We consider formulation and some elaboration of \( p \)-adic and adelic quantum cosmology. Adelic generalization of the Hartle-Hawking proposal does not work in models with matter fields. \( p \)-Adic and adelic minisuperspace quantum cosmology is well defined as an ordinary application of \( p \)-adic and adelic quantum mechanics. It is illustrated by a few of minisuperspace cosmological models in one, two and three dimensions. As a result of \( p \)-adic quantum effects and adelic approach, all these models exhibit some discreteness of the minisuperspace.

1. Introduction

The main task of quantum cosmology\(^1\) is description of the very early stage in the evolution of the universe. At this stage the universe is in a quantum state, which is described by a wave function. Usually one takes that this wave function is complex-valued and depends on some real parameters. Since quantum cosmology is related to the Planck scale phenomena there is a sense to reconsider its foundations. We will here maintain the standard point of view that the wave function takes complex values, but we will treat its arguments in a more complete way. Namely, we will regard space-time coordinates and matter fields to be adelic, i.e. they have real as well as \( p \)-adic properties simultaneously. This approach is motivated by the following reasons: (i) the field of rational numbers \( \mathbb{Q} \), which contains all observational and experimental numerical data, is a dense subfield not only in the

\(^1\) Quantum cosmology
field of real numbers $\mathbb{R}$ but also in the fields of $p$-adic numbers $\mathbb{Q}_p$ ($p$ is any prime number), (ii) there is a plausible analysis\(^2\) within and over $\mathbb{Q}_p$ as well as that one related to $\mathbb{R}$, (iii) general mathematical methods and fundamental physical laws should be invariant\(^3\) under an interchange of the number fields $\mathbb{R}$ and $\mathbb{Q}_p$, (iv) there is a quantum gravity uncertainty\(^4\) $\Delta x$ while measuring distances around the Planck length $\ell_0$,

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} \text{cm},$$

which restricts priority of archimedean geometry based on real numbers and gives rise to employment of nonarchimedean geometry related to $p$-adic numbers\(^3\), (v) it seems to be quite reasonable to extend compact archimedean geometries by the nonarchimedean ones in the path integral method, and (vi) adelic quantum mechanics\(^5\) applied to quantum cosmology provides realization of all the above statements.

The successful application of $p$-adic numbers and adeles in modern theoretical and mathematical physics started in 1987, in the context of string amplitudes\(^6,7\) (for a review, see Refs.\(^8,2,9\)). For a systematic research in this field it was formulated $p$-adic quantum mechanics\(^10,11\) and adelic quantum mechanics\(^5,12\). They are quantum mechanics with complex-valued wave functions of $p$-adic and adelic arguments, respectively. In the unified form, adelic quantum mechanics contains ordinary and all $p$-adic quantum mechanics.

$p$-Adic gravity and the wave function of the universe were considered in the paper\(^13\) published in 1991. An idea of the fluctuating number fields at the Planck scale was introduced and it was suggested to restrict the Hartle-Hawking\(^14\) proposal to summation only over algebraic manifolds. It was shown that the wave function for the de Sitter minisuperspace model can be treated in the form of an infinite product of $p$-adic counterparts.

Another approach to quantum cosmology, which takes into account $p$-adic effects was proposed in 1995\(^15\). Like in adelic quantum mechanics, adelic eigenfunction of the universe is a product of the corresponding eigenfunctions of real and all $p$-adic cases. $p$-Adic wave functions are defined by $p$-adic generalization of the Hartle-Hawking\(^14\) path integral proposal. It was shown that in the framework of this procedure one obtains an adelic wave function for the de Sitter minisuperspace model. However, this procedure with the Hartle-Hawking $p$-adic prescription does not work when matter fields are included into consideration. The solution of this problem was found\(^16\) treating minisuperspace cosmological models as models of adelic quantum mechanics.

In this paper we consider adelic quantum cosmology as an application of adelic quantum mechanics to the minisuperspace models. It will be illustrated by one-, two- and three-dimensional minisuperspace models. As a result of $p$-adic effects and adelic approach, in all these models there is some discreteness of minisuperspace.

In the next section we give some basic facts on $p$-adic and adelic mathematics. Section 3 is devoted to a brief review of $p$-adic and adelic quantum mechanics. $p$-
Adic and adelic quantum cosmology are formulated in the Section 4. Sections 5 and 6 contain some concrete minisuperspace models. At the end, we give some concluding remarks.

2. $p$-Adic Numbers and Adeles

We give here a brief survey of some basic properties of $p$-adic numbers and adeles, which we exploit in this work.

Completion of $\mathbb{Q}$ with respect to the standard absolute value ($|\cdot|_\infty$) gives $\mathbb{R}$, and an algebraic extension of $\mathbb{R}$ makes $\mathbb{C}$. According to the Ostrowski theorem any non-trivial norm on the field of rational numbers $\mathbb{Q}$ is equivalent to the absolute value $|\cdot|_\infty$ or to the $p$-adic norm $|\cdot|_p$, where $p$ is a prime number. $p$-Adic norm is the nonarchimedean (ultrametric) one and for a rational number, $0 \neq x \in \mathbb{Q}$, $x = p^\nu \frac{m}{n}$, $0 \neq n, \nu, m \in \mathbb{Z}$, has a value $|x|_p = p^{-\nu}$. Completion of $\mathbb{Q}$ with respect to the $p$-adic norm for a fixed $p$ leads to the corresponding field of $p$-adic numbers $\mathbb{Q}_p$. Completions of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ and all $|\cdot|_p$ exhaust all possible completions of $\mathbb{Q}$.

$p$-Adic number $x \in \mathbb{Q}_p$, in the canonical form, is an infinite expansion

$$x = p^\nu \sum_{i=0}^{\infty} x_i p^i, \quad x_0 \neq 0, \quad 0 \leq x_i \leq p - 1. \quad (2)$$

The norm of $p$-adic number $x$ in (2) is $|x|_p = p^{-\nu}$ and satisfies not only the triangle inequality, but also the stronger one

$$|x + y|_p \leq \max(|x|_p, |y|_p). \quad (3)$$

Metric on $\mathbb{Q}_p$ is defined by $d_p(x, y) = |x - y|_p$. This metric is the nonarchimedean one and the pair $(\mathbb{Q}_p, d_p)$ presents locally compact, topologically complete, separable and totally disconnected $p$-adic metric space.

