Planar Dirac Electron in Coulomb and Magnetic Fields:

a Bethe ansatz approach

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Abstract

The Dirac equation for an electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields is an example of the so-called quasi-exactly solvable models. The solvable parts of its spectrum was previously solved from the recursion relations. In this work we present a purely algebraic solution based on the Bethe ansatz equations. It is realised that, unlike the corresponding problems in the Schrödinger and the Klein-Gordon case, here the unknown parameters to be solved for in the Bethe ansatz equations include not only the roots of wave function assumed, but also a parameter from the relevant operator. We also show that the quasi-exactly solvable differential equation does not belong to the classes based on the algebra $sl_2$.

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Recently a new type of spectral problem, so-called quasi-exactly solvable model (QESM), was discovered by physicists and mathematicians [1]-[8]. This is a special class of quantum-mechanical problems for which analytical solutions are possible only for parts of the energy spectra and for particular values of the fundamental parameters. The reason for such quasi-exactly solvability is usually the existence of a hidden Lie-algebraic structure [2]-[6]. More precisely, quasi-exactly solvable (QES) Hamiltonian can be reduced to a quadratic combination of the generators of a Lie group with finite-dimensional representations.

The first physical example of QESM in atomic physics is the system of two electrons moving in an external oscillator potential discussed in [9, 10]. The authors of these works apparently were unaware of the mathematical development in QESM. Later, several physical QESM were discovered, which include the two-dimensional Schrödinger [11], the Klein-Gordon [12], and the Dirac equations [13] of an electron moving in an attractive/repulsive Coulomb field and a homogeneous magnetic field. The essential features shared by all these above examples are as follows. The differential equations are solved according to the standard procedure. After separating out the asymptotic behaviors of the system, one obtains an equation for the part which can be expanded as a power series of the basic variable. But instead of the two-step recursion relations for the coefficients of power series so often encountered in exactly solvable problems, one gets three-step recursion relations. The complexity of the recursion relations does not allow one to determine the energy spectrum exactly from the normalisability of the eigenfunctions. However, one can impose a sufficient condition for normalisability by terminating the series at a certain order of power of the variable; i.e. by choosing a polynomial. By doing so one could obtain exact solutions to the original problem, but only for certain energies and for specific values of the parameters of the problem. These parameters, namely, are the frequency of the oscillator potential and the external magnetic fields.

In [14] a systematic and unified algebraic treatment was given to the above-mentioned systems, with the exception of the Dirac case. This was made possible by realising that these systems are governed essentially by the same basic equation, which is quasi-exactly
solvable owing to the existence of a hidden $sl_2$ algebraic structure. This algebraic structure was first realised by Turbiner for the case of two electrons in an oscillator potential [15]. In this algebraic approach, analytic expressions of the solvable parts of the energy spectrum and the allowed parameters were expressible in terms of the roots of a set of Bethe ansatz equations.

In this paper we would like to extend the method of [14] to the planar Dirac equation of an electron in the Coulomb and magnetic fields. It turns out the a set of Bethe ansatz equation can also be set up in this case. However, unlike the systems considered in [14], here the unknown variables in the Bethe ansatz equations involved not only the roots of the wave functions assumed, but also a parameter from the relevant operator. We also demonstrate that the Bethe ansatz approach yields the same spectrum as that obtained by solving recursion relations. Finally, we show that the quasi-exactly solvability of this system is not related to the $sl_2$ algebra.

2. In 2+1 dimensions the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^\mu\nu, \quad g^\mu\nu = \text{diag}(1, -1, -1)$$

may be represented in terms of the Pauli matrices as $\gamma^0 = \sigma_3$, $\gamma^k = i\sigma_k$, or equivalently, the matrices $(\alpha_1, \alpha_2) = \gamma^0(\gamma^1, \gamma^2) = (-\sigma_2, \sigma_1)$ and $\beta = \gamma^0$. Then the Dirac equation for an electron minimally coupled to an external electromagnetic field has the form (we set $c = \hbar = 1$)

$$(i\partial_t - H_D)\Psi(t, \mathbf{r}) = 0,$$  \hspace{1cm} (2)

where

$$H_D = \alpha \mathbf{P} + \beta m - eA^0 \equiv \sigma_1 P_2 - \sigma_2 P_1 + \sigma_3 m - eA^0$$

