak mixings. The relation \( M_{d22} \approx M_{d23} \) is sensitive to the value of the phase \( \delta \) in the CKM matrix, and holds for \( \delta = 60^\circ - 90^\circ \).

Another important (but not minimal) basis for quark mass matrices is the nearest neighbour interaction (NNI) basis [29],

\[
M_u = \begin{pmatrix} 0 & A' & 0 \\ A & 0 & B' \\ 0 & B & C \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & D' & 0 \\ D & 0 & E' \\ 0 & E & F \end{pmatrix}. \] (32)

The famous Fritzsch ansatz [30] is obtained if \( M_u \) and \( M_d \) are also hermitian, in which case we have the approximate form

\[
M_u = \begin{pmatrix} 0 & \sqrt{m_u m_c} & 0 \\ \sqrt{m_u m_c} & 0 & \sqrt{m_c m_t} \\ 0 & \sqrt{m_c m_t} & m_t \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & \sqrt{m_d m_s} & 0 \\ \sqrt{m_d m_s} & 0 & \sqrt{m_s m_b} \\ 0 & \sqrt{m_s m_b} & m_b \end{pmatrix}. \] (33)

This ansatz is now ruled out mainly by the huge value of the top quark mass combined with the small value of \( V_{cb} \). In fact the Fritzsch ansatz implies two mass-mixing relations,

\[
V_{us} \approx \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} \] (34)

\[
V_{cb} \approx \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} \] (35)

While the first relation is consistent with experiments, the second one is ruled out for \( V_{cb} \approx 0.04 \) and \( m_t \approx 180 \text{ GeV} \).

Within the basis with \( M_u \) and \( M_d \) hermitian, it is always possible to choose the mass matrices to have vanishing 1-3 and 3-1 elements [31]. Then the further vanishing of the
1-1 element (but not also the 2-2) in both matrices is still consistent with phenomenology [32,33], and $M_u$, $M_d$ can be written in the approximate form

$$M_u = \begin{pmatrix} 0 & \sqrt{m_u m_c} & 0 \\ \sqrt{m_u m_c} & m_c & \sqrt{m_u m_t} \\ 0 & \sqrt{m_u m_t} & m_t \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & \sqrt{m_d m_s} & 0 \\ \sqrt{m_d m_s} & m_s & \sqrt{m_d m_b} \\ 0 & \sqrt{m_d m_b} & m_b \end{pmatrix},$$

(36)
yielding the relations (34) and

$$V_{cb} \simeq \sqrt{\frac{m_d}{m_b}} - \sqrt{\frac{m_u}{m_t}}.$$

(37)

Note that filling up position 2-2 in $M_d$ and $M_u$ we provide a direct contribution to the mass of quarks $s$ and $c$, thus allowing element 2-3, and hence $V_{cb}$, to be lowered (compare Eqn.(33) with Eqn.(36)).

Hermitian minimal parameter bases (with no zeros in the diagonal) have been classified in Ref. [34]. There are 18 such bases. They have one zero in $M_u$ ($M_d$) and two in $M_d$ ($M_u$) (since these matrices are hermitian we count two symmetric zeros as one). Phenomenologically viable hermitian matrices with five and four zeros have been recently studied in Refs. [35–37]. For five zeros, only one form is actually viable [32,35]. It can be approximated as

$$M_u = \begin{pmatrix} 0 & 0 & \sqrt{m_u m_t} \\ 0 & m_c & 0 \\ \sqrt{m_u m_t} & 0 & m_t \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & \sqrt{m_d m_s} & 0 \\ \sqrt{m_d m_s} & m_s & \sqrt{m_d m_b} \\ 0 & \sqrt{m_d m_b} & m_b \end{pmatrix},$$

(38)
yielding the relations

$$V_{us} \simeq \sqrt{\frac{m_d}{m_s}}, \quad V_{cb} \simeq \sqrt{\frac{m_d}{m_b}}, \quad V_{ub} \simeq \sqrt{\frac{m_u}{m_t}}.$$

(39)

It is important to notice that within the SM only the 1-1 zeros in (36) and (38) are physically meaningful, since the other zeros are basis zeros (see Ref. [34]). All minimal parameter bases are equivalent in the SM.

The three matrix expressions (30) (with $M_u$ diagonal), (36) and (38), in their different (approximate) form, have almost the same physical contents on quark masses and mixings in the SM, within the available experimental informations.\(^4\) In particular, matrices (38) and (30) are derived from two minimal parameter bases. Of course, matrices (38) give slightly different predictions for mixings, with respect to matrices (36), but if one fills up position 1-1 with very small entries, they become nearly equivalent. The approximate vanishing of elements 1-1 is in correspondence with the approximate relations that we found for asymmetric matrices.

\(^4\)Mixings in (33), (36) and (38) are given by the formula $V_{ij} \simeq M_{uij}/M_{uji} - M_{dij}/M_{dji}$, $j > i$. In fact, for a symmetric $2 \times 2$ mass matrix $M$ the mixing angle $\theta$ is given by $\tan 2\theta = 2M_{12}/(M_{22} - M_{11})$. For a hierarchical matrix $\sin \theta \simeq M_{12}/M_{22}$. Moreover, $V_{ij} \simeq V_{uij} - V_{dij}$. If $M$ is not symmetric (for example in Eqn.(30)), the same calculation applies to $MM^\dagger$. The expression of mass-mixing relations depends on mass matrix model.
In the hermitian cases the largest element is by far the 3-3. Moreover, $M_{d33} \simeq m_b$ and $M_{u33} \simeq m_t$, that is $Y_{d33} \simeq Y_{u33} \simeq 1$ in the supersymmetric model with large $\tan \beta$ may hold. This suggests that mixings and lighter generation masses are generated on a different footing with respect to the third generation mass, which is in some sense more fundamental. This fact should be related to the Froggatt-Nielsen mechanism. We see that the structure of matrices in Eqns.(36) and (38) allows for both the hierarchies (1) and the CKM form (12). However, we stress again that many other symmetric and asymmetric forms are possible. These forms become no more equivalent only when the overall gauge symmetry is enlarged, as we will discuss in the following sections.