In the metric space $\mathbb{Q}_p$, $p$-adic ball $B_\nu(a)$, with the centre at the point $a$ and the radius $p^\nu$ is the set

$$B_\nu(a) = \{ x \in \mathbb{Q}_p : |x - a|_p \leq p^\nu, \nu \in \mathbb{Z} \}. \quad (4)$$

$p$-Adic sphere $S_\nu(a)$ with the centre $a$ and the radius $p^\nu$ is

$$S_\nu(a) = \{ x \in \mathbb{Q}_p : |x - a|_p = p^\nu, \nu \in \mathbb{Z} \}. \quad (5)$$

It holds:

$$B_\nu(a) = \bigcup_{\nu' \leq \nu} S_{\nu'}(a),$$

$$S_\nu(a) = B_\nu(a) \backslash B_{\nu - 1}(a), \quad B_\nu(a) \subset B_{\nu'}(a), \quad \nu < \nu',$$

$$\bigcap_{\nu} B_\nu(a) = \{ a \}, \quad \bigcup_{\nu} B_\nu(a) = \bigcup_{\nu} S_\nu(a) = \mathbb{Q}_p. \quad (6)$$
Elementary \( p \)-adic functions\(^{18}\) are given by the series of the same form as in the real case, e.g.,

\[
\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!},
\]

\[
\sinh x = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!}, \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!},
\]

\[
\tanh x = \sum_{k=2}^{\infty} \frac{2^k (2^k - 1) B_k x^{2k-1}}{k!}, \quad \coth x = \frac{1}{x} + \sum_{k=2}^{\infty} \frac{2^k B_k x^{2k-1}}{k!},
\]

where \( B_k \) are Bernoulli’s numbers. These functions have the same domain of convergence \( G_p = \{ x \in Q_p : |x|_p < |2|_p \} \). Note the following \( p \)-adic norms of the hyperbolic functions: \(|\sinh x|_p = |x|_p\) and \(|\cosh x|_p = 1\).

Real and \( p \)-adic numbers are unified in the form of the adeles\(^{19}\). An adele is an infinite sequence

\[
a = (a_\infty, a_2, \ldots, a_p, \ldots),
\]

where \( a_\infty \in Q_\infty \), and \( a_p \in Q_p \), with restriction that \( a_p \in Z_p \) (\( Z_p = \{ x \in Q_p : |x|_p \leq 1 \} \)) for all but a finite set \( S \) of primes \( p \).

If we introduce \( \mathcal{A}(S) = Q_\infty \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p \) then the space of all adeles is \( \mathcal{A} = \bigcup_S \mathcal{A}(S) \), which is a topological ring. Namely, \( \mathcal{A} \) is a ring with respect to the componentwise addition and multiplication. A principal adele is a sequence \((r, r, \ldots, r, \ldots) \in \mathcal{A} \), where \( r \in Q \). Thus, the ring of principal adeles, which is a subring of \( \mathcal{A} \), is isomorphic to \( Q \).

An important function on \( \mathcal{A} \) is the additive character \( \chi(x), x \in \mathcal{A} \), which is a continuous and complex-valued function with basic properties:

\[
|\chi(x)|_\infty = 1, \quad \chi(x + y) = \chi(x)\chi(y).
\]

This additive character may be presented as

\[
\chi(x) = \prod_v \chi_v(x_v) = \exp(-2\pi i x_\infty) \prod_p \exp(2\pi i \{x_p\}_p),
\]

where \( v = \infty, 2, \ldots, p, \ldots \), and \( \{x\}_p \) is the fractional part of the \( p \)-adic number \( x \).

Map \( \varphi : \mathcal{A} \to C \), which has the form

\[
\varphi(x) = \varphi_\infty(x_\infty) \prod_{p \in S} \varphi_p(x_p) \prod_{p \notin S} \Omega(|x_p|_p),
\]

where \( \varphi_\infty(x_\infty) \in D(Q_\infty) \) is an infinitely differentiable function on \( Q_\infty \) and falls to zero faster than any power of \(|x_\infty|_\infty\) as \(|x_\infty|_\infty \to \infty\), \( \varphi_p(x_p) \in D(Q_p) \) is a locally constant function with compact support, and

\[
\Omega(|x_p|_p) = \begin{cases} 1, & |x_p|_p \leq 1, \\ 0, & |x_p|_p > 1, \end{cases}
\]
is called an elementary function on $\mathcal{A}$. Finite linear combinations of elementary functions (13) make the set of the Schwartz-Bruhat functions $\mathcal{D}(\mathcal{A})$. The existence of $\Omega$-function is unavoidable for a construction of any adelic model. The Fourier transform is
\[ \hat{\varphi}(\xi) = \int_{\mathcal{A}} \varphi(x) \chi(\xi x) dx \] (15)
and it maps one-to-one $\mathcal{D}(\mathcal{A})$ onto $\mathcal{D}(\mathcal{A})$. It is worth noting that $\Omega$-function is a counterpart of the Gaussian in the real case, since it is invariant with respect to the Fourier transform.

The integrals of the Gauss type over the $p$-adic sphere $S_\nu$, $p$-adic ball $B_\nu$ and over any $Q_v$ are:
\[ \int_{S_\nu} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} \lambda_p(\alpha)|2\alpha|^{-1/2} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), & |\frac{\beta}{2\alpha}| = p^\nu, \\ 0, & |\frac{\beta}{2\alpha}| \neq p^\nu, \end{cases} \] (16)
\[ \text{for } |4\alpha| \geq p^{2-2\nu}, \]
\[ \int_{B_\nu} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} p^\nu \Omega(p^\nu |\beta|_p), & |\alpha|_p p^{2\nu} \leq 1, \\ \frac{\lambda_v(\alpha)}{|2\alpha|_p^2} \chi_p\left(-\frac{\beta^2}{4\alpha}\right) \Omega\left(p^{-\nu} |\frac{\beta}{2\alpha}|_p\right), & |\alpha|_p p^{2\nu} > 1, \end{cases} \] (17)
\[ \int_{Q_v} \chi_p(\alpha x^2 + \beta x) dx = \lambda_v(\alpha)|2\alpha|^{-1/2} \chi_v\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0. \] (18)
The arithmetic functions $\lambda_v(\alpha) : Q_v \mapsto C$, where $v = \infty, 2, 3, 5, \ldots$, have the following properties:
\[ |\lambda_v(\alpha)|_\infty = 1, \quad \lambda_v(0) = 1, \quad \lambda_v(ab^2) = \lambda_v(a), \] (19)
\[ \lambda_v(a)\lambda_v(b) = \lambda_v(a+b)\lambda_v(a^{-1} + b^{-1}), \] (20)
where $a \neq 0, \ b \neq 0$.

