The Most General and Renormalizable Maximal Abelian Gauge

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Abstract

We construct the most general gauge fixing and the associated Faddeev-Popov ghost term for the $SU(2)$ Yang-Mills theory, which leaves the global $U(1)$ gauge symmetry intact (i.e., the most general Maximal Abelian gauge). We show that the most general form involves eleven independent gauge parameters. Then we require various symmetries which help to reduce the number of independent parameters for obtaining the simpler form. In the simplest case, the off-diagonal part of the gauge fixing term obtained in this way is identical to the modified maximal Abelian gauge term with two gauge parameters which was proposed in the previous paper from the viewpoint of renormalizability. In this case, moreover, we calculate the beta function, anomalous dimensions of all fields and renormalization group functions of all gauge parameters in perturbation theory to one-loop order. We also discuss the implication of these results to obtain information on low-energy physics of QCD.

Key words: maximal Abelian gauge, Abelian dominance, renormalizability, non-Abelian gauge theory
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1 Introduction

The gauge fixing is an indispensable procedure in quantizing the continuum gauge theory. It is believed that the physically meaningful results do not depend on the gauge fixing condition. Therefore we can adopt any favorite gauge fixing condition for obtaining physical quantities. We often adopt the Lorentz gauge in order to simplify the calculation. In this gauge, the Lorentz covariance is explicitly preserved. Especially, the Landau gauge is very efficient to simplify the calculations in the gauge theory.

In this paper, nevertheless, we consider the maximal Abelian (MA) gauge. This gauge is very useful to clarify the low energy physics of quantum chromodynamics (QCD), since in the low energy region of QCD, the Abelian projection procedure [1] or a hypothesis of Abelian dominance [2] has been justified by the recent research mainly based on numerical simulations [3]. Especially, quark confinement can be explained based on the dual superconductor picture of QCD vacuum [4] at least qualitatively. Dual superconductivity of QCD is expected to be described by the dual Ginzburg-Landau (DGL) theory. However, the DGL theory is an Abelian gauge theory, while QCD is an $SU(3)$ non-Abelian gauge theory. Therefore, for the dual superconductor picture to be responsible for the low energy physics in QCD such as quark confinement and spontaneous breakdown of chiral symmetry, the Abelian projection procedure turns out to be useful. Thus we expect that the maximal Abelian gauge [5] is the most useful gauge for describing the low energy physics of QCD.

In a series of papers [6, 7, 8, 9, 10, 11], we have tried to give an analytical framework which enables us to explain the Abelian dominance in QCD under the MA gauge from the viewpoint of renormalizability. The MA gauge is a nonlinear gauge fixing condition, in sharp contrast with the conventional gauge fixing of the Lorentz type which is a linear gauge. Due to this non-linearity, we must introduce the quartic ghost–antighost self-interaction to maintain the renormalizability. The modified MA gauge fixing term [7, 9] was devised to incorporate such a self-interaction term in a natural way. We have pointed out a possibility of dynamical mass generation of off-diagonal gluons and off-diagonal ghosts due to the ghost–antighost condensation. The fact that the off-diagonal fields become massive while the diagonal fields remain massless or have smaller masses gives an analytical explanation of Abelian dominance in the low energy region.