C. Neutrino masses and mixings

Recent experiments show that neutrinos mix just like quarks do. However, complications arise due to the possible additionnal presence of the Majorana mass term. Let us consider the part of the SM Lagrangian related to lepton masses and mixings,

$$\mathcal{L} = \bar{e}_L M_e e_R + \bar{\nu}_L M_D \nu_R + g \bar{\nu}_L e_L W + \left(\nu^c\right)_L \frac{M_R}{2} \nu_R,$$

where $M_e$ is the mass matrix of charged leptons. The effective Lagrangian in the seesaw mechanism is instead

$$\mathcal{L}_{ss} = \bar{e}_L M_e e_R + \bar{\nu}_L \frac{M_L}{2} \left(\nu^c\right)_R + g \bar{\nu}_L e_L W + \left(\nu^c\right)_L \frac{M_R}{2} \nu_R,$$

which shows the decoupling of heavy $\nu_R$ from light $\nu_L$. Diagonalization of $M_e$ and $M_L$, with positive eigenvalues [38], by means of (bi)unitary transformations $V_{eL}^\dagger M_e V_{eR}$ and $V_{L}^\dagger M_L V_{L}^*$, respectively, and of $M_R$, gives

$$\mathcal{L}_{ss} = \bar{e}_L D_e e_R + \bar{\nu}_L \frac{D_L}{2} \left(\nu^c\right)_R + g V_{MNS} \bar{\nu}_L e_L W + \left(\nu^c\right)_L \frac{D_R}{2} \nu_R,$$

where $V_{MNS} = V_{eL}^\dagger V_{\nu L}$ is the (Maki-Nakagawa-Sakata) lepton mixing matrix [39]. Contrary to the CKM matrix, neutrino oscillation experiments allow for large and even maximal mixings in the MNS matrix. It is easy to see that in Eqn.(41) we can go to a basis where $M_e$ is diagonal, without changing masses and mixings appearing in Eqn.(42). The relation between the weak (flavor) eigenstates $\nu_\alpha$ ($\alpha = e, \mu, \tau$) of Eqn.(41) and the mass eigenstates $\nu_i$ ($i = 1, 2, 3$) of Eqn.(42) is given by the MNS matrix, $\nu_\alpha = U^\dagger_{\alpha i} \nu_i = U_{\alpha i} \nu_i$, and, if $M_e$ is diagonal, $M_L$ can be obtained from effective neutrino masses and mixings, $M_L = U^\dagger D_L U^*$, with $D_L = \text{diag}(m_1, m_2, m_3)$. The MNS matrix can be parametrized as the standard form of the CKM matrix (Eqn.(11)) times a diagonal phase matrix $D = \text{diag}(e^{i\phi_1/2}, e^{i\phi_2/2}, 1)$. Thus, two additional (Majorana) phases appear in the MNS matrix, due to the fact that only the phases of charged leptons are free. Majorana phases do not affect neutrino oscillations.

In order to determine the scale and structure of the mass matrix $M_R$, we follow the (bottom-up) approach of Ref. [40]. If there is no mixing, then $M^{-1}_L = \text{diag}(1/m_1, 1/m_2, 1/m_3)$. Instead, assuming a hierarchical spectrum, $m_1 \ll m_2 \ll m_3$, for single maximal mixing ($U_{e2} = 0$, $U_{\mu 3} = 1/\sqrt{2}$, $U_{e3} \simeq 0$) one has
\[
M_L^{-1} \simeq \frac{1}{2} \begin{pmatrix}
\frac{2}{m_1} & 0 & 0 \\
0 & \frac{1}{m_2} & -\frac{1}{m_2} \\
0 & -\frac{1}{m_2} & \frac{1}{m_2}
\end{pmatrix},
\]
and for double maximal mixing (\(U_{e2} = 1/\sqrt{2}, U_{\mu 3} = 1/\sqrt{2}, U_{e3} \simeq 0\))
\[
M_L^{-1} \simeq \frac{1}{2} \begin{pmatrix}
\frac{1}{m_1} & \frac{1}{\sqrt{2m_1}} & \frac{1}{\sqrt{2m_1}} \\
-\frac{1}{\sqrt{2m_1}} & \frac{1}{2m_1} & -\frac{1}{2m_1} \\
\frac{1}{\sqrt{2m_1}} & -\frac{1}{2m_1} & \frac{1}{2m_1}
\end{pmatrix}.
\]
Note that in matrix (44) all entries are of the same order. This happens also for the 2-3 sector of matrix (43). In both cases the leading form for \(M_L\) was
\[
M_L \sim \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix},
\]
where a dominant block appears [41]. Actually, in order to get the form of \(M_R\) we have to assume something about Dirac mass matrices. In many unified models (see for example Refs. [37,42]), the Dirac mixing is small, because lepton mass matrices are related to quark mass matrices,\(^5\) so that \(M_e \simeq D_e\) and
\[
M_D \simeq \frac{m_\tau}{m_b} \text{diag}(m_u, m_c, m_t).
\]
The leading form is \(M_D \sim \text{diag}(0, 0, 1)\). The factor \(m_\tau/m_b\) takes into account the running of quark masses with respect to lepton masses. Also quark mixings run with the energy scale. However, it has been shown [43,44] that \(V_{us}\) and \(V_{ub}/V_{cb}\) are almost unchanged, while \(V_{cb}\) and \(V_{ab}\) may change at most by 30 per cent, so that Dirac mixings remain small. For the present general analysis we need not to perform a detailed calculation of running effects. Then, from the inverted seesaw formula \(M_R \simeq -M_D M_L^{-1} M_D^T\) we get the scales (the largest element in \(M_R\))
\[
M_{R33} \simeq \left(\frac{m_\tau}{m_b}\right)^2 \frac{m_1^2}{2m_2},
\]
and
\[
M_{R33} \simeq \left(\frac{m_\tau}{m_b}\right)^2 \frac{m_1^2}{4m_1},
\]
for single and double maximal mixing, respectively. From neutrino oscillation experiments we know that the mixing of atmospheric (\(\nu_\mu\)) neutrinos, related to \(U_{\mu 3}\), is nearly maximal,