3. $p$-Adic and Adelic Quantum Mechanics
In foundations of standard quantum mechanics (over $R$) one usually starts with a representation of the canonical commutation relation
\[ [\hat{q}, \hat{k}] = i\hbar, \] (21)
where $q$ is a spatial coordinate and $k$ is the corresponding momentum. It is well known that the procedure of quantization is not unique. In formulation of $p$-adic quantum mechanics$^{10,11}$ the multiplication $\hat{q}\psi \rightarrow x\psi$ has no meaning for $x \in Q_p$ and $\psi(x) \in C$. Also, there is no possibility to define $p$-adic ”momentum” or ”Hamiltonian” operator. In the real case they are infinitesimal generators of space and time translations, but, since $Q_p$ is disconnected field, these infinitesimal transformations
become meaningless. However, finite transformations remain meaningful and the corresponding Weyl and evolution operators are $p$-adically well defined.

Canonical commutation relation in $p$-adic case can be represented by the Weyl operators ($\hbar = 1$)

$$\hat{Q}_p(\alpha)\psi_p(x) = \chi_p(\alpha x)\psi_p(x)$$

$$\hat{K}_p(\beta)\psi(x) = \psi_p(x + \beta).$$

Now, instead of the relation (21) in the real case, we have

$$\hat{Q}_p(\alpha)\hat{K}_p(\beta) = \chi_p(\alpha \beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha)$$

in the $p$-adic one. It is possible to introduce the family of unitary operators

$$\hat{W}_p(z) = \chi_p(-\frac{1}{2}qk)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad z \in Q_p \times Q_p,$$

that is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements $(z, \alpha)$ with the group product

$$(z, \alpha) \cdot (z', \alpha') = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z')),$$

where $B(z, z') = -kq' + qk'$ is a skew-symmetric bilinear form on the phase space.

Dynamics of a $p$-adic quantum model is described by a unitary operator of evolution $U(t)$ without using the Hamiltonian. Instead of that, the evolution operator has been formulated in terms of its kernel $\mathcal{K}_t(x, y)$

$$U_p(t)\psi(x) = \int_{Q_p} \mathcal{K}_t(x, y)\psi(y)dy.$$ \hspace{1cm} (27)

In this way $^{10}$ $p$-adic quantum mechanics is given by a triple

$$(L_2(Q_p), \mathcal{W}_p(z_p), U_p(t_p)).$$ \hspace{1cm} (28)

Keeping in mind that standard quantum mechanics can be also given as the corresponding triple, ordinary and $p$-adic quantum mechanics can be unified in the form of adelic quantum mechanics$^6,12$

$$(L_2(A), W(z), U(t)).$$ \hspace{1cm} (29)

$L_2(A)$ is the Hilbert space on $A$, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(A)$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(A)$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_A \mathcal{K}_t(x, y)\psi(y)dy = \prod_v \int_{Q_v} \mathcal{K}^{(v)}_t(x_v, y_v)\psi^{(v)}(y_v)dy_v.$$ \hspace{1cm} (30)

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_{\alpha}t)\psi_{\alpha\beta}(x),$$ \hspace{1cm} (31)
where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = (E_\infty, E_2, ..., E_p, ...)$ is the corresponding energy, indices $\alpha$ and $\beta$ denote energy levels and their degeneration. Note that any adelic eigenfunction has the form

$$\Psi(x) = \Psi_\infty(x_\infty) \prod_{p \in S} \Psi_p(x_p) \prod_{p \not\in S} \Omega(|x_p|_p), \quad x \in A,$$

(32)

where $\Psi_\infty \in L^2(R)$, $\Psi_p \in L^2(Q_p)$.

A suitable way to calculate $p$-adic propagator $K_p(x''; x', t')$ is to use Feynman’s path integral method, i.e.

$$K(x'', t''; x', t') = \int_{x'', t''}^{x', t'} \chi_p \left( -\frac{1}{\hbar} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) Dq.$$

(33)

It has been evaluated\textsuperscript{21,22} for quadratic Lagrangians in the same way for real and $p$-adic case and the following exact general expression is obtained:

$$K_v(x'', t''; x', t') = \lambda_v \left( -\frac{1}{2\hbar} \frac{\partial^2 S}{\partial x'' \partial x'} \right) \left| \frac{\partial^2 S}{\partial x'' \partial x'} \right| \frac{1}{2} \chi_v \left( -\frac{1}{\hbar} \tilde{S}(x'', t''; x', t') \right),$$

(34)

where $\lambda_v$ functions satisfy relations (19) and (20). When one has a system with more than one dimension with uncoupled spatial coordinates, then the total propagator is the product of the corresponding one-dimensional propagators.

As an illustration of $p$-adic and adelic quantum-mechanical models, the following one-dimensional systems with the quadratic Lagrangians were considered: a free particle and harmonic oscillator\textsuperscript{2,5,12}, a particle in a constant field\textsuperscript{23}, a free relativistic particle\textsuperscript{20} and a harmonic oscillator with time-dependent frequency\textsuperscript{24}.

Adelic quantum mechanics takes into account ordinary as well as $p$-adic quantum effects and may be regarded as a starting point for construction of a more complete superstring and M-theory. In the low-energy limit adelic quantum mechanics becomes the ordinary one\textsuperscript{20}.

