Abstract

A general form of the polarization matrix valid for any type of electromagnetic radiation (plane waves, multipole radiation etc.) is defined in terms of a certain bilinear form in the field-strength tensor. The quantum counterpart is determined as an operator matrix with normal-ordered elements with respect to the creation and annihilation operators. The zero-point oscillations (ZPO) of polarization are defined via difference between the anti-normal and normal ordered operator polarization matrices. It is shown that ZPO of the multipole field are stronger than those described by the model of plane waves and are concentrated in a certain neighborhood of a local source.

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I. INTRODUCTION

It is well known that the polarization measurements play very important role in optics and spectroscopy for a long time (e.g., see [1]). At present, the polarization entanglement of photon twins is widely discussed in connection with the fundamental problems of quantum theory such as the nonlocality and existence of hidden parameters. In spite of the fact that there is still a certain question in regard to fundamental issues of the entangled states of quantum systems, quantum entanglement has been recognized as an important tool in quantum information processing and quantum computation (e.g., see [2] and references therein).

The description usually given of the polarization is a classical one, defining the polarization as a measure of transversal anisotrophy of the plane electromagnetic waves [1,3]. This definition is based on the fact that the classical radiation field measured in a small volume at far distance from a local source can be approximated by the plane waves. In this case, the field strengths $\vec{E}$ and $\vec{B}$ have only two symmetric spatial components $E_x = B_y$ and $E_y = -B_x$ transversal with respect to the direction of propagation chosen to be the z-axis. Since these complex components may have different magnitude and phases, the quantitative description of polarization is provided by the so-called polarization matrix with the elements [3]

$$P_{\sigma \sigma'} = E_{\sigma}^* E_{\sigma'}, \quad \sigma, \sigma' = x, y. \quad (1)$$

Here $E_{\sigma}$ denotes the component of the positive-frequency part of the classical electric field strength and $\vec{B}$ is the magnetic induction. The quantum counterpart of (1) is represented by the operator polarization matrix which is obtained from (1) by a formal substitution of corresponding operators instead of the complex field strengths [4].

The plane-wave approximation used in the definition of (1) and its quantum counterpart is valid only at far distances from a local source such as an atom or molecule. In the intermediate and near zones, the field strengths can oscillate in any direction in spite of the fact that $\vec{E} \cdot \vec{B} = 0$ and $|E|^2 - |B|^2 = 0$ everywhere. The atomic transitions emit the multipole radiation represented by the spherical waves either classical or quantum [5]. In contrast to the plane waves, the field strengths of the multipole field can have all three spatial components. For example, the electric-type multipole radiation always has all three components of the electric field strength, while the magnetic induction obeys the condition of transversality $\vec{B}(\vec{r}) \cdot \vec{r} = 0$. In contrast, the magnetic multipole radiation has the three components of magnetic induction at completely transversal electric field [6]. Therefore, the polarization of either multipole field is the three-dimensional rather than two-dimensional property of the radiation. This means that, instead of the $(2 \times 2)$ polarization matrix (1), we have to consider a more general $(3 \times 3)$ polarization matrix [7]. In the plane wave approximation, this general polarization matrix should be reduced to (1).

The three-dimensional structure of the polarization can be illustrated as follows. The quantum theory interprets the polarization as a given spin state of photons [5]. The spin of a photon is known to be 1. Thus, it has just the three independent spin states. In the case of plane waves of photons, the third state with zero projection of spin on the axis of quantization is forbidden and the polarization picture is reduced to the transversal anisotrophy of the field. On the contrary, all the three spin states are allowed in the case of
multipole radiation [5]. Hence, in general case, we have to consider the spatial anisotropy of
the field rather than the transversal one. It is also clear that both the electric field strength
and magnetic induction can have nontrivial spatial properties and therefore both of them
should be taken into account [8].