\(^5\)Note that, even in the supersymmetric SM, \(M_D\) and \(M_u\) (\(M_e\) and \(M_d\)) are generated by the same Higgs doublet, \(H_2\) (\(H_1\)), and a similarity, \(M_D \sim M_u, M_e \sim M_d\), can be assumed.
while the mixing of solar ($\nu_e$) neutrinos, related to $U_{e2}$, may be small or large. Therefore, from Eqns.(47),(48) we conclude that for small mixing of solar neutrinos the scale of $M_R$ depends on $1/m^2_2$, while for large mixing it depends on $1/m_1$. We obtain $M_R \sim 10^{15}$ GeV for the small mixing Mikheyev-Smirnov-Wolfenstein (MSW) solution to the solar neutrino problem, $M_R \gtrsim 10^{16}$ GeV for large mixing MSW (the favoured solution), $M_R \gtrsim 10^{17}$ GeV for (large mixing) low-$\Delta m^2$ MSW, and $M_R \gtrsim 10^{18}$ GeV for (large mixing) vacuum oscillations. The structure of the matrix $M_R$ is hierarchical and nearly diagonal, with leading form $M_R \sim \text{diag}(0,0,1)$, that is similar to $M_D$, but with an enhanced hierarchy of masses and mixings [46]. Strong mass hierarchy in the heavy neutrino mass matrix is indeed one of the possible conditions in order to get a seesaw enhancement of lepton mixing [47]. For a general classification of models with large lepton mixing, see Ref. [48]. In particular, for double large mixing we have found $M_{Rij} \simeq m_{Di}/m_1$. This last simple situation does not occur for the inverse hierarchy $m_1 \simeq m_2 \gg m_3$, when however $M_R \sim \text{diag}(0,0,1)$ again. The degenerate spectrum $m_1 \simeq m_2 \simeq m_3$ is unnatural in the seesaw framework, if one assumes a hierarchical $M_D$. In fact, in Eqn.(22), $M_R$ should work in such a way as to cancel almost exactly the hierarchy of $M_D$.

At this stage, in the SM, the scale of $M_R$ is a new parameter of the theory. However, in some extensions of the SM this scale is related to a SSB, and thus, especially in the $SO(10)$ model, it is constrained. We will briefly discuss this important issue at the end of section VI. Two distinct theoretical patterns are related to it.

### III. LEFT-RIGHT MODEL

The SM is a successful theory. However, enlarged gauge symmetries have been proposed, which are broken down to the SM symmetry at some high energy scale. The simplest gauge extension of the SM involving a left-right analogy is based on the gauge group $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ [49,50]. The groups $SU(2)_L$ and $SU(2)_R$ generate left-handed ($V-A$) and right-handed ($V+A$) interactions, respectively, with coupling constants $g_{2L}$ and $g_{2R}$. The $V+A$ interactions are suppressed by the high mass of $SU(2)_R$ gauge bosons.

The classification of left-handed and right-handed fermions is the following:

\[
\begin{align*}
\begin{pmatrix} u \\ d \end{pmatrix}_L & \sim (3, 2, 1, 1/3), \\
\begin{pmatrix} u \\ d \end{pmatrix}_R & \sim (3, 1, 2, 1/3) \\
\begin{pmatrix} \nu \\ e \end{pmatrix}_L & \sim (1, 2, 1, -1), \\
\begin{pmatrix} \nu \\ e \end{pmatrix}_R & \sim (1, 1, 2, -1)
\end{align*}
\]

with $Q = T_{3L} + T_{3R} + (B - L)/2$. The generator $B - L$ is the difference between baryon and lepton numbers. Charge-conjugate states belong to the conjugate representations. Gauge bosons are the gluons $(8, 1, 1, 0)$, the $W_L$ bosons $(1, 3, 1, 0)$, the $W_R$ bosons $(1, 1, 3, 0)$, and the

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6For a recent account of the solar neutrino problem and its oscillation solutions, see Ref. [4]. MSW solutions refer to oscillations in matter [45].
singlet \((1,1,1,0)\). The Higgs fields needed to achieve the SSB and the seesaw mechanism are \(\varphi \sim (1,2,2,0)\) and \(\Delta_R \sim (1,1,3,2)\). For example the bidoublet \(\varphi\), written as

\[
\varphi = \begin{pmatrix}
\varphi^0_1 & \varphi^+_1 \\
\varphi^-_2 & \varphi^0_2
\end{pmatrix},
\]

breaks the SM group and gives Dirac masses to quarks and leptons through the Yukawa terms

\[
(3, 2, 1, 1/3)_f \times (\bar{3}, 1, 2, -1/3)^c \times (1, 2, 2, 0)_{H, \bar{H}} = (1, 1, 1, 0) + ...
\]

\[
(1, 2, 1, -1)_f \times (1, 1, 2, 1)^c \times (1, 2, 2, 0)_{H, \bar{H}} = (1, 1, 1, 0) + ....
\]

with \(H = \varphi\) and \(\bar{H} = \sigma_2 H^* \sigma_2\). This happens because of its VEV \(\varphi^0_1 = k_1, \varphi^0_2 = k_2\). As a consequence, the Dirac neutrino mass is generated on the same footing as the other fermion masses, in particular the charged lepton mass. Four Yukawa terms yield Dirac masses, as in the SM, but with a different pattern. Quark mass matrices are written as \(M_u = r k_1 + s k^*_2\), \(M_d = r k_2 + s k^*_1\), where \(r, s\) are Yukawa coupling matrices. The triplet \(\Delta_R\) breaks the left-right model to the SM and gives a Majorana mass to the right-handed neutrino through the Yukawa term

\[
(1, 1, 2, -1)_f \times (1, 1, 2, -1)_f \times (1, 1, 3, 2)_H = (1, 1, 1, 0) + ....
\]

