Conserved quantities in a black hole collision

S. Dain and J. A. Valiente-Kroon
Max-Planck-Institut für Gravitationsphysik, Albert Einstein Institut
Am Mühlenberg 1, 14476 Golm bei Potsdam, Germany

May 29, 2001

Abstract

The Newman-Penrose constants of the spacetime corresponding to the development of the Brill-Lindquist initial data are calculated by making use of a particular representation of spatial infinity due to H. Friedrich. The Brill-Lindquist initial data set represents the head-on collision of two non-rotating black holes. In this case one non-zero constant is obtained. Its value is given in terms of the product of the individual masses of the black holes and the square of a distance parameter separating the two black holes. This constant retains its value all along null infinity, and therefore it provides information about the late time evolution of the collision process. In particular, it is argued that the magnitude of the constants provides information about the amount of residual radiation contained in the spacetime after the collision of the black holes.

PACS: 04.20Ha, 04.20Bw, 04.20Ex

Introduction. Friedrich and Kánnár [5] have calculated an expression for the so-called Newman-Penrose[9] constants of time symmetric spacetimes in terms of the initial data on a spacelike Cauchy hypersurface. The Newman-Penrose constants are a set of 10 (5 complex) in general non-trivial conserved quantities defined as integrals of the form

\[ G_m = \int_{S^2} \Psi_0^6 2Y_{2,m} dS, \]

over cuts of future null infinity (\(S^2\)), where \(\Psi_0 = \Psi_0^5 r^{-5} + \Psi_0^6 r^{-6} + \cdots\) is the fastest decaying component of the Weyl tensor along the null geodesics with affine parameter \(r\), \(2Y_{2,m}\) are spin-2 weighted spherical harmonics, and \(m = -2, \ldots, 2\). Generalisations of these constants can be constructed to include the cases when null infinity is not smooth but polyhomogeneous [10, 11].

The interpretation and physical meaning of these constants have been a source of debate and controversy up to this day. They have been for long time regarded as a mere mathematical curiosities, devoid of any physical application. An explicit evaluation of these constants has been carried out for stationary spacetimes [9]. In this case they were found to have the following structure:

\[ G_m = (\text{dipole})^2 - (\text{monopole}) \times (\text{quadrupole}). \]

This expression shows that the Newman-Penrose have a non-trivial physical content even in the case of non-radiative spacetimes. Note however, that the Newman-Penrose constants are all zero in the case of the Schwarzschild spacetime. On the other hand, calculations of the Newman-Penrose constants for non-stationary spacetimes are rather scarce. The constants have been calculated for a class of type D boost-rotation spacetimes [7].

The Friedrich-Kánnár formula allows the evaluation of the Newman-Penrose constants of a broad class of physically interesting spacetimes, in particular certain space times describing the head on collision of two non-rotating black holes. This highly relativistic phenomenon is usually studied by means of the numerical evolution of conformally flat, time symmetric initial
The initial 3-geometries are the Brill-Lindquist data [3] and the Misner data [8]. The difference between the two initial data sets is essentially topological. The Brill-Lindquist initial data possesses three different asymptotically flat regions connected by two Einstein-Rosen bridges, whereas the in the Misner data there are only two asymptotically flat regions connected by a pair of bridges. The physical implications of this difference of topology between the two initial data sets is still not clear. Nevertheless, the expressions in the Brill-Lindquist case are mathematically much simpler to handle than those in the Misner case. Therefore, we concentrate our attention on the Brill-Lindquist data. However, a similar calculations can be in principle carried out for the Misner data. The evaluation of the Newman-Penrose constants of the Brill-Lindquist initial data has a twofold objective. First, it seeks to gain some insight on the physical interpretation of the Newman-Penrose constants in the case of radiative spacetimes by relating them to initial data quantities with a clear geometrical meaning. And second, due to the fact that the Newman-Penrose constants retain their value all along null infinity together with the possibility of knowing their value directly from the initial data, one may be able to extract information about the late time behaviour of the complicate process of the collision of two black holes.