(3)

is the Dirac Hamiltonian, $P_k = -i\partial_k + eA_k$ is the operator of generalized momentum of the electron, $A_\mu$ the vector potential of the external electromagnetic field, $m$ the rest mass of the electron, and $-e \ (e > 0)$ is its electric charge. The Dirac wave function $\Psi(t, \mathbf{r})$
is a two-component function. In an external Coulomb field and a constant homogeneous magnetic field \( B > 0 \) along the \( z \) direction, the potential \( A_\mu \) assume the following forms in the symmetric gauge

\[
A^0(r) = Ze/r \quad (e > 0), \quad A_x = -By/2, \quad A_y = Bx/2 .
\]  

(4)

We assume the wave functions to have the form

\[
\Psi(t, x) = \frac{1}{\sqrt{r}} \exp(-iEt)\psi_l(r, \varphi) ,
\]

(5)

where \( E \) is the energy of the electron, and

\[
\psi_l(r, \varphi) = \begin{pmatrix} F(r)e^{il\varphi} \\ G(r)e^{i(l+1)\varphi} \end{pmatrix}
\]

with integral number \( l \). The function \( \psi_l(r, \varphi) \) is an eigenfunction of the conserved total angular momentum \( J_z = L_z + S_z = -i\partial/\partial\varphi + \sigma_3/2 \) with eigenvalue \( j = l + 1/2 \). It should be reminded that \( l \) is not a good quantum number. Only the eigenvalues \( j \) of the conserved total angular momentum \( J_z \) are physically meaningful.

By putting Eq.(5) and (6) into (2), and taking into account of the equations

\[
P_x \pm iP_y = -ie^{\pm i\varphi} \left( \frac{\partial}{\partial r} \pm \left( \frac{i}{r}\frac{\partial}{\partial \varphi} - \frac{eBr}{2} \right) \right),
\]

(7)

we obtain

\[
\frac{dF}{dr} - \left( \frac{l + \frac{1}{2}}{r} + \frac{eBr}{2} \right) F + \left( E + m + \frac{Z\alpha}{r} \right) G = 0 ,
\]

(8)

\[
\frac{dG}{dr} + \left( \frac{l + \frac{1}{2}}{r} + \frac{eBr}{2} \right) G - \left( E - m + \frac{Z\alpha}{r} \right) F = 0 ,
\]

(9)

where \( \alpha \equiv e^2 = 1/137 \) is the fine structure constant. In a strong magnetic field the asymptotic solutions of \( F(r) \) and \( G(r) \) have the forms \( \exp(-eBr^2/4) \) at large \( r \), and \( r^\gamma \) with \( \gamma = \sqrt{(l + 1/2)^2 - (Z\alpha)^2} \) for small \( r \). One must have \( Z\alpha < 1/2 \), otherwise the wave function will oscillate as \( r \to 0 \) when \( l = 0 \) and \( l = -1 \).

Let us assume

\[
F(r) = r^\gamma \exp(-eBr^2/4) Q(r), \quad G(r) = r^\gamma \exp(-eBr^2/4) P(r) .
\]

(10)
In [13] we showed that parts of the spectrum could be analytically solved for by imposing the sufficient condition that $Q(r)$ and $P(r)$ be polynomials, thus showing that the system belongs to the QESM. The spectrum was solved in [13] from the recursion relations for the coefficients in the series expansion in $Q$ and $P$. In this paper, we will show that the same spectrum can also be obtained in a purely algebraic way. This is achieved by the method of factorisation which leads to a set of Bethe ansatz equations [13, 14].