4. $p$-Adic and Adelic Quantum Cosmology

Adelic quantum cosmology is an application of adelic quantum theory to the universe as a whole\textsuperscript{15}. Adelic quantum theory unifies both, $p$-adic and standard quantum theory\textsuperscript{5}. In the path integral approach to standard quantum cosmology, the starting point is Feynman’s idea that the amplitude to go from one state with intrinsic metric $h'_{ij}$, and matter configuration $\phi'$ on an initial hypersurface $\Sigma'$, to another state with metric $h''_{ij}$, and matter configuration $\phi''$ on a final hypersurface $\Sigma''$, is given by a functional integral of the form

$$\langle h''_{ij}, \phi''; \Sigma''| h'_{ij}, \phi', \Sigma' \rangle = \int D(g_{\mu\nu})_\infty D(\Phi)_\infty \chi_\infty (-S_\infty[g_{\mu\nu}, \Phi]),$$

(35)

over all four-geometries $g_{\mu\nu}$, and matter configurations $\Phi$, which interpolate between the initial and final configurations\textsuperscript{1}. In this expression $S[g_{\mu\nu}, \Phi]$ is an Einstein-
Hilbert action for the gravitational and matter fields (which can be massless, minimally or conformally coupled with gravity). This action can be calculated if we use metric in the standard 3+1 decomposition

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j, \]

where \( N \) and \( N_i \) are the lapse and shift functions. To perform \( p \)-adic and adelic generalization we make first \( p \)-adic counterpart of the action using form-invariance under change of real to the \( p \)-adic number fields. Then we generalize (35) and introduce \( p \)-adic complex-valued cosmological amplitude

\[ (h''_{ij}, \phi'', \Sigma'' | h', \phi', \Sigma')_p = \int D(g_{\mu\nu})_p D(\Phi)_p \chi_p (-S_p[g_{\mu\nu}, \Phi]). \] (37)

Since the space of all three-metrics and matter field configurations on a three-surface (superspace), has infinitely many dimensions, one takes some approximation. A useful approximation is to truncate the infinite degrees of freedom to a finite surface (superspace), has infinitely many dimensions, one takes some approximation.

\[ \text{ usual metric is a Robertson-Walker one, in which spatial section has the form} \]

\[ h_{ij} dx^i dx^j = a^2(t) d\Omega_2^3 = a^2(t) \left[ d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \] (38)

If we use also a single scalar field \( \phi \), as a matter content of the model, minisuperspace coordinates are \( \{a, \phi\} \). More generally, models can be homogeneous but also anisotropic ones, and they will be here also considered. All such models can be classified in: (i) Kantowski-Sachs models with spatial sections of topology \( S^1 \times S^2 \)

\[ h_{ij} dx^i dx^j = a^2(t) d\varphi^2 + b^2(t) d\Omega_2^3, \] (39)

where \( d\Omega_2^3 \) is the metric on the two-sphere, and minisuperspace coordinates are \( \{a, b, \phi\} \); (ii) Bianchi models, which are the most general homogeneous cosmologies with a three-dimensional group of isometries. The three-metric of each of these models can be written in the form \( h_{ij} dx^i dx^j = h_{ij}(t) \omega^i \otimes \omega^j \), where \( \omega^i \) are the invariant 1-forms associated with the isometry group. The simplest example is the Bianchi I model with \( \omega^1 = dx, \omega^2 = dy \) and \( \omega^3 = dz \),

\[ h_{ij} dx^i dx^j = a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2, \] (40)

and minisuperspace coordinates are \( \{a, b, c, \phi\} \). For the minisuperspace models, functional integrals in (35) and (37) are reduced to functional integrals over three-metric and configuration of matter fields, and to another usual integral over the lapse function \( N \). For the boundary condition \( q_\alpha(t_2) = q''_\alpha, q_\alpha(t_1) = q'_\alpha \) in the gauge \( \tilde{N} = 0 \), we have \( \nu \)-adic minisuperspace propagator

\[ \langle q''_\alpha ; q'_\alpha \rangle_\nu = \int dN K_\nu(q''_\alpha; N; q'_\alpha, 0), \] (41)
where

\[ K_\nu(q''_\alpha, N; q'_\alpha, 0) = \int Dq_{\alpha} \chi_\nu(-S_\nu[q_{\alpha}]), \quad (42) \]

is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates \((q'_\alpha, q''_\alpha)\) in a fixed time \(N\). \(S_\nu\) is the \(\nu\)-adic action of the minisuperspace model, i.e.

\[ S_\nu[q_{\alpha}] = \int_0^1 dt N \left[ \frac{1}{2N^2} f_{\alpha\beta}(q) q^\alpha q^\beta - U(q) \right], \quad (43) \]

where \(f_{\alpha\beta}\) is a minisuperspace metric \((ds^2_m = f_{\alpha\beta} dq^\alpha dq^\beta)\) with an indefinite signature \((-1, +1, +1, \ldots)\). This metric includes spatial (gravitational) components and also matter variables for the given model. The standard minisuperspace ground state wave function in the Hartle-Hawking (no-boundary) proposal\(^{14}\), will be got if one performs functional integration in the Euclidean version of

\[ \Psi_\infty[h_{ij}] = \int D(g_{\mu\nu})_\infty D(\Phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]), \quad (44) \]

over all compact four-geometries \(g_{\mu\nu}\) which induce \(h_{ij}\) at the compact three-manifold. This three-manifold is the only boundary of the all four-manifolds. If we generalize Hartle-Hawking proposal to the \(p\)-adic minisuperspace, then an adelic Hartle-Hawking wave function is an infinite product

\[ \Psi[h_{ij}] = \prod_\nu \int D(g_{\mu\nu})_\nu D(\Phi)_\nu \chi_\nu(-S_\nu[g_{\mu\nu}, \Phi]), \quad (45) \]

where path integration must be performed over both, archimedean and nonarchimedean geometries. If after calculation of the corresponding functional integrals we obtain as a result \(\Psi[h_{ij}]\) in the form (32), we will say that such cosmological model is adelic one.