If \(k_{1,2}\) and \(v_R\) are the VEVs of \(\varphi^0_{1,2}\) and \(\Delta_R\), respectively, the full neutrino mass matrix is in the form

\[
M \sim \begin{pmatrix}
0 & k_{1,2} \\
k_{1,2} & v_R
\end{pmatrix}
\]

and, since \(k_{1,2} \ll v_R\), the seesaw mechanism holds. All fermion masses are generated by the SSB and the mass of the right-handed neutrino is related to the scale of left-right breaking [51]. This is in contrast with the SM, where the right-handed neutrino mass is produced by a bare mass term, unless a singlet Higgs field is included on purpose.

In LRM, if \(k_1 = k_2\), the Dirac neutrino masses are expected to be similar to charged lepton masses (and the up quark masses similar to the down quark masses, up-down symmetry) and as a consequence the scale for \(M_R\) is lowered by three or four orders with respect to the case (46) (quark-lepton symmetry). However, up quark masses must be different from down quark masses, so that in a minimal model \(k_1 \neq k_2\) is required. In general, mass matrices are arbitrary. If \(k_2 = 0\) we have \(r\) and \(s\) very different from each other.

A. Right-handed mixings

Let us discuss how the inclusion of \(SU(2)_R\) in the gauge group allows one to select, in principle, viable forms for quark mass matrices among the several SM bases. The key input would be some knowledge of right-handed interactions. For the quark mass and charged weak current terms we have the Lagrangian

\[
\mathcal{L}_M = \bar{u}_L M_u u_R + \bar{d}_L M_d d_R + g_L \bar{u}_L d_L W_L + g_R \bar{u}_R d_R W_R
\]
and after diagonalization of quark mass matrices

\[ \mathcal{L}_M = \bar{u}_L D_u u_R + \bar{d}_L D_d d_R + g_L \bar{u}_L V L d_L W_L + g_R \bar{u}_R V R d_R W_R, \]

(57)

with \( V_L = V_{CKM} \), \( V_R = V_{uR}^\dagger V_{dR} \). Right-handed currents appear on the same footing of the left-handed currents, except for the fact that \( m_{W_R} \gg m_{W_L} \). The mixing matrix \( V_R \) contains three angles and six phases, since the phases of right-handed quarks are fixed because they must cancel with the left-handed phases in diagonalized mass matrices. The transformations that do not change the observables in Eqn.(57) are the following:

\[ u_L \rightarrow U u_L, \quad d_L \rightarrow U d_L, \]

(58)

\[ u_R \rightarrow V u_R, \quad d_R \rightarrow V d_R. \]

(59)

Left-handed states and also right-handed states transform together, according to the classification (49). Therefore, it is possible to have \( M_1 \) diagonal by \( U^\dagger M_1 V \) but \( M_2 \) is fixed to become \( U^\dagger M_2 V \) and cannot be hermitian or with three zeros in general as it happens in the SM. The bases considered within the SM become strong ansätze in the LRM, giving the same \( V_L \) but different \( V_R \). Thus, a true test of mass matrices should be done in a left-right extension of the SM. In fact, a direct way to study quark mass matrices is to look at right-handed mixings [52]. For example, matrix (28) gives small right-handed mixings, while matrix (29) gives a large \( V_{cb}^R \). A systematic analysis of quark mass matrices with \( M_u \) diagonal and \( M_d \) containing three zeros, within LRM, has been performed in Ref. [27]. Right-handed mixings are obtained through the relation \( M_d^\dagger M_d = V_R D_R^2 V_R^\dagger \). It is worth noting that for the hermitian models considered in subsection IIB the right-handed mixings are equal to the left-handed ones. This condition occurs in the case of manifest left-right symmetry [53], when \( g_{2L} = g_{2R} \) above the left-right scale and \( k_1, k_2 \) are real. In that case, also a Higgs triplet \( \Delta_L \), with VEV \( v_L \ll k_{1,2} \), has to be introduced, which gives a direct contribution to the mass of left-handed neutrinos [54].

By using transformations (58),(59) the form of both \( M_u \) and \( M_d \) may be changed. However, we need other observable parameters, for example new mixings, to select viable forms for such models of mass matrices. These new physical parameters exist in some extensions of the LRM, notably the Pati-Salam model (section IV) and the \( SO(10) \) model (section VI).

### IV. PARTIAL UNIFICATION

Lepton number is considered as the fourth color in the Pati-Salam model, based on the gauge group \( SU(4)_C \times SU(2)_L \times SU(2)_R \) [49]. This model preserves the left-right gauge analogy of LRM and in addition the color \( SU(3) \) symmetry is extended to \( SU(4) \) in order to include lepton number. Fermions belong to the \((4,2,1)\) and \((4,1,2)\) and their conjugate representations,

\[ \begin{pmatrix} u & \nu \\ d & e \end{pmatrix}_L \sim (4,2,1), \quad \begin{pmatrix} u & \nu \\ d & e \end{pmatrix}_R \sim (4,1,2), \]