In spite of such an advantage in the low energy region of QCD, the MA gauge is rarely adopted in contrast with the Lorentz gauge. One of the reasons is that the calculation in the MA gauge is very complicated because of the nonlinearity of MA gauge fixing condition. Another reason is that the MA gauge partially fixes the non-Abelian gauge symmetry leaving the residual $U(1)^{N-1}$ gauge symmetry so that the global color symmetry is partially broken and we must distinguish diagonal components and off-diagonal components of fields in the MA gauge. Therefore, the detailed investigation of the MA gauge has not yet been performed even in the perturbative level except for pioneering works [12]. In this paper, therefore, we present perturbative results, i.e., calculations of the beta-function, anomalous dimensions of fields, and renormalization group (RG) function of gauge parameters in the MA gauge.
Before performing concrete calculations, we obtain the most general form of $SU(2)$ Yang-Mills action in the MA gauge. We show that the most general form of the MA gauge involves eleven gauge parameters. We classify the gauge parameter space from the viewpoint of symmetries. A detailed consideration of such a gauge fixing term in the case of $SU(2)$ has already been attempted by Min, Lee and Pac[12] or Hata and Niigata[13] and some of RG functions of gauge parameters were calculated there. However, in this paper, we give thorough analyses of the symmetries imposed on the possible action in the MA gauge. In the simplest case, the off-diagonal part of the gauge fixing terms obtained in this way is identical to the modified maximal Abelian gauge terms with two gauge parameters proposed in the previous paper from the viewpoint of renormalizability. Moreover, we calculate anomalous dimensions of all fields and RG functions of gauge parameters appearing in our action to one-loop order of perturbation theory in the scheme of the dimensional regularization.

Even though our main interest lies in the investigation of the low energy physics of QCD and the perturbative approach is not valid there, the perturbative calculations are the first step toward the non-perturbative studies of the low energy physics governed by the strong coupling dynamics. This is because the high energy behavior could be related to the low energy one by renormalization group equation and analyticity.[14] In fact, the anomalous dimensions and RG functions calculated by perturbative method give indispensable ingredients for the nonperturbative approaches, for instance, the truncated coupled Dyson-Schwinger equations, superconvergence relations[15, 16] and numerical simulations on a lattice[17].

This paper is organized as follows. In section 2, we give a general consideration on the renormalizable gauge fixing and FP ghost term respecting the global $U(1)$ gauge symmetry in the $SU(2)$ non-Abelian gauge theory. By taking account of the symmetries, we can fix some of the parameters without spoiling the renormalizability. Then we restrict our consideration to a fixed parameter subspace. It is possible to choose a minimal version of the maximal Abelian gauge by restricting the parameter space to three independent parameters. In section 3, we calculate quantum corrections to all the remaining parameter in the minimal choice of the most general MA gauge, although some of the anomalous dimensions have already been obtained in the previous papers[10, 11]. The renormalizability of the modified MA gauge is confirmed to one-loop order of perturbation theory. We give the conclusion and discussion in the final section. In Appendix A, we discuss the rescaling of the fields preserving BRST transformation and its connection to the renormalization.

2 The most general gauge fixing terms

In this section, we construct the most general gauge fixing (GF) and the associated Faddeev-Popov (FP) ghost term for the Maximal Abelian gauge in the $SU(2)$ Yang-Mills theory. Note that we require only the global $U(1)$ symmetry for the gauge fixing term, not the global $SU(2)$ gauge symmetry. The most general FP+GF term
is obtained in the BRST exact form,

\[ S_{\text{GF}+\text{FP}} = -i \int d^4 x \delta_B G, \]  

where \( G \) is a functional of gluons \( A^A_{\mu} = (A^a_{\mu}, a_{\mu}) \), ghosts \( C^A = (C^a, C^3) \), antighosts \( \bar{C}^A = (\bar{C}^a, \bar{C}^3) \) and Nakanishi-Lautrup fields \( B^A = (B^a, B^3) \) with \( a = 1, 2 \).

First, the functional \( G \) must satisfy the following requirements.

1) The functional \( G \) must be of mass dimension 3. Since \( A^A_{\mu} = (A^a_{\mu}, a_{\mu}) \), \( C^A = (C^a, C^3) \), \( \bar{C}^A = (\bar{C}^a, \bar{C}^3) \) and \( B^A = (B^a, B^3) \) has respectively the mass dimension 1, 1, 1 and 2, a monomial in the functional \( G \) consists of at most three fields.

2) The functional \( G \) must have the global \( U(1) \) symmetry.