As we shall see, a more successful \(p\)-adic generalization of the minisuperspace cosmological models can be performed in the framework of \(p\)-adic and adelic quantum mechanics\(^{16}\) without use of the Hartle-Hawking proposal. In such case, we examine conditions under which exist some eigenstates of the evolution operator (21).

5. \(p\)-Adic Models in the Hartle-Hawking Proposal

The Hartle-Hawking proposal for the wave function of the universe is generalized to \(p\)-adic case in Refs\(^{15,25}\). In this approach, \(p\)-adic wave function is a solution of the integral

\[ \Psi_p(q^\alpha) = \int \mathcal{D}N \mathcal{K}_p(q^\alpha, N; 0, 0), \quad (46) \]

where \(p\)-adic integration has to be performed over the \(p\)-adic ball \(B_0\).

5.1. Models of the de Sitter Type
Models of the de Sitter type are models with cosmological constant \( \Lambda \) and without matter fields. We consider two minisuperspace models of this type, with \( D = 4 \) and \( D = 3 \) space-time dimensions. The corresponding Einstein-Hilbert action is

\[
S = \frac{1}{16\pi G} \int_M d^Dx \sqrt{-g} (R - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{h} K,
\]

where \( R \) is the scalar curvature of \( D \)-manifold \( M \), \( K \) is the trace of the extrinsic curvature \( K_{ij} \) of the boundary \( \partial M \) of the \( D \)-manifold \( M \). The metric for this model\(^{26}\) is of the Robertson-Walker type

\[
ds^2 = \sigma^2 [ -N^2 dt^2 + a^2(t) d\Omega^2_{D-1} ].
\]

In this expression \( d\Omega^2_{D-1} \) denotes the metric on the unit \( (D-1) \)-sphere, \( \sigma = \frac{8\pi G}{V_D - 1} \frac{1}{(D-1)(D-2)} \) is the volume of the unit \( (D-1) \)-sphere. In the standard \( D = 3 \) case, this model is related to the multiple sphere configuration and wormhole solutions. \( \nu \)-Adic classical action for this model is

\[
\bar{S}_\nu(a'', N; a', 0) = \frac{\lambda}{2\sqrt{\lambda}} \left[ N\sqrt{\lambda} + \lambda \left( \frac{2a''a'}{\sinh(N\sqrt{\lambda})} - a'^2 + a''^2 \right) \right].
\]

Let us note that \( \lambda \) denotes here appropriately rescaled the cosmological constant \( \Lambda \).

Using (34) for the propagator of this model we have

\[
\mathcal{K}_\nu(a'', N; a', 0) = \lambda \left( \frac{2\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right) \left( \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right)^{1/2} \chi_\nu(-\bar{S}_\nu(a'', N; a', 0)).
\]

The \( p \)-adic Hartle-Hawking wave function is

\[
\Psi_p(a, \lambda) = \int_{|N|_p \leq 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left( \frac{\sqrt{\lambda} \coth(N\sqrt{\lambda})}{2} a^2 \right),
\]

which after \( p \)-adic integration becomes

\[
\Psi_p(a, \lambda) = \begin{cases} 
\Omega(|a|_p), & |\lambda|_p \leq p^{-2} \\
\frac{1}{2} \Omega(|a|_2), & |\lambda|_2 \leq 2^{-4}.
\end{cases}
\]

The de Sitter model in \( D = 4 \) space-time dimensions is described by the metric\(^{27}\)

\[
dds^2 = a^2 \left( -\frac{N^2}{q(t)} dt^2 + q(t) d\Omega^2_3 \right).
\]

For the \( \nu \)-adic classical action

\[
\bar{S}_\nu(q'', T; q', 0) = \frac{\lambda^2 T^3}{24} - [\lambda(q' + q'') - 2] \frac{T^4}{4} - \frac{(q'' - q')^2}{8T}
\]

the corresponding propagator is

\[
\mathcal{K}_\nu(q'', T|q', 0) = \frac{\lambda \nu(-8T)}{|4T|^{1/4}} \chi_\nu(-\bar{S}_\nu(q'', T|q', 0)).
\]
We obtain the $p$-adic Hartle-Hawking wave function by the integral
\[
\Psi_p(q, \lambda) = \int_{|T|_p \leq 1} dT \frac{\chi_p(-8T)}{|4T|_p^{1/2}} \chi_p \left( -\frac{\lambda^2 T^3}{24} + \frac{(\lambda q - 2) T}{4} + \frac{q^2 T}{8T} \right),
\] (56)
and as a result we get \cite{15,25} also $\Omega(|q|_p)$ function with the condition $\lambda = 4 \cdot 3 \cdot l$, $l \in \mathbb{Z}$.

5.2. Model with a Homogeneous Scalar Field

To deal with the models of the de Sitter type is very instructive. Although these models are without matter content, they are in quantum cosmology of such significance as the model of harmonic oscillator in quantum mechanics. However, it is also important to consider models with some matter content. If we use metric in the form \cite{28}
\[
dS^2 = \sigma^2 \left( -N^2(t) \frac{dt^2}{a^2(t)} + a^2(t) d\Omega_3^2 \right),
\] (57)
the gravitational part of the action in the form (47) (with $D = 4$), and the corresponding action for a scalar field as
\[
\mathcal{S}_{\text{matter}} = -\frac{1}{2} \int_M d^4x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right],
\] (58)
then after some substitutions, we get the classical action and propagator as follows:
\[
\begin{align*}
\bar{S}_p(x'', y'', N|x', y', 0) &= \frac{\alpha^2 - \beta^2}{24} N^3 + \frac{1}{4} (2 - \alpha (x' + x'') - \beta (y' + y'')) N \\
&\quad - \frac{-(x'' - x')^2 + (y'' - y')^2}{8N}, \\
K_p(x'', y'', N|x', y', 0) &= \frac{1}{|4N|_p} \chi_p(-\bar{S}_p(x'', y'', N|x', y', 0)).
\end{align*}
\] (59)

As we have shown \cite{16} for this model, a $p$-adic Hartle-Hawking wave function in the form of $\Omega$-function does not exist. This leads to the conclusion that either the above model is not adelic, or that $p$-adic generalization of the Hartle-Hawking proposal is not an adequate one. However, if in the action (59) we take $\beta = 0$, $y = 0$, then we get classical action for the de Sitter model (54), and such model, as we showed it, is the adelic one. The similar conclusion holds also for some other models in which minisuperspace is not one-dimensional. This is a reason to regard $p$-adic and adelic minisuperspace quantum cosmology just as the corresponding application of $p$-adic and adelic quantum mechanics without the Hartle-Hawking proposal.