Substituting Eq.(10) into Eq.(8) and (9) and eliminating $P(r)$ from the coupled equations, we have

\[
\left\{ \frac{d^2}{dr^2} + \left[ \frac{2\gamma}{r} - eBr + \frac{Z\alpha/r^2}{E + m + Z\alpha/r} \right] \frac{d}{dr} + E^2 - m^2 \right.
\]

\[
\left. + \frac{2EZ\alpha}{r} + \frac{l + 1/2}{r^2} - \frac{\gamma}{r^2} - eB(\Gamma + 1) \right. \]

\[
\left. + \frac{Z\alpha/r^2}{E + m + Z\alpha/r} \left[ \frac{\gamma}{r} - eBr - \frac{l + 1/2}{r} \right] \right\} Q(r) = 0 ,
\]

where $\Gamma = l + 1/2 + \gamma$. Once $Q(r)$ is solved, the form of $P(r)$ is obtainable from Eqs.(8) and (10). If we let $x = r/l_B$, $l_B = 1/\sqrt{eB}$, Eq.(11) becomes

\[
\left\{ \frac{d^2}{dx^2} + \left[ \frac{2\beta}{x} - x + \frac{Z\alpha}{x((E + m)l_Bx + Z\alpha)} \right] \frac{d}{dx} \right.
\]

\[
\left. + (E^2 - m^2)l_B^2 + \frac{2EZl_B\alpha}{x} + \frac{(l + 1/2 - \gamma)}{x^2} - (\Gamma + 1) \right. \]

\[
\left. - \frac{Z\alpha(l + 1/2 - \gamma)}{x^2[(E + m)l_Bx + Z\alpha]} - \frac{Z\alpha}{(E + m)l_Bx + Z\alpha} \right\} Q(x) = 0 .
\]

Eq.(12) can be rewritten as

\[
\left\{ \frac{d^2}{dx^2} + \left[ \frac{2\beta}{x} - x - \frac{1}{x + x_0} \right] \frac{d}{dx} + \epsilon + \frac{b}{x} - \frac{c}{x + x_0} \right\} Q(x) = 0 .
\]

Here $\beta = \gamma + 1/2$, $x_0 = Z\alpha/[(E + m)l_B]$, $\epsilon = (E^2 - m^2)l_B^2 - (\Gamma + 1)$, $b = b_0 + L/x_0$, $b_0 = 2EZ\alpha l_B$, $L = (l + 1/2 - \gamma)$, and $c = x_0 + L/x_0$. On expressing $l_B$ in the expression of $\epsilon$ in terms of $x_0$, we get

\[
\epsilon = \frac{E - m}{E + m} \left( \frac{Z\alpha}{x_0} \right)^2 - (\Gamma + 1) .
\]

It is obvious that the energy $E$ is determined once we know the values of $\epsilon$ and $x_0$. The corresponding value of the magnetic field $B$ is then obtainable from the expression $l_B = Z\alpha/[(E + m)x_0]$. Solution of $x_0$ is achieved below by means of the Bethe ansatz equations.
3. We observe that the problem of finding the spectrum for Eq. (13) is equivalent to determining the eigenvalues of the operator

\[ H = -\frac{d^2}{dx^2} - \left( \frac{2\beta}{x} - x - \frac{1}{x + x_0} \right) \frac{d}{dx} - \frac{b}{x} + \frac{c}{x + x_0} . \]  

We want to factorise the operator (15) in the form

\[ H = a^+ a + \epsilon . \]  

The eigenfunctions of the operator \( H \) at \( \epsilon = 0 \) must satisfy the equation

\[ aQ(x) = 0 . \]  

Suppose polynomial solution exist for Eq. (13), say \( Q \) equals a non-vanishing constant, or \( Q = \prod_{k=1}^{n} (x - x_k) \), where \( x_k \) are the zeros of \( Q \), and \( n \) is the degree of \( Q \). In the case where \( Q \) is a constant (which may be viewed as corresponding to \( n = 0 \)), the operators \( a \) and \( a^+ \) have the form

\[ a = \frac{d}{dx} , \quad a^+ = -\frac{d}{dx} - \left( \frac{2\beta}{x} - x - \frac{1}{x + x_0} \right) . \]  