The main scope of this note is to discuss a new general form of the polarization matrix
which can be defined directly in terms of the field-strength tensor. We show, that the general
polarization matrix (GPM) fits the polarization of plane and multipole waves adequately.
We also construct the quantum counterpart of GPM and discuss the zero-point oscillations
of polarization.

II. GENERAL POLARIZATION MATRIX

It is well known that the general description of electromagnetic field is based on the
field-strength tensor defined as follows [6]

\[
F(\vec{r}) = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix}
\]  

(2)

For example, the set of Maxwell’s equations can easily be expressed in terms of (2). Here
the field strengths are determined at the observation point \( \vec{r} \).

To construct GPM, we assume that (2) consists of the positive-frequency parts of the
field strengths. In fact, \( F \) is additive with respect to the contributions coming from the
positive- and negative frequency parts. Consider the simplest bilinear form

\[
R(\vec{r}) = F^+(\vec{r})F(\vec{r}),
\]

(3)

that can be constructed through the use of (2). It is clear that this form differs from the
energy-momentum tensor of the field by a scalar. In some sense, (3) is similar to Ricci tensor
considered in general relativity [9]. By construction, (3) is the (4 \times 4) block matrix of the
form

\[
R(\vec{r}) = \begin{pmatrix}
W_E & \vec{S} \\
\vec{S}^+ & P
\end{pmatrix}
\]

Here \( W_E = \vec{E}^* \cdot \vec{E} \) is the scalar, defining the electric-field contribution into the energy density
and \( \vec{S} = \vec{E}^* \times \vec{B} \) is proportional to the Poynting vector. We choose to interpret \( P \) in (3) as
GPM.

It is clear that \( P(\vec{r}) \) is the Hermitian (3 \times 3) matrix additive with respect to contributions
coming from the electric field and magnetic induction:

\[
P(\vec{r}) = P_E(\vec{r}) + P_B(\vec{r}),
\]

(4)

where
\[
    P_E(\vec{r}) = \begin{pmatrix}
    E_x^* E_x & E_x^* E_y & E_x^* E_z \\
    E_y^* E_x & E_y^* E_y & E_y^* E_z \\
    E_z^* E_x & E_z^* E_y & E_z^* E_z
    \end{pmatrix}
\]

and
\[
    P_B(\vec{r}) = \begin{pmatrix}
    B_y^* B_y + B_z^* B_z & -B_x^* B_x & -B_y^* B_x \\
    -B_z^* B_y & B_x^* B_x + B_z^* B_z & -B_x^* B_y \\
    -B_y^* B_z & B_y^* B_z & B_x^* B_x + B_y^* B_y
    \end{pmatrix}
\]  

By construction, the two terms in (4) describe the spatial anisotropy of the electric and magnetic fields respectively. The diagonal terms give contribution of corresponding field strengths into the intensity in given directions. The off-diagonal terms provide the "phase information" about the phase difference between the spatial components. For example, the off-diagonal terms in (5) are specified by the two out of three phase differences \(\Delta_{\mu\mu'} = \arg E_{\mu'} - \arg E_{\mu}\) such that

\[
    \Delta_{xy} + \Delta_{yz} + \Delta_{zx} = 0.
\]

In the case of plane waves described by the polarization matrix (1), there is only one polarization phase difference [3,6]. The matrix (5) has been examined in [7] in the case of electric-type multipole radiation. In turn, (6) is similar to the object has been proposed in [8] for the magnetic-type radiation.

Let us show that GPM (4) is reduced to (1) in the plane wave representation. Assume that the waves propagate along the \(z\)-direction. Then (5) takes the form

\[
    P_E = \begin{pmatrix}
    \mathcal{P} & 0 \\
    0 & 0
    \end{pmatrix}
\]

where \(\mathcal{P}\) is the conventional polarization matrix (1). In turn, (6) takes the form

\[
    P_B = \begin{pmatrix}
    \mathcal{P} & 0 \\
    0 & W_E
    \end{pmatrix}
\]

where \(W_E = \vec{E}^* \cdot \vec{E}\). Thus, in the case of plane waves, GPM (4) fits the conventional theory of polarization of plane waves adequately.