(60)

and gauge bosons to \((15,1,1), (1,3,1), (1,1,3)\). Left-handed states belong to one multiplet. New interactions between quarks and leptons, and hence new mixings, arise. In analogy with
the previous model, the minimal Higgs contents is $(1, 2, 2)$, $\left( \overline{10}, 1, 3 \right)$, giving Dirac masses by the two Yukawa terms
\[
\begin{align*}
(4, 2, 1)_f \times (\overline{4}, 1, 2)_\mathbb{T} \times (1, 2, 2)_{H, \overline{H}} &= (1, 1, 1) + \ldots 
\end{align*}
\] (61)
and a Majorana mass to the right-handed neutrino by
\[
\begin{align*}
(4, 1, 2)_f \times (4, 1, 2)_f \times (\overline{10}, 1, 3)_H &= (1, 1, 1) + \ldots . 
\end{align*}
\] (62)
Due to the fact that quarks and leptons belong to the same multiplets, there is no more the freedom in fermion mass matrices. The different forms for quark mass matrices considered above correspond now to physically distinguished mechanisms, even for the hermitian cases. For example, $V_{ub}$ has a direct component in model (38), while has an indirect origin in model (36). If we use transformations (58),(59), some mixing between quarks and leptons will change. Also the zeros appearing in SM bases are physically meaningful.

Moreover, term (61), which involves the $SU(4)$ singlet, gives the mass relations
\[
M_e = M_d, \quad M_\nu = M_u 
\] (63)
(from now on we set $M_\nu \equiv M_D$), while analogous terms with $(15, 2, 2)_{H, \overline{H}}$ would give
\[
M_e = -3M_d, \quad M_\nu = -3M_u, 
\] (64)
due to the form diag$(1, 1, 1, -3)$ for the VEV of the adjoint $15$. Therefore, the $SU(4)$ symmetry leads to simple relations between quark and lepton mass matrices, that is to quark-lepton symmetry. But if we combine the effect of the singlet and the adjoint, simple relations are lost in general. Quarks and leptons are unified in the Pati-Salam model, but in general not the coupling constants. This is achieved in the $SO(10)$ model. Before, we discuss the prototype of unification models, that is the $SU(5)$ model.

V. SU(5)

Grand unification assumes that the gauge group describing the fundamental interactions is simple and that it is broken down to the SM group in one or more steps, each of them related to a mass scale and a residual symmetry group. Therefore, the evolution of coupling constants must converge to one point at the unification scale. The simplest grand unification theory (GUT) is based on the unitary group $SU(5)$ [55], broken in one step down to the SM group. The minimal nonsupersymmetric $SU(5)$ model does not agree with experimental data, since the three coupling constants $g_3, g_2, g_1$ do not meet at a single point [56], proton lifetime is predicted too short and $\sin^2 \theta_W$ too small. However, the supersymmetric version [57] is reliable [58], with a predicted scale for superpartners, $M_S$, around $10^3$ GeV. Proton lifetime results to be $1 \times 10^{35 \pm 1}$ yr, and the current experimental lower limit by SuperKamiokande is $3.3 \times 10^{33}$ yr [59].

Left-handed fermions belong to the $\overline{5}$ and $10$ representations, because under the group $SU(3) \times SU(2) \times U(1)$ these representations are decomposed as
\[
\begin{align*}
\overline{5} &= (\overline{3}, 1, 1/3) + (1, 2, -1/2) \\
10 &= (3, 2, 1/6) + (\overline{3}, 1, -2/3) + (1, 1, 1), 
\end{align*}
\] (65)
(66)
in such a way that $d^c$, $e$, $\nu$ belong to the $\mathbf{5}$ and $u$, $d^c$, $e^c$ to the $\mathbf{10}$. Gauge fields belong to the adjoint $\mathbf{24}$,

$$24 = (8, 1, 0) + (1, 3, 0) + (1, 1, 0) + (3, 2, -5/6) + (\mathbf{3}, 2, 5/6),$$

(67)

whose terms refer to gluons, electroweak bosons and lepto-quarks. The breaking of $SU(5)$ down to the SM group is obtained by a $\mathbf{24}$ of Higgs fields and the further breaking to $SU(3) \times U(1)$ by a $\mathbf{5}$.

In the simplest case, fermion masses are generated by the Yukawa coupling with the $\mathbf{5}$ of Higgs, while the $\mathbf{24}$ does not contribute. Since

$$\mathbf{5} \times \mathbf{5} = \mathbf{10} + \mathbf{15}$$

(68)

$$\mathbf{5} \times \mathbf{10} = 5 + 45$$

(69)

$$\mathbf{10} \times \mathbf{10} = \mathbf{5} + \mathbf{\bar{15}} + \mathbf{50},$$

(70)

we have two main terms

$$\mathbf{5}_f \times \mathbf{10}_f \times \mathbf{5}_H = 1 + ...$$

(71)

$$\mathbf{10}_f \times \mathbf{10}_f \times \mathbf{5}_H = 1 + ...,$$

(72)

which give mass to down quarks and charged leptons, and up quarks, respectively, with the relation

$$M_e = M_d^T,$$

(73)

while $M_u$ is independent. Also the inclusion of the $\mathbf{45}$ of Higgs fields can contribute to fermion masses by the Yukawa terms

$$\mathbf{5}_f \times \mathbf{10}_f \times \mathbf{45}_H = 1 + ...$$

(74)

$$\mathbf{10}_f \times \mathbf{10}_f \times \mathbf{45}_H = 1 + ...,$$

(75)

yielding the relation

$$M_e = -3M_d^T.$$  

(76)