If \( Q = \prod_{k=1}^{n} (x - x_k) \), \( a \) and \( a^+ \) will assume the form

\[ a = \frac{d}{dx} - \sum_{k=1}^{n} \frac{1}{x - x_k} \]  

(19)

and

\[ a^+ = -\frac{d}{dx} - \left( \frac{2\beta}{x} - x - \frac{1}{x + x_0} \right) - \sum_{k=1}^{n} \frac{1}{x - x_k} . \]  

We now substitute the forms of \( a \) and \( a^+ \) into Eq. (16) and compare the result with Eq. (15). This leads to conditions that must be satisfied by the various parameters and the roots \( x_k \)’s. For constant \( Q \) \((n = 0)\), one has

\[ \epsilon = b = c = 0 . \]  

(21)

The fact that \( c = 0 \) implies

\[ x_0^2 = -L . \]  

(22)
For $n \geq 1$, one gets

\begin{align}
\frac{b_0 + L}{x_0} &= 2\beta \sum_{k=1}^{n} \frac{1}{x_k}, \quad \epsilon = n, \\
x_0 + \frac{L}{x_0} &= \sum_{k=1}^{n} \frac{1}{x_k + x_0}, \\
\frac{2\beta}{x_k} - x_k - \frac{1}{x_k + x_0} - 2 \sum_{j \neq k}^{n} \frac{1}{x_j - x_k} &= 0, \quad k = 1, \ldots, n.
\end{align}

(23)

(24)

(25)

Eqs.(22), (24) and (25) constitute the set of $n + 1$ Bethe ansatz equations relevant to this Dirac system, which involve $n + 1$ unknown parameters $\{x_0, x_1, \ldots, x_n\}$. It is worthwhile to note that, unlike the corresponding equations in the Schrödinger and the Klein-Gordon case discussed in [14], this set of Bethe ansatz equations involved not only the roots $x_k$'s, but also a parameter $x_0$ from $H$. From the second equation in Eq.(23) we get

\begin{equation}
E^2 - m^2 = \frac{1}{l_B^2} (\Gamma + n + 1).
\end{equation}

(26)

Since $-1/2 \leq \Gamma \leq 0$ for $Z\alpha < 1/2$ [13], we see from Eq.(26) that the solvable parts of the spectrum must satisfy $|E| \geq m$.

So we see that the solution of the solvable parts of the spectrum $E$ boils down to solving the Bethe ansatz equations for $x_0$ in the differential operator, and the roots $x_k$ ($k = 1, \ldots, n$) of $Q(x)$. Once the value of $x_0$ for each order $n = \epsilon$ is known, the energy $E$ is given by Eq.(14). The corresponding magnetic field $B$ is then determined from the definition of $b_0$, or from Eq.(26). The Bethe ansatz equations thus provides a systematic solutions of the QES spectrum. Of course, as the order of the degree of $Q$ increases, analytical solutions of the Bethe ansatz equations becomes difficult, and one must resort to numerical methods.

**4.** In what follows we shall show the consistency of the solutions by the Bethe ansatz approach and that by the recursion relations presented in [13] for the first three lowest orders ($n = 0, 1, 2$) in $Q$. Instead of solving for $x_0$, our strategy is to eliminate it in Eq.(14) by means of the equations (22)-(25) so as to obtain an equation obeyed by $E$ for each order of $Q$. This equation is then compared with the corresponding equation obtained from the recursion relations as presented in [13].
From Eq.(21) and (22) we have $x_0^2 = -L$ and $\epsilon = 0$ when $Q$ is a constant. Substitute these values of $x_0$ and $\epsilon$ into Eq.(14), and using the fact that $\Gamma L = (Z\alpha)^2$, we obtain the corresponding value of $E$ as

$$E = -\frac{m}{2(l + \gamma + 1)}.$$  \hspace{1cm} (27)

This is the result presented in [13]. The corresponding allowed value of the magnetic field $B$ is then obtained from Eq.(26) and (27).