3) The functional \( G \) must have the ghost number \(-1\). Note that \( A^A_{\mu} = (A^a_{\mu}, a_{\mu}) \), \( C^A = (C^a, C^3) \), \( \bar{C}^A = (\bar{C}^a, \bar{C}^3) \) and \( B^A = (B^a, B^3) \) has the ghost number 0, 1, \(-1\) and 0, respectively.

From the above requirements 1) and 2), the possible form of the monomials in \( G \) can be classified into seven groups:

\[
\begin{align*}
&\delta^{ab} X^a Y^b Z^3, \quad \epsilon^{ab3} X^a Y^b Z^3, \quad X^3 Y^3 Z^3, \\
&\delta^{ab} X^a Y^a, \quad \epsilon^{ab} X^a Y^b, \quad X^3 Y^3, \\
&X^3.
\end{align*}
\]

Taking account of the fact that the functional \( G \) is of the form,

\[ G = \bar{C} \Phi, \]

apart from the index, we find that one of \( X \), \( Y \) and \( Z \) must be an antighost \( \bar{C}^A = (\bar{C}^a, \bar{C}^3) \) and that \( \Phi \) must be of dimension 2 and of ghost number zero from the requirement 3).

Second, we consider the global \( SU(2) \) symmetry which is broken by the MA gauge fixing. The invariants under the global \( SU(2) \) rotation are

\[ \epsilon^{ABC} X^A Y^B Z^C, \quad \delta^{AB} X^A X^B. \]

Therefore, the three groups of the seven groups belong to this type:

\[ \epsilon^{ab3} X^a Y^b Z^3, \quad \delta^{ab} X^a Y^b, \quad X^3 Y^3. \]

The remaining four groups,

\[
\begin{align*}
&\delta^{ab} X^a Y^b Z^3, \quad X^3 Y^3 Z^3, \quad \epsilon^{ab} X^a Y^b, \quad X^3,
\end{align*}
\]

are incompatible with the global \( SU(2) \) symmetry if they exist in the functional \( G \). They are called the \textit{exceptional} terms. Thus, the possible form of the functional is rewritten as

\[ S_{\text{GF}+\text{FP}} = -i \int d^4 x \delta_B \left( G^{(a)} + G^{(i)} + G^{\text{ex}} \right), \]
where we have decomposed the terms belonging to the first group (7) into two functionals \( G^{(a)} \) and \( G^{(i)} \) according to their forms,

\[
G^{(a)} = \bar{C}a \Phi^a, \quad G^{(i)} = \bar{C}^3 \Psi^3, \quad (10)
\]

and \( G^{\text{ex}} \) denotes the exceptional terms of the form,

\[
G^{\text{ex}} = \bar{C}a \Phi^a + \bar{C}^3 \Psi^3. \quad (11)
\]

The first functional \( G^{(a)} \) plays the role of partially fixing the \( SU(2) \) gauge symmetry to \( U(1) \). The possible form of monomials in \( G^{(a)} \) is either \( \epsilon^{ab3} C^a Y^b Z^3 \) or \( \delta^{ab} \bar{C}^a Y^b \). It is easy to see that the possible choices are given as \( \epsilon^{ab3} \bar{C}^a C^b \), \( \bar{C}^b C^3 \), \( \bar{C}^a A^b a_\mu \) and \( \delta^{ab} \bar{C}^a Y^b \sim \delta^{ab} \bar{C}^a \). Thus the most general form of \( G^{(a)} \) is given by

\[
G^{(a)} := \bar{C}a \left[ (\partial^\mu \delta^{ab} - \xi g \epsilon^{ab} a_\mu) A^b_\mu + \frac{\alpha}{2} B^a + \frac{C}{2} g \epsilon^{ab} \bar{C}^b C^3 + i \eta g \epsilon^{ab} \bar{C}^3 C^b \right]. \quad (12)
\]

It turns out that the off-diagonal component of the Nakanishi-Lautrup field \( B^a \) is generated from this functional after performing the BRST transformation explicitly.