Consider now the case of electric-type multipole radiation when \(\forall \vec{r}, B_z(\vec{r}) = 0\). Then, the matrix (6) takes the form

\[
    P_B(\vec{r}) = \begin{pmatrix}
    \mathcal{P}^{(B)} & 0 \\
    0 & W_B
    \end{pmatrix}
\]

where

\[
    \mathcal{P}_{\sigma\sigma'}^{(B)}(\vec{r}) \equiv B_\sigma^* B_{\sigma'}, \quad \sigma, \sigma' = x, y,
\]

and \(W_B = \vec{B}^* \cdot \vec{B}\) is the magnetic contribution into the energy density. Thus, in the case of electric-type multipole radiation, the matrix (5) describes the spatial anisotropy of the
electric field strength at the point \( \vec{r} \), while the magnetic contribution corresponds to the transversal anisotropy. Due to the orthogonality condition \( \vec{E}(\vec{r}) \cdot \vec{B}(\vec{r}) = 0 \), there are only two independent polarization phase differences \( \Delta_{\alpha\beta} \) in this case. The polarization of magnetic-type multipole radiation can be described in terms of (4) in similar way.

It is well known that the contribution of radial component of the multipole radiation decreases rapidly with the distance from a local source [6]. Therefore, at far distances, the terms with \( E_z \) and \( B_z \) in (5) and (6) become negligible, and GPM (4) is reduced to (1). In other words, the additional terms in (4) can be important in the case of classical radiation at the intermediate and near zones only.

**III. OPERATOR POLARIZATION MATRIX OF QUANTUM FIELD**

Within the quantum domain, the field strengths in (5) and (6) should be changed by the creation and annihilation operators of photons with given polarization \( E_\mu, B_\mu \rightarrow a_\mu, E_\mu^+, B_\mu^+ \rightarrow a_\mu^+ \), where the subscript \( \mu \) takes the free different values corresponding to the circular (transversal) polarizations with different helicities and to the linear polarization in the radial (longitudinal) direction [5,7]. Leaving aside the known structure of the position dependent mode functions, we can conclude that the operator matrix \( \hat{R}(\vec{r}) \) (3) corresponds to the normal-ordered form in the creation and annihilation operators:

\[
R \rightarrow \hat{R}_n = \hat{R}(a^+ a).
\]

In addition, one can define the anti-normal operator matrix

\[
R \rightarrow \hat{R}_{an} = \hat{R}(aa^+) = FF^+
\]

through the change of order of factors in the classical matrices (5) and (6) and successive quantization. Although within the classical picture \( F^+ F = FF^+ \), the difference of the two operators

\[
\hat{R}_{an} - \hat{R}_n \equiv R_0
\]

defines, by construction, the zero-point oscillations (ZPO) of the electromagnetic field. In fact

\[
\langle 0|\hat{R}_{an} - \hat{R}_n|0\rangle = \langle 0|R([a, a^+])|0\rangle = R(\langle 0|[a, a^+]|0\rangle).
\]

Due to the commutation relation \([a_\alpha, a_\beta^+] = \delta_{\alpha\beta}\), the difference (7) is represented by the \( c \)-number matrix

\[
R^{(0)} = \left( \begin{array}{cc} [\vec{E}, \vec{E}^+] & \vec{E} \times \vec{B}^+ - \vec{E}^+ \times \vec{B} \\ \vec{E}^+ \times \vec{B} - \vec{E} \times \vec{B}^+ & P_{an} - P_n \end{array} \right)
\]

(8)