For $n = 1$, we find from Eqs.(14), (23), (24) and (25) that

$$\Gamma + 2 = \frac{E - m (Z\alpha)^2}{E + m \frac{x_0^2}{x_0^2}} , \hspace{1cm} (28)$$

$$b_0 + \frac{L}{x_0} = \frac{2\beta}{x_1} , \hspace{1cm} (29)$$

$$\frac{1}{x_1 + x_0} = x_0 + \frac{L}{x_0} , \hspace{1cm} (30)$$

$$\frac{2\beta}{x_1 - x_1} - \frac{1}{x_1 + x_0} = 0 . \hspace{1cm} (31)$$

Eq.(29), (30), and (31) imply $x_1 = b_0 - x_0$. Substituting $x_1$ into Eq.(30), we obtain

$$x_0^2 = L \left[ \frac{E + m}{2E(Z\alpha)^2} - 1 \right]^{-1} . \hspace{1cm} (32)$$

Then from Eq.(32) and Eq.(28), we get

$$\left[ 4(\Gamma + 1) - \frac{\Gamma}{2\alpha^2} \right] E^2 + 4Em + \frac{\Gamma}{(Z\alpha)^2}m^2 = 0 . \hspace{1cm} (33)$$

The energy $E$ can be solved from Eq.(33) by the standard formula, after which the magnetic field is determined from Eq.(26). Eq.(33) does not resemble the one obtained from recursion relation in [13]. However, on multiplying Eq.(33) by $\Gamma + 1$ and making use of the fact that $(Z\alpha)^2 = \Gamma(\Gamma - 2\gamma)$, we can show, after some algebra, that Eq.(33) is equivalent to the corresponding equation given in [13].

Finally we consider the case for $n = 2$. We have Eq.(14) with $\epsilon = 2$, together with Eqs.(23), (24) and (25) in the forms

$$\Gamma + 3 = \frac{E - m (Z\alpha)^2}{E + m \frac{x_0^2}{x_0^2}} , \hspace{1cm} (34)$$
\[ b_0 + \frac{L}{x_0} = \frac{2\beta}{x_1} + \frac{2\beta}{x_2}, \quad (35) \]
\[ \frac{1}{x_1 + x_0} + \frac{1}{x_2 + x_0} = x_0 + \frac{L}{x_0}, \quad (36) \]
\[ \frac{2\beta}{x_1} - x_1 - \frac{1}{x_1 + x_0} - \frac{2}{x_2 - x_1} = 0, \quad (37) \]
\[ \frac{2\beta}{x_2} - x_2 - \frac{1}{x_2 + x_0} - \frac{2}{x_1 - x_2} = 0. \quad (38) \]

From these equations we find \( x_1 + x_2 = b_0 - x_0 \) and \( x_1x_2 = 2\beta x_0 (b_0 - x_0) / (b_0x_0 + L) \). Putting these expressions into Eq.(36) and using the fact that \( \Gamma = 2\beta + L - 1 \), we arrive at

\[ (b_0^2 - 2\beta)x_0^2 + b_0\Gamma x_0 + \left[b_0^2 (L - 1) - L (2\beta + 1)\right] + \frac{b_0\Gamma L}{x_0} = 0. \quad (39) \]

Now multiplying Eq.(39) by \( \Gamma \), using \( \Gamma L = (Z\alpha)^2 \), and expressing \( b_0 \), \( l_B \), and \( 1/x_0^2 \) in terms of \( E \), we get finally

\[ \left\{ 4(2\Gamma + 3) - \frac{1}{(Z\alpha)^2} \left[ 6\Gamma + 2(\gamma + 1) + \frac{(2\gamma + 1)\Gamma}{\Gamma + 3} \right] \right\} E^3 \]
\[ + \left\{ 12 - \frac{1}{(Z\alpha)^2} \left[ 2(\gamma + 1) - \frac{(2\gamma + 1)\Gamma}{\Gamma + 3} \right] \right\} E^2 m \]
\[ + \frac{1}{(Z\alpha)^2} \left[ 6\Gamma + 2(\gamma + 1) + \frac{(2\gamma + 1)\Gamma}{\Gamma + 3} \right] E m^2 \]
\[ + \frac{1}{(Z\alpha)^2} \left[ 2(\gamma + 1) - \frac{(2\gamma + 1)\Gamma}{\Gamma + 3} \right] m^3 = 0. \quad (40) \]

Again, this equation does not look the same as that obtained from the recursion relations. But we can show they are in fact equivalent as they differ only by a multiplicative factor \((\Gamma + 1)(\Gamma + 2)\).