By making use of the anti-BRST transformation, this functional is recast into

\[
G^{(a)} \equiv -\bar{\delta}_B \left[ \frac{1}{2} A^a A^a_\mu - \frac{\zeta}{2} i C^a C^a \right] + \bar{C}a \left[ (1 - \xi) \eta g \epsilon^{ab} a_\mu A^b_\mu + \frac{\alpha}{2} B^a + \frac{C}{2} g \epsilon^{ab} \bar{C}^3 C^b \right]. \quad (13)
\]

The first term of the right hand side of (13) is both BRST and anti-BRST exact, and give rise to the modified MA gauge fixing term proposed in the previous papers [7, 9]. After performing the BRST transformation, we obtain

\[
S^{(a)} := -i \int d^4x \delta_B G^{(a)} = \int d^4x \left\{ \frac{\alpha}{2} B^a B^a + B^a D^\xi A^a_\mu - i \epsilon^{ab} B^a C^b C^3 + i \eta g \epsilon^{ab} B^a C^3 C^b \right. \\
- i \eta g \epsilon^{ab} B^3 C^a C^b + i \bar{C}a D^\xi D_\mu C^a + i \epsilon^{ab} \bar{C}a D^\xi A^a_\mu C^3 + i g(1 - \xi) \epsilon^{ab} \bar{C}a A^b_\mu \delta\mu C^3 - i g^2 \xi \epsilon^{ad} \epsilon^{cb} \bar{C}a C^b A^c A^d \\
+ g^2 \frac{\zeta}{4} \epsilon^{ad} \epsilon^{cb} \bar{C}a C^b C^c C^d + g^2 \eta C^3 C^3 C^a C^a \right\}, \quad (14)
\]

where we have defined a covariant derivative \( D^\xi_\mu \) in terms of the Abelian gluon \( a_\mu \) as

\[
D^\xi_\mu \phi^a := (\partial_\mu \delta^{ab} - \xi g \epsilon^{ab} a_\mu) \phi^b, \quad (15)
\]

which is abbreviated in the special case of \( \xi = 1 \) as

\[
D_\mu \phi^a := D^{\xi=1}_\mu \phi^a = (\partial_\mu \delta^{ab} - g \epsilon^{ab} a_\mu) \phi^b. \quad (16)
\]
The second functional $G^{(i)}$ is used to fix the residual $U(1)$ gauge symmetry. The possible monomials are of two types: $\epsilon^{abc}\bar{C}^aY^b Z^c$ and $\delta^{33}\bar{C}^3Y^3$. Therefore, we obtain

$$\epsilon^{abc}\bar{C}^3 Y^a Z^b \sim \epsilon^{abc}\bar{C}^3 (\bar{C}^a C^b, A^\mu_A A^\mu_B = 0)$$

and

$$\delta^{33}\bar{C}^3 Y^3 \sim \delta^{33}\bar{C}^3 (B^3, \partial^\mu a_\mu).$$

It should be remarked that the term proportional to $\bar{C}^3 e^{ab}\bar{C}^a C^b$ is a candidate for the terms in this functional. However, such a term has already been included in Eq. (12) as the last term. Thus the general form of $G^{(i)}$ is given by

$$G^{(i)} := \bar{C}^3 \left[ \kappa \partial^\mu a_\mu + \frac{\beta}{2} B^3 \right].$$

(17)

After performing the BRST transformation, we obtain

$$S^{(i)} := -i \int d^4 x \delta_B G^{(i)} = \int d^4 x \left\{ \frac{\beta}{2} B^3 B^3 + \kappa B^3 \partial^\mu a_\mu + i \kappa \bar{C}^3 \partial^2 C^3 + ig \kappa \bar{C}^3 \epsilon^{ab} \partial^\mu (A^\mu_a C^b) \right\}.$$  

