Spectral Zeta Functions in Non-Commutative Spacetimes

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Formulas for the most general case of the zeta function associated to a quadratic+linear+constant form (in \( Z \)) are given. As examples, the spectral zeta functions \( \zeta_\alpha(s) \) corresponding to bosonic (\( \alpha = 2 \)) and to fermionic (\( \alpha = 3 \)) quantum fields living on a noncommutative, partially toroidal spacetime are investigated. Simple poles show up at \( s = 0 \), as well as in other places (simple or double, depending on the number of compactified, noncompactified, and noncommutative dimensions of the spacetime). This poses a challenge to the zeta-function regularization procedure.

A fundamental property shared by all zeta functions is the existence of a reflection formula. It allows for its analytic continuation in an easy way [1]. But much better than that is a formula that yields exponentially quick convergence and everywhere, not just in the reflected domain of convergence. We here provide explicit, Chowla-Selberg-like [2] extended formulas for all possible cases involving forms of the very general type: quadratic+linear+constant [3]-[6]. Then we move to specific applications of these formulas in noncommutative field theory.

Consider the zeta function (Re \( s > p/2 \)):

\[
\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{m} \in \mathbb{Z}_{1/2}^p} \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}.
\]

The prime on a summation sign means that the point \( \vec{n} = \vec{0} \) is to be excluded from the sum. This is irrelevant when \( q \) or some component of \( \vec{c} \) is non-zero but, on the contrary, it becomes an inescapable condition in the case when \( c_1 = \cdots = c_p = q = 0 \). We can view the expression inside the square brackets of the zeta function as a sum of a quadratic, a linear, and a constant form, namely, \( Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \tilde{q} \).

(i) For \( q \neq 0 \), we obtain [5]

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\text{det } A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2s/2 + p/4 - s/2 + p/4}{\sqrt{\text{det } A}} \frac{\Gamma(s)}{\Gamma(s)}
\]

\[
\times \sum_{\vec{m} \in \mathbb{Z}_{1/2}^p} \cos(2\pi \vec{m} \cdot \vec{c}) (\tilde{m}^T A^{-1} \tilde{m})^{s/2 - p/4}
\]

\[
\times K_{p/2-s} \left( 2\pi \sqrt{2q \tilde{m}^T A^{-1} \tilde{m}} \right),
\]

where \( K_{p/2-s} \) is the modified Bessel function of the second kind and the subindex \( 1/2 \) in \( \mathbb{Z}_{1/2}^p \) means that in this sum, only half of the vectors \( \vec{m} \in \mathbb{Z}^p \) enter. We have denoted this formula, Eq. (2), by the acronym ECS1. It is notorious how the only pole of this inhomogeneous Epstein zeta function appears, explicitly, at \( s = p/2 \), where it belongs. Its residue is given by:

\[
\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\text{det } A)^{-1/2}.
\]

(ii) In the case \( q = 0 \) but \( c_1 \neq 0 \), we obtain...
\[ \zeta_{A_p;\pi,0}(s) = \frac{2^s}{\Gamma(s)} \left( \det A_{p-1} \right)^{-1/2} \sum_{n=0}^{p-1} \left( a_{11} - \bar{a}_{p-1}^T A_{p-1}^{-1} \bar{a}_{p-1} \right) \right]^{(p-1)/2-s} \times \Gamma(s - (p - 1)/2) \left[ \zeta_H(2s - p + 1, c_1) + \zeta_H(2s - p + 1, 1 - c_1) \right] \times 4 \pi^s \left( a_{11} - \bar{a}_{p-1}^T A_{p-1}^{-1} \bar{a}_{p-1} \right) \right]^{(p-1)/4-s/2} \times \sum_{n=1}^{\infty} \left( \bar{m}_j A_{j-1}^{-1} \right) s/2-j/4 \right) \times K_{(p-1)/2-s} \left( 2\pi n \sqrt{a_{p-j} m_j A_{j-1}^{-1} \bar{m}_j} \right), \right. \]
responding to bosonic and fermionic operators in this system are of a different kind, never considered before. And, moreover, they can be most conveniently written in terms of the zeta functions above. What is also nice is the fact that a unified treatment (with just one zeta function) can be given for both cases, the nature of the field appearing there as a simple parameter, together with those corresponding to the numbers of compactified, noncompactified, and noncommutative dimensions of the spacetime.

The spectral zeta function for the corresponding (pseudo-)differential operator can be written in the form [7]

\[ \zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{\Gamma(s - (d + 1)/2)}{\Gamma(s)} \times \sum_{\vec{n} \in \mathbb{Z}^d} \frac{I(\vec{n})^{(d+1)/2} \left[ 1 + \Lambda \theta^{2-2\alpha} Q(\vec{n})^{-\alpha} \right]^{(d+1)/2-s}}{\left( d + 1 - 2s \right) R^{d+1-2s}} \]

where \( V = \text{Vol}(\mathbb{R}^{d+1}) \), the volume of the non-compact part, and \( Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2 \), a diagonal quadratic form, being the compactification radii \( R_j = a_j^{-1/2} \). The spectral zeta function for the compactified, noncompactified, and noncommutative systems are of a different kind, never considered before, with the result:

\[ \zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{\Gamma(s - (d + 1)/2)}{\Gamma(s)} \times \sum_{\vec{n} \in \mathbb{Z}^d} \frac{I(\vec{n})^{(d+1)/2} \left[ 1 + \Lambda \theta^{2-2\alpha} Q(\vec{n})^{-\alpha} \right]^{(d+1)/2-s}}{\left( d + 1 - 2s \right) R^{d+1-2s}} \]

being now the quadratic form: \( I(\vec{n}) = \sum_{j=1}^p n_j^2 \).