In the minimal $SU(5)$, just like in the MSM, the neutrino is massless. However, if a singlet $(\nu^c)_L$ is introduced, a Dirac mass is produced by the Yukawa term

$$\mathbf{5}_f \times \mathbf{1}_f \times \mathbf{15}_H = 1 + ...,$$

(77)

and a Majorana mass for the right-handed neutrino by the bare term $\mathbf{1}_f \times \mathbf{1}_f$, or, if also a singlet Higgs is introduced, by the Yukawa term $\mathbf{1}_f \times \mathbf{1}_f \times \mathbf{1}_H$. Then a seesaw mechanism can work. One can also have a direct contribution to the mass of left-handed neutrinos through the Yukawa term $\mathbf{5}_f \times \mathbf{5}_f \times \mathbf{15}_H$. Note that, since $M_u$ is generated by a different term with respect to $M_{d,e}$, one can expect, as is the case, the up quark mass hierarchy to be different from the down quark and charged lepton mass hierarchies, which are similar to each other. This effect has some analogy with that we have seen in supersymmetric SM, but is enforced by the quadratic presence of $\mathbf{10}_f$ in Eqn.(72).
The mass matrix relation (73) gives \( m_d = m_c, \ m_s = m_\mu, \ m_b = m_\tau \) at the unification scale and \( m_d/m_e = m_s/m_\mu = m_b/m_\tau \simeq 3 \) at the low scale [60]. Thus only \( m_b = m_\tau \) at the unification scale is reliable. However, appropriate couplings with both \( \overline{5}_H \) and \( 4\overline{5}_H \) give the suggestive Georgi-Jarlskog ansatz [61]

\[
M_u = \begin{pmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & B & C \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & D & 0 \\ D & E & 0 \\ 0 & 0 & F \end{pmatrix}, \quad M_e = \begin{pmatrix} 0 & D & 0 \\ D & -3E & 0 \\ 0 & 0 & F \end{pmatrix},
\]

where the relation \( m_b = m_\tau \) is retained, and \( m_\mu \simeq 3m_s, \ m_d \simeq 3m_e \). Hence, at the low scale, the good relations \( m_b \simeq 3m_\tau, \ m_s \simeq m_\mu, \ m_d \simeq 9m_e \) are obtained. The matrix \( M_u \) has the Fritzsch form. The agreement with data is much better in the supersymmetric case [62]. However, a more confident choice for the mass matrices would be, for example, that of filling up element 2-2 in \( M_u \) and element 2-3 in \( M_{d,e} \), in order to have the viable form with mass matrices having similar structure, considered in subsection IIB. Yukawa unification \( y_b = y_\tau \), coming out from Eqn.(74), is reliable. The transpose in Eqn.(73) or (76) can nicely be used to obtain large lepton mixing retaining small quark mixing, if one takes asymmetric matrices. Some models that use this feature, in the NNI basis, are in Refs. [63,44].

Since \( u_R \) and \( d_R \) belong to different multiplets, in \( SU(5) \) we still have a freedom in quark mass matrices similar to the SM case, so that without loss of generality \( M_u \) can be taken diagonal (and \( M_d \) with three zeros). This is not true in the \( SO(10) \) model. Moreover, while in the \( SU(4) \times SU(2) \times SU(2) \) model both \( M_e \) and \( M_\nu \) are related to \( M_d \) and \( M_u \), respectively, in the \( SU(5) \) model \( M_e \) is related to \( M_d \), but \( M_u \) and \( M_\nu \) are independent from each other. By using the Georgi-Jarlskog trick and the quark mass matrix (30), one is led to propose [64]

\[
M_e = \frac{m_\tau}{m_b} \begin{pmatrix} 0 & \sqrt{m_d m_s} & 0 \\ \sqrt{m_d m_s} & -3m_s & \frac{m_b}{\sqrt{3}} \\ 0 & m_s & \frac{2m_b}{\sqrt{3}} \end{pmatrix},
\]

while \( M_\nu \) could be diagonal, for example \( M_\nu \sim \text{diag}(m_u, m_c, m_t) \sim \text{diag}(\lambda^8, \lambda^4, 1)m_t \), or \( M_\nu \sim \text{diag}(m_d, m_s, m_b) \sim \text{diag}(\lambda, \lambda^2, 1)m_b \), or \( M_\nu \sim \text{diag}(\lambda^2, \lambda, 1)m_b \). A possible way to select the form of \( M_\nu \) is by means of its impact within the baryogenesis via leptogenesis mechanism [64] (see also Ref. [65]), which is based on the decay of heavy neutrinos [66]. The \( SU(5) \) model retains more freedom for mass matrices with respect to the \( SO(10) \) model that we are going to discuss.

**VI. SO(10)**

As we said in the previous section, the nonsupersymmetric \( SU(5) \) model is not supported by experiment, while supersymmetric \( SU(5) \) is reliable. Another alternative to the simplest \( SU(5) \) is the \( SO(10) \) model [67], which has also some advantages. The orthogonal group \( SO(10) \) includes both \( SU(5) \) and \( SU(4) \times SU(2) \times SU(2) \). Under \( SU(5) \), the fundamental spinorial 16 representation of \( SO(10) \) is decomposed as

\[
16 = \overline{5} + 10 + 1,
\]

(80)
that is all left-handed fermions, including the charge-conjugate states, belong to a single representation and there is a natural place for \((\nu^c)_L\) in the \(SU(5)\) singlet. This fact makes the \(SO(10)\) model more compelling, and possibly more predictive for fermion masses, with respect to the \(SU(5)\) model. Gauge bosons are in the adjoint \(45 = 24 + 10 + 10\), where \(24\) are the \(SU(5)\) bosons. Under \(SU(4) \times SU(2) \times SU(2)\) the \(16\) is decomposed as

\[
16 = (4, 2, 1) + (\bar{4}, 1, 2). \tag{81}
\]

The breaking to the SM can occur through one, two, or more steps. In a typical breaking chain \([68]\) the \(SO(10)\) symmetry is broken down to \(SU(4) \times SU(2) \times SU(2)\) at the unification scale \(M_U \sim 10^{16}\) GeV by a \(210_H\), then to the SM at the intermediate scale \(M_I \sim 10^{11}\) GeV by a \(126_H\), and the final breaking is due to a \(10_H\). Only the SM (or \(SU(5)\)) singlet in \(126\), which corresponds to \((10, 1, 3)\) in the Pati-Salam group, takes the VEV.