6. Minisuperspace Models in $p$-Adic and Adelic Quantum Mechanics

In this approach we investigate conditions under which quantum-mechanical $p$-adic ground state exists in the form of $\Omega$-function and some other eigenfunctions. This approach leads to the desired result and it enables adelization of all exactly soluble minisuperspace cosmological models, usually with some restrictions on the
parameters of the models. The necessary condition for the existence of an adelic model is an existence of \( p \)-adic quantum-mechanical ground state \( \Omega(q_{\alpha}|_p) \), i.e.

\[
\int_{|q_{\alpha}'|_p \leq 1} K_p(q_{\alpha}'', N; q_{\alpha}', 0)dq_{\alpha}' = \Omega(|q_{\alpha}'|_p).
\]

(61)

Analogously, if a system is in the state \( \Omega(p''|q_{\alpha}|_p) \), then its kernel must satisfy

\[
\int_{|q_{\alpha}'|_p \leq p^{-\nu}} K_p(q_{\alpha}'', N; q_{\alpha}', 0)dq_{\alpha}' = \Omega(p''|q_{\alpha}|_p).
\]

(62)

If \( p \)-adic ground state is of the form of the \( \delta \)-function, we will investigate conditions under which the corresponding kernel of the model satisfies equation

\[
\int_{Q_p} K_p(q_{\alpha}'', T; q', 0)\delta(p'' - |q_{\alpha}|_p)dq_{\alpha}' = \delta(p'' - |q_{\alpha}'|_p),
\]

(63)

with zero energy \( E = 0 \).

In the following, we apply (61), (62) and (63) to the some minisuperspace models.

6.1. Models of the de Sitter type

6.1.1. The de Sitter Model in \( D = 3 \) Dimensions

By application of the above exposed formalism of \( p \)-adic quantum mechanics, for this model we found \( \Psi_p(a, N) = \{ \right\}

\[
\Omega(|a|_p), \quad |N|_p \leq 1, \quad p \neq 2,
\]

\[
\Omega(|a|_2), \quad |N|_2 \leq \frac{1}{2}, \quad p = 2,
\]

(64)

with conditions \( |\lambda|_p \leq 1 \) and \( |\lambda|_2 \leq 2 \). We also found

\[
\Psi_p(a, N) = \left\{ \right.

\[
\Omega(p''|a|_p), \quad |N|_p \leq p^{-2\nu}, \quad |\lambda|_p \leq p^{2\nu},
\]

\[
\Omega(2''|a|_2), \quad |N|_2 \leq 2^{-2 - 2\nu}, |\lambda|_2 \leq 2^{1 + 4\nu},
\]

(65)

where \( \nu = 1, 2, \ldots \). The existence of the ground state in the form of the \( \delta \)-function may be investigated by the equation (63), i.e.

\[
\int_{Q_p} K_p(q_{\alpha}'', T; q', 0)\delta(p'' - |a'|)dq_{\alpha}' = \delta(p'' - |a''|),
\]

(66)

with the kernel (50), what leads to the equation

\[
\lambda_p \left( -\frac{\sqrt{\lambda}}{2 \sinh(N\sqrt{\lambda})} \right) \left[ \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right]^{1/2}_p \chi_p \left( -\frac{N}{2} + \frac{\sqrt{\lambda}}{2 \tanh(N\sqrt{\lambda})} q''^2 \right)
\]

\[
\times \int_{|q''|_p = p''} \chi_p \left( \frac{\sqrt{\lambda}}{2 \tanh(N\sqrt{\lambda})} q''^2 - \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} a''a' \right) dq_{\alpha}' = \delta(p'' - |a''|_p).
\]

(67)
The above integration is performed over $p$-adic sphere with the radius $p^\nu$ and for $|N/T|_p \leq p^{2\nu-2}$, $\nu = 1, 0, -1, \ldots$. As a result, on the left hand side we have

$$\chi_p \left( \frac{N}{2} + \frac{\sqrt{\lambda}}{2} \tanh(N\sqrt{\lambda})a''^2 \right).$$

To have an equality, the argument of the additive character must be equal or less than unity. This requirement leads to the condition

$$\left| \frac{\sqrt{\lambda} \tanh(N\sqrt{\lambda})a''^2}{2} \right|_p = p^{2\nu-2} |\lambda|_p \leq 1, \quad \Leftrightarrow \quad |\lambda|_p \leq p^{2-4\nu}.$$  

This (for the $p$-adic norms of $N$ and $\lambda$) is also related to the domain of convergence of the analytic function $\tanh$.

$$|N\sqrt{\lambda}|_p \leq |N|_p |\lambda|^{1/2}_p \leq p^{2\nu-2} \cdot p^{1-2\nu} = p^{-1}, \quad \forall \nu.$$  

If $p = 2$, then condition $|N|_2 \leq 2^{2\nu-3}$ holds for $\nu = 1, 0, -1, -2, \ldots$, and we are in the domain of convergence. Finally, we conclude that also $p$-adic ground state

$$\Psi_p(a, N) = \left\{ \begin{array}{ll} \delta(p^\nu - |a|_p), & |N|_p \leq p^{2\nu-2}, \quad |\lambda|_p \leq p^{2-4\nu}, \\ \delta(2^\nu - |a|_2), & |N|_2 \leq 2^{2\nu-3}, \quad |\lambda|_2 \leq 2^{-4\nu}. \end{array} \right. \quad (68)$$

exists for $\nu = 1, 0, -1, -2, \ldots$.