5. We now demonstrate that the QES equation (13) cannot be represented as bilinear combination of the generators of the \( sl_2 \) algebra. The question of whether there exists non-\( sl_2 \)-based one-dimensional QESM was first posed in [2] in which all \( sl_2 \)-based QESM are classified. The first example of such a kind was given in [16], which presents a potential arising in the context of the stability analysis around the kink solution for \( \phi^4 \)-type field theory in \( 1 + 1 \) dimensions.

To show that Eq.(13) is also not generated by the \( sl_2 \) algebra, let us rewrite it as

\[ \left\{ - \left( x^2 + x_0x \right) \frac{d}{dx} + \left[ x^3 + x_0x^2 + (1 - 2\beta)x - 2\beta x_0 \right] \frac{d}{dx} \right\} d \]
\[-\varepsilon x^2 + (c - b - \varepsilon x_0) x - bx_0 \} Q(x) = 0 . \quad (41)\]

Turbiner \cite{2} has shown that all $sl_2$-based second order QES differential equations can be cast into the form

\[-P_4(x) \frac{d^2 Q}{dx^2} + P_3(x) \frac{dQ}{dx} + (P_2(x) - \lambda) Q = 0 , \quad (42)\]

where

\[
\begin{align*}
P_4(x) &= a_{++} x^4 + a_{+0} x^3 + (a_{+-} + a_{00}) x^2 + a_{0-} x + a_{--} , \\
P_3(x) &= 2 (2j - 1) a_{++} x^3 + [(3j - 1) a_{+0} + b_+] x^2 \\
&+ [2j (a_{+-} + a_{00}) + a_{00} + b_0] x + ja_{0-} + b_- , \\
P_2(x) &= 2j (2j - 1) a_{++} x^2 + 2j (ja_{+0} + b_+) x + a_{00}j^2 + b_0 j .
\end{align*}
\]

Here $a_{kl}$’s and $b_k$’s ($k, l = +, 0, -$) are constants, and $j$ is a non-negative integer or half-integer. Eq.(42) corresponds to the eigenvalue problem

\[
HQ = \lambda Q , \quad H = - \sum_{k,l=+,-} a_{kl} J^k J^l + \sum_{k=+,-} b_k J^k , \quad (44)
\]

which has a polynomial solution of power $2j$ in $x$. Here $J^k$’s are the generators of $sl_2$:

\[
J^+ = x^2 \frac{d}{dx} - 2j x , \quad J^0 = x \frac{d}{dx} - j , \quad J^- = \frac{d}{dx} .
\]

Comparing Eqs.(41) and (42) we find that the two equations are inconsistent with each other. For instance, the coefficient of $x^4$ in $P_4$ requires $a_{++} = 0$, whereas the coefficient of $x^3$ in $P_3$ implies $2(2j - 1) a_{++} = 1$, which gives a non-vanishing $a_{++}$ for positive integral and half-integral values of $j$. This shows that Eq.(13) is not $sl_2$-based.

6. In conclusion, we have given an algebraic solution to the planar Dirac equation of an electron in the Coulomb and magnetic fields. The relevant Bethe ansatz equations are presented. Unlike the corresponding equations in the Schrödinger and the Klein-Gordon case discussed in \cite{14}, the unknown variables in this set of Bethe ansatz equations include not only the roots of the polynomial assumed, but also a parameter from the QES differential operator.
Equivalence between this approach and that by the recursion relations is demonstrated. Finally, we show that the QES equation for this problem does not belong to any of the classes based on the $sl_2$ algebra.

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References