Looking at the product \(16 \times 16 = 10 + 126 + 120\), with \(10 + 126\) the symmetric part and \(120\) the antisymmetric part, we realize that, since representations \(10\) and \(120\) are real, and the \(126\) is complex, the Yukawa terms which can give mass to the fermions are

\[
\begin{align*}
16_f \times 16_f \times 10_H &= 1 + ... \tag{82} \\
16_f \times 16_f \times \overline{126}_H &= 1 + ... \tag{83} \\
16_f \times 16_f \times 120_H &= 1 + .... \tag{84}
\end{align*}
\]

However, we can use also two \(10\)s or two \(120\)s. In such a case we obtain from term (82) the relations (63), from term (83) the relations (64), and no relation from term (84). These relations come out because of the \((1, 2, 2)\) component of \(10\), and the \((15, 2, 2)\) component of \(\overline{126}\). Since the \(120\) contains both components, it yields no relation in the general case. The situation is similar to the Pati-Salam model. If one \(10\) is used, then in the nonsupersymmetric case we have \(|M_e| = |M_\nu| = |M_u| = |M_d|\), relations not viable. In general, the \(10\) and \(\overline{126}\) contributions are symmetric and the \(120\) contribution is antisymmetric. A Majorana mass for the right-handed neutrino is obtained from the coupling with the \(SU(5)\) singlet contained in \(\overline{126}\), related to the scale of intermediate symmetry breaking. The simplest seesaw mechanism is then in the form

\[
M \sim \begin{pmatrix} 0 & 10 \\ 10 & \overline{126} \end{pmatrix}, \tag{85}
\]

but \(120_H\) and \(\overline{126}_H\) can contribute to Dirac masses and \(\overline{126}_H\) can have a component contributing to light neutrino mass. The Georgi-Jarlskog ansatz is obtained by means of \(10_H\) and \(\overline{126}_H\) \([69, 37]\), with mass matrices necessarily symmetric. In this case the Dirac neutrino mass matrix is related to the up quark mass matrix, in such a way that the calculation of subsection IIC applies. Yukawa unification \(y_b = y_\tau\), coming out from Eqn.(82), is reliable both in the nonsupersymmetric case, when it is realized at the intermediate scale, and in the supersymmetric case, when it holds at the unification scale. The further unification \(y_t = y_\nu = y_\tau\) is reliable only for large \(v_2/v_1\) or \(\tan \beta\) \([70]\). In any case, Eqn.(82) gives also \(y_t = y_\nu\). Therefore, a minimal model of fermion masses includes only \(10\)s and \(126\)s. Two \(10\)s and one \(126\) provide Dirac masses and another \(126\) gives a mass to right-handed neutrinos. For a study of right-handed mixings in \(SO(10)\), when asymmetric matrices and \(120\)s are used, again in the NNI form, see Ref. [71].

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We notice that in nonsupersymmetric \( SO(10) \) it is \( M_R \sim M_I \), while as seen in subsection IIC the calculation gives \( M_R \gtrsim M_U \). There are two main ways to solve this problem. First, if the matrix \( M_R \) has a roughly off-diagonal form and \( M_{R33} \sim 0 \) \([40,72]\), which can be obtained by some suitable cancellations among effective neutrino parameters. Second, in supersymmetric \( SO(10) \), which can be broken in one step to the SM, so that \( M_R \sim M_U \). We notice also that for vacuum oscillations the scale of \( M_R \), as calculated in subsection IIC, is well above the unification scale. This solution is not favoured from the theoretical point of view. On the other hand the MSW solutions are consistent with the unification scale. The large mixing MSW solution is also favoured by data fits \([73]\).

**VII. OUTLOOK**

Understanding the pattern of fermion masses and mixings is a key subject in modern particle physics. However, despite many efforts, few advances have been obtained. The values of quark masses and mixings are sufficiently well known, but we do not have a confirmed theory about the origin of their spectrum. On the other hand, neutrino masses and lepton mixings are a new field of research both for experiment and for theory. In extensions of the SM, the lepton sector is often linked to the quark sector. Nevertheless, new observables can appear. To date we have no experimental evidence about them. Roughly speaking, the LRM leads to up-down symmetry, and the Pati-Salam model to quark-lepton symmetry. The \( SO(10) \) model preserves the features of the Pati-Salam model, while the \( SU(5) \) model only relates \( M_d \) to \( M_e \).

An indication towards a theory for fermion masses and mixings could be obtained through the union of the Froggatt-Nielsen mechanism with the Georgi-Jarlskog ansatz, namely hierarchical entries in mass matrices (see for example Eqn.(36)) together with simple relations between the quark and lepton sectors. In this case the zeros appearing in mass matrices are approximate and in fact high powers of some small parameter \([74]\). Then, a mass matrix is roughly in the form

\[
M_{ij} \sim \epsilon^{\theta(f_i) + \eta(J_j)}
\]  

(86)

where \( q \) are charges related to the horizontal symmetry, and the breaking parameter \( \epsilon \) should be related to \( \lambda \). Usually \( \epsilon \) is the ratio between the VEV of a heavy scalar field, which breaks the horizontal symmetry, and the mass of very heavy fermions. Several choices for this symmetry are possible. For example, abelian \( U(1) \) or nonabelian \( U(2) \). For some general considerations on broken horizontal symmetries, see Ref. \([75]\). The explicit realization of a compelling model can be much involved \([76]\). Nevertheless, through this paper we are led to two main general scenarios for fermion mass and mixing parameters, preferably within the \( SO(10) \) model.

In the nonsupersymmetric scenario the hierarchy \( m_t \gg m_b \) is due to a double Higgs contribution, the relation \( m_b \simeq m_\tau \) holds at the intermediate scale so that the values of \( m_b \) and \( m_\tau \) at the low scale are understood as a running effect, Dirac matrices are nearly diagonal, \( M_R \) has a nearly off-diagonal form at the intermediate scale.