6.1.2. The de Sitter Model in D=4 Dimensions

As it was already shown, the ground states for this model exist in the forms

$$\Psi_p(q, T) = \left\{ \begin{array}{ll} \Omega(|q|_p), & |T|_p \leq 1, \quad \lambda = 4 \cdot 3 \cdot l, \quad l \in Z \\ \Omega(|q|_2), & |T|_2 \leq \frac{1}{2}, \quad \lambda = 4 \cdot 3 \cdot l, \quad l \in Z. \end{array} \right. \quad (69)$$

$$\Psi_p(q, T) = \left\{ \begin{array}{ll} \Omega(p^\nu|q|_p), & |T|_p \leq p^{-2\nu}, \quad |\lambda|_p \leq 3|\lambda|^{1/2}_p p^{3\nu}, \quad \nu = 0, 1, 2, \ldots \\ \Omega(2^\nu|q|_2), & |T|_2 \leq 2^{-2\nu}, \quad |\lambda|_2 \leq 2^{3\nu-1}, \quad \nu = 1, 2, 3, \ldots. \end{array} \right. \quad (70)$$

Looking for the existence of the $p$-adic ground state in the form of the $\delta$-function, we have to solve the integral equation

$$\frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p \left( -\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q^\nu T}{4} + \frac{q^{2\nu}}{8T} \right)$$

$$\times \int_{|q'|_p = p^\nu} \chi_p \left( \frac{q^2}{8T} + \left( \frac{\lambda T}{4} - \frac{q^{\nu}}{4T} \right) q' \right) dq' = \delta(p^\nu - |q'|_p). \quad (71)$$

After the corresponding integration, for the left hand side of the previous equation, we obtain

$$\chi_p \left( -\frac{\lambda^2 T^3}{6} - \frac{T}{2} + \frac{\lambda q^\nu}{2T} \right).$$
By the very similar analysis for the parameter $\lambda$, we get $|\lambda|_p \leq p^{2-3\nu}$, and finally

$$\Psi_p(q,T) = \left\{ \begin{array}{ll}
\delta(p^\nu - |q|_p), & |T|_p \leq p^{2\nu-2}, \quad |\lambda|_p \leq p^{2-3\nu}, \\
\delta(2^\nu - |q|_2), & |T|_2 \leq 2^{2\nu-1}, \quad |\lambda|_2 \leq 2^{-3\nu},
\end{array} \right. \tag{72}$$

where $\nu = 1, 0, -1, -2, \cdots$ if $p \neq 2$, and $\nu = 0, -1, -2, \cdots$ if $p = 2$.

6.2. Model with a Homogeneous Scalar Field

This is two-dimensional minisuperspace model with two decoupled degrees of freedom. Its ground state is of the form$^{16}$ $\Omega(|x|_p)\Omega(|y|_p)$, i.e.

$$\Psi_p(x,y,N) = \left\{ \begin{array}{ll}
\Omega(|x|_p)\Omega(|y|_p), & |N|_p \leq 1, \\
\Omega(|x|_2)\Omega(|y|_2), & |N|_2 \leq \frac{1}{2},
\end{array} \right. \tag{73}$$

with $\alpha = 4 \cdot 3 \cdot l_1$, $\beta = 4 \cdot 3 \cdot l_2$, $l_1, l_2 \in \mathbb{Z}$, and also

$$\Psi_p(x,y,N) = \left\{ \begin{array}{ll}
\Omega(p^\nu |x|_p)\Omega(p^\mu |y|_p), & |\alpha|_p \leq |3|_{p}^{1/2} p^{3\nu}, \quad |\beta|_p \leq |3|_{p}^{1/2} p^{3\mu}, \\
\Omega(2^\nu |x|_2)\Omega(2^\mu |y|_2), & |\alpha|_2 \leq 2^{3\nu-1}, \quad |\beta|_2 \leq 2^{3\mu-1}.
\end{array} \right. \tag{74}$$

As in the previous cases, we also investigate the existence of the vacuum state of the form $\delta(p^\nu - |x|_p)\delta(p^\mu - |y|_p)$. After the very similar calculations as in the subsection 6.1, we find $p$-adic wave function for the ground state

$$\Psi_p(x,y,N) = \left\{ \begin{array}{ll}
\delta(p^\nu - |x|_p)\delta(p^\mu - |y|_p), & |N|_p \leq p^{2\nu,\mu-2}, \quad |\alpha, \beta|_p \leq p^{2-3\nu,\mu}, \\
\delta(2^\nu - |x|_2)\delta(2^\mu - |y|_2), & |N|_2 \leq 2^{2\nu,\mu-1}, \quad |\alpha, \beta|_2 \leq 2^{-3\nu,\mu},
\end{array} \right. \tag{75}$$

where $\nu, \mu = 0, -1, -2, \ldots$.

6.3. Anisotropic Bianchi Model with Three Scale Factors

In this case we start with metric$^{29}$

$$ds^2 = a^2 \left[ -\frac{N^2(t)}{a^2(t)} dt^2 + a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2 \right]. \tag{76}$$

It leads to the action

$$S[a,b,c] = \frac{1}{2} \int_0^1 dt \left[ -\frac{a}{N} (\dot{abc} + \dot{ab} \dot{c} + \dot{a} \dot{b} \dot{c}) - Nbc\lambda \right]. \tag{77}$$

By means of the substitution

$$x = \frac{bc + a^2}{2}, \quad y = \frac{bc - a^2}{2}, \quad z^2 = a^2 \dot{b} \dot{c} \tag{78}$$

we obtain the classical action and propagator in the form

$$S_v(x'', y'', z'', N; x', y', z', 0) = -\frac{1}{4N} \left[ (x'' - x')^2 - (y'' - y')^2 + 2(z'' - z')^2 \right] - \frac{\lambda N}{4} \left[ (x' + x'') + (y' + y'') \right], \tag{79}$$

14
\[ \mathcal{K}_v(x'',y'',z'',N;x',y',z',0) = \frac{\lambda_v(-2N)}{|4^{1/3}N|^{3/2}} \chi_v \left( -\tilde{S}_v(x'',y'',z'',N;x',y',z',0) \right). \]  

(80)

In the above way, one gets the \( p \)-adic eigenstates

\[ \Psi_p(x,y,z,N) = \begin{cases} \Omega(|x|_p)\Omega(|y|_p)\Omega(|z|_p), & |N|_p \leq 1, |\lambda|_p \leq 1, \\ \Omega(|x|_2)\Omega(|y|_2)\Omega(|z|_2), & |N|_2 \leq \frac{1}{2}, |\lambda|_2 \leq 2, \end{cases} \]

(81)

and

\[ \Psi_p(x,y,z,N) = \begin{cases} \prod_{i=1}^{2} \Omega(p^{\nu_i}|x|_p), & |N|_p \leq p^{-2\nu_i}, |\lambda|_p \leq p^{3\nu_i}, p^{3\nu_2}, \\ \prod_{i=1}^{3} \Omega(2^{\nu_i}|x|_2), & |N|_2 \leq 2^{-2\nu_1,2-1} 2^{-2\nu_3-2}, \end{cases} \]

(82)

where \( |\lambda|_2 \leq 2^{3\nu_1,2+1} \).