The scalar term here describes the contribution of ZPO of the electric field into the energy density since [10]

\[
\langle 0|\vec{E}^+ + \vec{E}\rangle^2|0\rangle = [\vec{E}, \vec{E}^+].
\]
In turn, the off-diagonal vector terms in (8) describe ZPO of the Pointing vector, while the \((3 \times 3)\) matrix
\[
P^{(0)} = P_{an} - P_n
\]

It seems to be very interesting to compare ZPO of quantum electromagnetic field in the two representations, i.e. for the plane and spherical waves of photons. The energy of free field is described by the following well-known formulas [5]
\[
H_{\text{plane}} = \sum_{k, \sigma} \hbar \omega_k (a_{k\sigma}^+ a_{k\sigma} + 1/2),
\]
\[
H_{\text{sph}} = \sum_k \sum_{\lambda, j, m} (a_{k\lambda jm}^+ a_{k\lambda jm} + 1/2),
\]
(9)
for the plane and spherical waves of photons respectively. Here \(\sigma = 1, 2\) is the index of polarization of plane waves, \(\lambda\) labels the type of multipole radiation either electric or magnetic, \(j = 1, 2, \cdots\), is the angular momentum of photons, and \(m = -j, \cdots, j\). The energy of ZPO in the whole volume of quantization is provided by the averaging of (9) over the vacuum state that gives
\[
H_{\text{plane}}^{(0)} = \sum_k \hbar \omega_k,
\]
\[
H_{\text{sph}}^{(0)} = \sum_{\lambda} \sum_k \sum_{j=1}^{\infty} (2j + 1) \hbar \omega_k / 2.
\]
(10)

At first sight, the two expressions in (10) are equivalent because both give the infinite energy of the vacuum state. At the same time, it is known that this infinity is inessential because any measurement used to recognize ZPO implies an averaging over a finite exposition time of detector [12,13]. Such an averaging is equivalent to a filtration, leading to a separation of a finite transmission frequency band. It is clear that the second expression in (10) strongly exceeds the first one in the case of the same transmission frequency band. Even if we assume that filtration separates only the electric dipole radiation (\(\lambda = E, j = 1\)), it is seen that
\[
H_{\text{sph}}^{(0)} / H_{\text{plane}}^{(0)} = 3/2.
\]
This means that the measuring level of ZPO of multipole field exceeds that of the plane waves of photons. This result follows from the fact that the multipole radiation has much more quantum degrees of freedom than the plane waves and each degree of freedom contributes into the vacuum fluctuations. Since the plane wave representation corresponds to the free field in empty space, while the spherical wave representation takes into account the existence of a singular point (emitter or absorber of the radiation field), this means that the presence of a local source (atom) leads to an increase of ZPO.

In a certain sense, the obtained picture of ZPO of spherical waves of photons results from the spatial symmetry of solution of the homogeneous wave equation used in the quantization [5]. It does not need any physical specification of the source considered as a singular point for the spherical waves. This means that ZPO described by \(H_{\text{sph}}^{(0)}\) in (10) are independent on whether the atom is used as a source or as a detector.
Let us note in this connection that possible influence of an atom on the electromagnetic vacuum state in the absence of radiation has been discussed in quantum electrodynamics for a long time (e.g., see [14]). In the usual picture, the so-called $A^2$-term arising from the atom-field interaction is considered is responsible for the renormalization of the vacuum fluctuations. In the case under consideration, the effect is caused by the geometry of space provided by the presence of atom independent of the atom field interaction. Therefore, in a certain sense, the effect is similar to the Casimir one [15].

Consider now the spatial structure ZPO. Independent of the representation, the density of ZPO can be described as follows

$$H^{(0)}(\vec{r}) = \frac{k^2}{8\pi}[\vec{A}(\vec{r}), \vec{A}^+(\vec{r})],$$

(11)

where $\vec{A}(\vec{r})$ is the positive-frequency part of vector potential at the point $\vec{r}$ and $\vec{E}(\vec{r}) = -ik\vec{A}(\vec{r})$ in the case of harmonic field. The mode function of plane waves has the form $\exp(i\vec{k} \cdot \vec{r})$. Then, the commutator in the right-hand side of (11) is independent of $\vec{r}$. In other words, ZPO of plane waves manifest the spatial homogeneity.