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In the supersymmetric scenario $m_t \gg m_b$ because $\tan \beta$ is large, the relation $m_b \simeq m_\tau$ holds at the unification scale running at the low scale, Dirac matrices are nearly diagonal, also $M_R$ has a nearly diagonal form at the unification scale. The last remark makes the supersymmetric scenario more homogeneous.

For an exact horizontal symmetry, only the third generation would be massive, and mixing angles vanish. Symmetry breaking terms gradually appear in mass matrices as powers of $\epsilon$ or $\lambda$, generating the hierarchy of masses and mixings, according to what we have seen in previous sections. Different sets of matrix entries arise at different orders in horizontal symmetry breaking and details depend on the assignment of horizontal charges. Since the heavy fields responsible for this mechanism can have masses above the unification scale, we see that a full account of fermion masses and mixings is likely to be realized towards the Planck scale ($M_P \sim 10^{19}$ GeV), where quantum gravity effects in particle physics become important. The leading theoretical framework for describing this kind of physics is string theory [78], which indeed includes horizontal symmetries. Therefore, as a conclusion, we can say that the subject of fermion masses and mixings has many phenomenological and theoretical implications.

**APPENDIX: DIRAC AND MAJORANA MASS TERMS**

In this appendix we write the Dirac and Majorana mass terms using both the bispinor and the (Weyl) spinor notations. Let us consider a Dirac bispinor

$$\varphi = \left( \begin{array}{c} \psi_L \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ \psi_R \end{array} \right) = \varphi_L + \varphi_R,$$

where the spinors $\psi_L$ and $\psi_R$ are the left-handed and right-handed chirality components. With this field $\varphi$ we can build the invariant term

$$\overline{\varphi} \varphi = \overline{\varphi_R} \varphi_L + \overline{\varphi_L} \varphi_R,$$

which corresponds to the Dirac mass term. The conjugate bispinor is

$$\varphi^c = \left( \begin{array}{c} (\psi^c)_L \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ (\psi^c)_R \end{array} \right) = \left( \begin{array}{c} \sigma_2 \psi^*_R \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ -\sigma_2 \psi^*_L \end{array} \right) = (\varphi^c)_L + (\varphi^c)_R,$$

where $\sigma_2$ is a Pauli matrix, and we have $\varphi^c = \gamma_2 \varphi^*$, so that $\overline{\varphi^c} \varphi^c = \overline{\varphi} \varphi$. Let us define also (autoconjugate) Majorana bispinors:

$$\varphi_L = \left( \begin{array}{c} \psi_L \\ 0 \end{array} \right) + \left( \begin{array}{c} -\sigma_2 \psi^*_L \\ 0 \end{array} \right) = \left( \begin{array}{c} \psi_L \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ (\psi^c)_R \end{array} \right) = \varphi_L + (\varphi^c)_R,$$

$$\varphi_R = \left( \begin{array}{c} \sigma_2 \psi^*_R \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ \psi_R \end{array} \right) = \left( \begin{array}{c} (\psi^c)_L \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ \psi_R \end{array} \right) = (\varphi^c)_L + \varphi_R.$$

7 The requirement of a large value for $\tan \beta$ has nontrivial consequences for supersymmetric phenomenology [77].
The left-handed (right-handed) field $\varphi_l$ ($\varphi_r$) can be expressed in terms of the left-handed (right-handed) component $\psi_L$ ($\psi_R$) only. There is an invariant term

$$\overline{\varphi_l}\varphi_l = (\varphi^c)_R\varphi_L + \overline{\varphi_L}(\varphi^c)_R,$$

which corresponds to the left-handed Majorana mass term, and another invariant term

$$\overline{\varphi_r}\varphi_r = \varphi_R(\varphi^c)_L + (\varphi^c)_L\varphi_R,$$

which corresponds to the right-handed Majorana mass term. Since $(\varphi_R)^c = (\varphi^c)_L$ and $(\varphi_L)^c = (\varphi^c)_R$ we have also $\overline{\varphi_l}\varphi_l = (\varphi^c)_L\varphi_L + \overline{\varphi_L}(\varphi^c)_L$ and $\overline{\varphi_r}\varphi_r = \varphi_R(\varphi^c)_L + (\varphi^c)_L\varphi_R$. In every mass term written above, the second term is the hermitian conjugate (h.c.) of the first one. Note that a left-handed bispinor is always coupled to a right-handed bispinor.

Previous mass terms are written using bispinors, but they can be expressed also by means of spinors. In fact, the Dirac invariant can be also written as

$$(\psi^c)^L_T\sigma_2\psi_L + \text{h.c.} \text{ or } \psi^T_L\sigma_2(\psi^c)_L + \text{h.c.},$$

and the two Majorana invariants as

$$\psi^T_L\sigma_2\psi_L + \text{h.c.}$$

for the left-handed neutrino and

$$(\psi^c)^T_L\sigma_2(\psi^c)_L + \text{h.c.}$$

for the right-handed neutrino. Remember that in $SU(2) \times SU(2)$, which acts as a representation of the Lorentz group, we have $(2, 1) \times (2, 1) = (1, 1) + (3, 1)$ and every mass term corresponds to the singlet $(1, 1)$. Here, a left-handed spinor is coupled to another left-handed spinor. Notice also that a factor $1/2$ must be included in the Majorana terms, due to the autoconjugation property of Majorana fields. A Dirac mass term conserves lepton number and electric charge, while a Majorana mass term violates lepton number by two units and is allowed only for neutral particles. For example, the neutrino may be either of the Dirac or of the Majorana type. If it is a Majorana particle, the neutrinoless double beta decay can occur. Therefore, the discovery of this decay would prove the Majorana nature for neutrinos. If both the Dirac and the Majorana mass terms are present, mass eigenstates are of the Majorana type.
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