After some calculations, this zeta function can be written in terms of the zeta functions that I have considered before, with the result:

\[ \zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - (d + 1)/2)}{l! \Gamma(s)} \times (-\Lambda \theta^{2-2\alpha})^l \zeta_{Q,0,0}(s + \alpha l - (d + 1)/2), \]

which reduces, in the particular case of equal radii, to

\[ \zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2} R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - (d + 1)/2)}{l! \Gamma(s)} \times (-\Lambda \theta^{2-2\alpha})^l \zeta_E(s + \alpha l - (d + 1)/2), \]

The pole structure of the resulting zeta function deserves a careful analysis. It differs, in fact, very much from all cases that were known in the literature till now. This is not difficult to understand, from the fact that the pole of the Epstein zeta function at \( s = p/2 - \alpha k + (d + 1)/2 = D/2 - \alpha k \), when combined with the poles of the gamma functions, yields a very rich pattern of singularities for \( \zeta_\alpha(s) \), on taking into account the different possible values of the parameters involved.

Having already given the formula (9) above — that contains everything needed to perform such calculation of pole position, residue and finite part — for its importance for the calculation of the determinant and the one-loop effective action from the zeta function, one start by specifying what happens at \( s = 0 \). Remarkably enough, a pole appears in many cases, depending on the values of \( \alpha \) for other values of \( s \). The general case will be considered later. It is convenient to classify all possible subcases according to the values of \( d \) and \( D = d + p + 1 \). The calculation has been carried out in [6] in detail, with the result:

For \( d = 2k \):

\[ \begin{cases} \text{if } D \neq 2\alpha & \Rightarrow \zeta(0) = 0, \\ \text{if } D = 2\alpha & \Rightarrow \zeta(0) = \text{finite}. \end{cases} \]

For \( d = 2k - 1 \):

\[ \begin{cases} \text{if } D \neq 2\alpha & \begin{cases} \text{finite, for } l \leq k \\ 0, \text{ for } l > k \end{cases} \Rightarrow \zeta(0) = \text{finite}, \\ \text{if } D = 2\alpha & \begin{cases} \text{pole, for } l \leq k \\ \text{finite, for } l > k \end{cases} \Rightarrow \zeta(0) = \text{pole}. \end{cases} \]

Table 1

| Pole structure of the zeta function \( \zeta_\alpha(s) \), at \( s = 0 \), according to the different possible values of \( d \) and \( D \) (2\alpha means multiple of 2\alpha.) |

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In the preceding section, for the Epstein zeta functions in Eq. (9), we obtain the following explicit analytic continuation of \( \zeta_s(\alpha) (\alpha = 2, 3) \), for bosonic and fermionic fields, to the whole complex \( s \)-plane:

\[
\zeta_s(\alpha) = \frac{2^{s-d}V}{(2\pi)^{(d+1)/2}\Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l! \Gamma(s+\alpha l-(d+1)/2)}
\times (-2^\alpha \lambda^2-2p)^l \sum_{j=0}^{p-1} (\det A_j)^{-1/2}
\times \left[ \pi^{d/2} \sum_{n=1}^{\infty} \frac{1}{n^{(d-j+1)/2-s-\alpha l}} \sum_{\vec{m} \in \mathbb{Z}^j} \right]
\times \left[ \left( \vec{m} \right)^{-(s+\alpha l)/2-(d+j+1)/4} \right]
\times K_{(d+j+1)/2-s-\alpha l} \left( \frac{2\pi n}{\sqrt{a_{p-j}\vec{m}^l A_j^{-1}\vec{m}j}} \right).
\]

The non-spurious poles of this zeta function are to be found in the terms corresponding to \( j = p-1 \). The pole structure is summarized in Table 2.

<table>
<thead>
<tr>
<th>( p ) odd</th>
<th>( p ) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a) pole/finite ( (l \geq l_1) )</td>
<td>(1b) double pole/pole ( (l \geq l_1, l_2) )</td>
</tr>
<tr>
<td>( D ) even</td>
<td>( D ) odd</td>
</tr>
<tr>
<td>(2a) pole/pole</td>
<td>(2b) pole/double pole ( (l \geq l_2) )</td>
</tr>
</tbody>
</table>

Table 2

General pole structure of the zeta function \( \zeta_s(\alpha) \), according to the different possible values of \( D \) and \( p \) being odd or even. In italics, the type of behavior corresponding to lower values of \( l \) is quoted, while the behavior shown in roman characters corresponds to larger values of \( l \).

Depending on \( D \) and \( p \) being even or odd, completely different situations arise, for different values of \( l \): from the disappearance of the pole, giving rise to a finite contribution, to the appearance of a simple or a double pole.

An application of these formulas to the calculation of the one-loop partition function corresponding to quantum fields at finite temperature, on a noncommutative flat spacetime, is given elsewhere [11].

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REFERENCES