The Brill-Lindquist data. As it has already mentioned, the Brill-Lindquist initial data is time symmetric. Therefore, it is solely in terms of the following negative-defined 3-metric:

\[ ds^2 = -\chi^4 (dr^2 + r^2 d\sigma^2), \]

with \( r = |x|, x \in \mathbb{R}^3 \), and

\[ \chi = 1 + \frac{m_1}{2|x - x_1|} + \frac{m_2}{2|x - x_2|}, \]

where \( x_1 \) and \( x_2 \) are two arbitrary points and \( | \cdot | \) denotes the euclidean distance. Without loss of generality we set the points \( x_1 \) and \( x_2 \) to lie along the z axis. The origin of the coordinate system is chosen be the middle point between the points \( x_1 \) and \( x_2 \). In the standard spherical coordinates,

\[ |x - x_1| = \left( r^2 + r_{12} r \cos \theta + (r_{12}/2)^2 \right)^{1/2}, \]
\[ |x - x_2| = \left( r^2 - r_{12} r \cos \theta + (r_{12}/2)^2 \right)^{1/2}, \]

where \( r_{12} = |x_1 - x_2| \). Thus, using the generating function of the Legendre polynomials one obtains directly the following expansions:

\[ (r^2 + r_{12} r \cos \theta + (r_{12}/2)^2)^{-1/2} = \frac{2^2 \pi r}{1} \sum_{n=0}^\infty (-1)^n \sqrt[4]{\frac{4\pi}{2n+1}} Y_{n,0} \left( \frac{r_{12}}{2r} \right)^n, \]
\[ (r^2 - r_{12} r \cos \theta + (r_{12}/2)^2)^{-1/2} = \frac{2^2 \pi r}{1} \sum_{n=0}^\infty \sqrt[4]{\frac{4\pi}{2n+1}} Y_{n,0} \left( \frac{r_{12}}{2r} \right)^n, \]

which valid for \( r > |x_1| \) and \( r > |x_2| \) respectively. Hence, the scalar field \( \chi \) defining the Brill-Lindquist 3-geometry is given by:

\[ \chi = 1 + \frac{1}{r} \sum_{n=0}^\infty \sqrt[4]{\frac{4\pi}{2n+1}} (m_2 + (-1)^n m_1) \frac{1}{2} \left( \frac{r_{12}}{2r} \right)^n Y_{n,0}. \]

We are mainly concerned with the examination of the behaviour of these initial data in the neighbourhood of the spatial infinity of one of the 3 asymptotically flat regions. To this end, we make use of Friedrich’s [4] representation of spatial infinity as a cylinder \( I = [-1,1] \times S^2 \). In order to do so, one has first to compactify the 3-geometry. After introducing a new radial coordinate \( \rho = 1/r \), one finds that the adequate conformal factor happens to be \( \theta^{-4} \) where

\[ \theta = \chi/\rho. \]

The resulting compactified 3-dimensional manifold is topologically equivalent to the 3-dimensional sphere \( S^3 \). By construction we have 3 distinguished points in this compact manifold. These
represent the infinities of the initial data. All three spatial infinities are equivalent. When the conformal metric is analytic, the conformal factor $\theta$ has near $\rho = 0$ the following form \[4\]:

$$\theta = \frac{U}{\rho} + W,$$

where $W$ and $U$ are analytic functions of the appropriate Cartesian coordinates. The function $U$ is determined completely in terms of the local geometry near infinity. Since the Brill-Lindquist data are conformally flat it follows that $U = 1$. The function $W$ contains information on the global geometry. In particular, the total mass of the data at this end is given by $2W(0)$. For the Brill-Lindquist data we have $W = \rho^{-1}(\chi - 1)$.

The cylinder of spatial infinity does not “live” in the unphysical spacetime manifold but in a bundle with base manifold the conformally compactified (unphysical) spacetime, and fibers given by $CSL(2, \mathbb{C}) = \mathbb{R}^+ \times SL(2, \mathbb{C})$. In order to make use of the results of \[5\] the angular dependence in the function $W$ is rewritten in terms of the functions $T_{m^j_k}$, which are a complete orthogonal set in $L^2(SU(2, \mathbb{C}))$. In particular, one has

$$Y_{n,0} = i^{2n} \sqrt{\frac{2n+1}{4\pi}} T_{2n}^n n.$$  

(12)

Whence the lift of the function $W$ is given by:

$$W = \sum_{n=0}^{\infty} \frac{1}{2} (m_1 + (-1)^n m_2) \left( \frac{r_{12}}{2} \right)^n \rho^n T_{2n}^n n.$$  

(13)

Therefore, we write

$$W = \sum_{n=0}^{\infty} \rho^n W_n,$$  

(14)

where

$$W_n = W_n;2n,n T_{2n}^n n,$$  

(15)

and

$$W_n;2n,n = \frac{1}{2n+1} (m_1 + (-1)^n m_2) r_{12}^n.$$  

(16)