For this model also exists ground state

\[ \Psi_p(x,y,z,N) = \begin{cases} \delta(p^{\nu_1} - |x''|_p)\delta(p^{\nu_2} - |y''|_p)\delta(p^{\nu_3} - |z''|_p), & |N|_p \leq p^{-2\nu_1,2} \\ \delta(2^{\nu_1} - |x''|_2)\delta(2^{\nu_2} - |y''|_2)\delta(2^{\nu_3} - |z''|_2), & |N|_2 \leq 2^{-2\nu_1,2}, \end{cases} \]

(83)

with conditions: \( |N|_p \leq p^{2\nu_1,2,3-2} \), if \( p \neq 2 \), and \( |N|_2 \leq 2^{2\nu_2-3} \), \( \nu_i = 1, 0, -1, -2, \ldots \) if \( p = 2 \).

6.4. Some two dimensional models

There is a class of two-dimensional minisuperspace models which, after some transformations, obtain the form of two oscillators. These models are: the isotropic Friedmann model with conformally and minimally coupled scalar field, and the anisotropic vacuum Kantowski-Sachs model. For all these three models the corresponding action may be written as

\[ S = \frac{1}{2} \int_0^1 dt N \left[ -\frac{\dot{x}^2}{N^2} + \frac{\dot{y}^2}{N^2} + x^2 - y^2 \right], \]

(84)

i.e. this is the action for two linear oscillators, but one of them has a negative energy. This classical action leads to the propagator

\[ \mathcal{K}_p(y'',x'',N;y',x',0) = \frac{1}{|N|_p} \chi_p \left( \frac{x''^2 + y''^2 - y'^2 - y'''^2}{2 \tan N} + \frac{y'y'' - x'x''}{\sin N} \right). \]

(85)

The linear harmonic oscillator was analyzed from \( p \)-adic, as well as from the adelic point of view. One can show that in the \( p \)-adic region of convergence of analytic functions \( \sin N \) and \( \tan N \), which is \( G_p = \{ N \in Q_p : |N|_p \leq 2|p|_p \} \), exist vacuum states \( \Omega(|x|_p) \Omega(|y|_p) \Omega(p^\nu|x|_p) \Omega(p^\nu|y|_p), \nu \in Z \), and also

\[ \Psi_p(x,y,N) = \begin{cases} \delta(p' - |x|_p)\delta(p'' - |y|_p), & |N|_p \leq p^{2\nu-2}, \\ \delta(2^\nu - |x|_2)\delta(2^\nu - |y|_2), & |N|_2 \leq 2^{2\nu-3}, \end{cases} \]

(86)

where \( \nu = 0, -1, -2, \ldots \).
7. Concluding Remarks

In this paper, we find applications of $p$-adic numbers in quantum cosmology very interesting. It gives new possibilities to investigate the structure of space-time at the Planck scale.

In the Hartle-Hawking approach the wave function of a spatially closed universe is defined by Feynman’s path integral method. The action is a function of the gravitational and matter fields, and integration is performed over all compact real four-metrics connecting two three-space states. According to Feynman’s integration over all real compact metrics, this approach generalizes to all corresponding compact $p$-adic metrics. However, it does not lead to the adequate adelic picture and generalization for a wide class of the minisuperspace models.

From the other side, the consideration of minisuperspace models in the framework of adelic quantum mechanics gives the appropriate adelic generalization. Moreover, we can conclude that all these models lead to the picture of space-time as a discrete one.

Namely, for all the above models there exists adelic wave function

$$\Psi(q^1, ..., q^n) = \prod_{\alpha=1}^{n} \Psi_{\infty}(q_{\infty}^\alpha) \prod_p \prod_{\alpha=1}^{n} \Omega(|q_{\alpha}^\alpha|_p),$$

(87)

where $\Psi_{\infty}(q_{\infty}^\alpha)$ are the corresponding wave functions of the universe in standard cosmology. Adopting the usual probability interpretation of the wave function (87) in rational points of $q^\alpha$, we have

$$|\Psi(q^1, ..., q^n)|_{\infty}^2 = \prod_{\alpha=1}^{n} |\Psi_{\infty}(q^\alpha)|_{\infty}^2 \prod_p \prod_{\alpha=1}^{n} \Omega(|q_{\alpha}^\alpha|_p),$$

(88)

because $(\Omega(|q_{\alpha}^\alpha|_p))^2 = \Omega(|q_{\alpha}^\alpha|_p)$. As a consequence of $\Omega$-function properties we have

$$|\Psi(q^1, ..., q^n)|_{\infty}^2 = \begin{cases} |\Psi_{\infty}(q^\alpha)|_{\infty}^2, & q^\alpha \in \mathbb{Z}, \\ 0, & q^\alpha \in \mathbb{Q}\setminus\mathbb{Z}. \end{cases}$$

(89)

This result leads to the discretization of minisuperspace coordinates $q^\alpha$, because probability is nonzero only in the integer points of $q^\alpha$. Keeping in mind that $\Omega$ function is invariant with respect to the Fourier transform, this conclusion is also valid for the momentum space. Note that this kind of discreteness depends on adelic quantum state of the universe. When system is in an excited state, then the sharp discrete structure disappears, and minisuperspace, as well as configuration space in quantum mechanics, demonstrate usual properties of real space.

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