In turn, the mode functions $V_{\lambda kjm}\mu(\vec{r})$ of the multipole radiation are represented by certain combinations of the spherical Bessel functions responsible for the radial dependence and spherical harmonics describing the angular distribution of radiation [5,6]. Therefore, the right-hand side of (11) is represented as follows

$$H^{(0)}(\vec{r}) = \sum_{\lambda} \sum_{k,j,m,\mu} |V_{\lambda kjm}\mu(\vec{r})|^2.$$  

(12)

The explicit form of $V_{\lambda kjm}\mu(\vec{r})$ can be found in [5,6,16]. It is a straightforward matter to show that (12) is independent of the angular variables. Due to the properties of spherical Bessel functions, the radial dependence of (12) shows an inhomogeneous behaviour. Viz, ZPO of the multipole field are concentrated in a certain vicinity of the source. At far distances, they tend to the level predicted by the representation of plane waves. The dipole ($j=1$) contribution into ZPO is shown in Fig. 1. It is seen that inside the spherical region of the radius $r_0 \sim 2/k = \Lambda/\pi$ ($\Lambda$ denotes the wavelength), surrounding the atom, the level of multipole ZPO strongly exceeds that of plane waves of photons.

Similar effect can be observed for ZPO of the polarization. Consider as an illustrative example the first term in (4) described by the matrix (5). It is seen that the corresponding contribution into (8) is provided by the matrix with the elements

$$P^{(0)}_{\mu\mu'}(\vec{r}) = k^2[A_\mu(\vec{r}), A^{\mu'}_\mu(\vec{r})].$$

It is straightforward to calculate these elements in the same way as (12). This yields the following. ZPO of polarization are independent of the angular variables, while strongly depend on the distance from the source. Then, ZPO of polarization are concentrated in some vicinity of the source as well as (12). Similar result can easily be obtained for (6).

IV. CONCLUSION

Let us briefly summarize the obtained results. It is shown that the general polarization matrix of an electromagnetic radiation (4) can be constructed directly from the field-strength
tensor (2). This matrix is additive with respect to the contributions coming from the electric and magnetic fields and quantitatively describes the anisotropy of the fields in terms of intensities of radiation in different directions and corresponding phase differences. In spite of the fact that (4) is examined here in the case of plane and spherical waves of photons, the same matrix can be used to specify the polarization of radiation in an arbitrary geometry, for example in the case of a circular cylindrical cavity. The quantization of electromagnetic field in such a cavity was considered in [17].

By construction, the general polarization matrix is a local object and the polarization of electromagnetic radiation may change from point to point. Only in the case of plane waves, the polarization is the spatially homogeneous property of the field (in the direction of propagation).

Similar to the field energy, the polarization manifests ZPO which can be calculated with the aid of the anti-normal and normal ordered operator polarization matrices. ZPO of polarization define the level of quantum noise in the polarization measurements. In the case of plane waves of photons, the level of ZPO has the same value at any point. The presence of a real local source (atom) leads to an increase of ZPO above the level predicted by the model of plane waves. This effect of concentration of ZPO near atom corresponds to the near and intermediate zones. To stress the importance of the obtained results, we note that the interatomic distances in a number of modern experiments on engineered entanglement in the system of trapped Ridberg atoms are of the order of \( r_0 \) (e.g., see [18] and references therein). Therefore, there should be a strong vacuum noise which can influence the precision of polarization measurements in such a system. The above effect can also be considered in the context of Casimir effect in atomic systems at short distances.

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FIG. 1. The radial dependence of the first term (at $j = 1$) in (12). The level of ZPO of the plane waves is shown by the dotted line.