Note that the total ADM mass $m$ satisfies $m = m_1 + m_2 = 2W(0) = 2W_{0,0,0}$. Friedrich \[4\] has shown that generic time symmetric data (like the Brill-Lindquist ones) give rise to logarithmic divergences at the sets where $I^+$ “touches” the cylinder $I$. In the same reference a regularity condition which allows to avoid such divergences has been put forward. This condition is given in terms of the Cotton tensor and its symmetrised derivatives at spatial infinity. Now, the Brill-Lindquist initial data is conformally flat, and thus it satisfies automatically the regularity condition. It should be remarked that it is still not known whether the Brill-Lindquist 3-geometry develops a smooth null infinity or not. Nevertheless, the fact that the initial data satisfy Friedrich’s regularity condition is a good hint that a smooth null infinity will be present. Here this will be assumed to be the case.

The NP constants of the BL data. If the spacetime is axially symmetric, then there is only one non-zero Newman-Penrose constant and the Friedrich-Kánnár formula yields:

$$G_0 = -\frac{1}{2 \sqrt{15\pi}} \left\{ 127 \left( \frac{1}{2} W_{0,0,0} W_{2;4,2} - 4 W_{1;2,1}^2 \right) - \frac{1}{2 \sqrt{6}} R_2 \right\},$$  

(17)

where $R_2 = (\sqrt{6}/2) D_{ab} D_{cd} R$, and $R$ is the Ricci scalar of the 3-metric on the initial hypersurface. If the initial data are locally conformally flat, then one can choose a conformal factor such that $R = 0$ in the neighbourhood of the reference asymptotic end. This is the case for the Brill-Lindquist data.
Substituting the result of equation (16) into equation (17) one readily obtains the following remarkable result:

\[ G_0 = -\frac{127\sqrt{15\pi}}{4}r_{12}^2m_1m_2. \]  

(18)

The quantity \( G_0 \) is clearly of a quadrupolar nature. Note that if either \( r_{12} = 0 \) or any of \( m_1, m_2 \) are zero, one recovers the initial data for the Schwarzschild spacetime, and consequently the constant \( G_0 \) vanishes. Even more interestingly, if the total mass \( m = m_1 + m_2 \) is kept fixed, then the constant maximizes its value in the case of the most symmetric configuration, i.e. whenever \( m_1 = m_2 = m/2 \). Similar results are expected to hold for the Misner initial data.

Assuming that the development of the Brill-Lindquist initial data given by equation (9) gives rise to a smooth complete null infinity then, the product given by equation (18) will be conserved along successive cuts of \( \mathcal{I}^+ \). It is important to observe that there are no other quantities known with such property. Thus, this result may be useful to check the accuracy of numeric simulations.

As a result of the head on collision of the two black holes, one should expect the system to settle down to a Schwarzschild black hole. The fact that the constant \( G_0 \) is zero only in the case where one has a single black hole right from the beginning shows that this final state of the evolution of the system is indeed an asymptotic state, and that it cannot be reached in a finite amount of time. In other words, there is always some amount of residual gravitation radiational. Hence, the Newman-Penrose constants contain information about the late time behaviour of the system. What is most remarkable is the fact that this information is readily available from the initial data! In spacetimes which contain the point \( i^+ \) (future timelike infinity, or equivalently past timelike infinity, \( i^- \)) the Newman-Penrose constants have been interpreted as the value of the Weyl tensor at \( i^+ \) [9, 6]. In a different context, the Newman-Penrose constants have appeared as coefficients in the leading term of late time expansions of the Bondi mass for some boost-rotation symmetric spacetimes [7], therefore giving a measure of “how quickly the system is settling down to a non-radiative state”. Due to the non-trivial topology of the Brill-Lindquist initial data, the conformal completion of the spacetime resulting from the evolution of the Brill-Lindquist initial data will not contain a regular point \( i^+ \), however, one expects a similar interpretation to hold. The formula (18) suggests that an initial configuration with the throats at a given separation should contain more residual radiation than other configuration with the same total mass but smaller separation. Analogously, a configuration with similar masses should contain more residual radiation than another with the same total mass but in which the ratio of the individual masses is quite different from one. These ideas are in agreement with recent numeric calculations for black-hole head on collisions [1, 2].

We thank Dr. H. Friedrich (MPI-AEI) and Dr. R. Lazkoz (Deustuko Unibertsitatea, Spain) for careful readings of the manuscript and for several suggestions which lead to its significant improvement.

References