(18)

The last functional $G^{ex}$ in Eq. (9) includes exceptional terms. The possible forms are classified into $\delta^{ab}X^a Y^b Z^3$, $X^3 Y^3 Z^3$, $\epsilon^{ab}X^a Y^b$ and $X^3$. The trilinear monomials are $\delta^{ab}C^a Y^b Z^3 \sim \delta^{ab} \bar{C}^a (A^b_\mu, C^b C^3 = 0, C^b \bar{C}^3)$ and $\delta^{ab} \bar{C}^3 X^a Y^b \sim \bar{C}^3 \delta^{ab} (A^a_\mu A^\mu_C, C^3 C^b)$. Moreover, $X^3 Y^3 Z^3 \sim \bar{C}^3 (a_\mu a^\mu, \bar{C}^3 C^3 = 0)$. The bilinear terms are $\epsilon^{ab}X^a Y^b \sim \epsilon^{ab} \bar{C}^a (B^3, \partial^\mu A^\mu_B)$. However, we have omitted the bilinear term $\epsilon^{ab} \bar{C}^a B^b$, since it gives a vanishing contribution after the BRST transformation, $\delta_B (\epsilon^{ab} \bar{C}^a B^b) = 0$. Now we require only the global $U(1)$ gauge symmetry for the gauge fixing terms so that the terms included in Eq. (19) are not forbidden in spite of the fact that the diagonal index is not contracted. The second term in the right hand side of (19) becomes a linear term in $B^3$ after carrying out the BRST transformation. We make use of the dimensional regularization in this paper so that the divergence coming from the tadpole of $B^3$ does not appear as a result of perturbative loop expansions. Therefore we can set the parameter $\omega = 0$ without spoiling the renormalizability. After performing the BRST transformation, we obtain

$$S^{ex} := -i \int d^4 x \delta_B G^{ex} = \int d^4 x \left\{ -ig \kappa B^a \bar{C}^3 C^a + \partial \epsilon^{ab} B^a D^\mu A^b + g(\omega - \partial) B^a a_\mu A^\mu_a \\
+ g \frac{\beta}{2} B^3 A^a_\mu A^\mu_a + g \frac{\kappa}{2} B^3 a_\mu a^\mu + ig \kappa B^3 \bar{C}^a C^a \\
+ i \epsilon^{ab} \bar{C}^a D^\mu D^\mu C^b + ig(\omega - \partial) \bar{C}^a a_\mu D^\mu C^a \\
- ig \kappa \bar{C}^a D^\mu A^\mu_a C^3 + ig(\omega - \partial) \bar{C}^a D^\mu C^3 A^\mu_a + ig^2(\omega - \partial) \epsilon^{ab} \bar{C}^a C^3 a_\mu A^\mu_b \\
+ ig \kappa \bar{C}^a D^\mu C^a + ig \epsilon^{ba} \bar{C}^a C^a A^\mu_A C^a \\
+ ig \kappa \bar{C}^a a_\mu D^\mu C^3 + ig^2 \omega e^{ba} \bar{C}^a A^\mu_a C^a A^\mu_a - g^2 \epsilon^{ab} \bar{C}^3 C^a A^\mu_C C^3 \right\}.$$  

(20)
Summing up three functionals and integrating out the Nakanishi-Lautrup fields, we obtain the most general form of the gauge fixing term with global $U(1)$ symmetry as

$$S_{GF+FP} = \int d^4x \left\{ + i \bar{C}^a D^\mu C^a + i \bar{\chi}^a D^\mu \bar{C}^a + i \kappa \bar{C}^a \partial^2 C^a 
+ i \frac{g_\kappa}{\beta} (\eta \epsilon^{ab} - \zeta \delta^{ab}) \bar{C}^a C^b \partial^\mu a_\mu + ig(\bar{\omega} - \bar{\nu}) \bar{C}^a a_\mu (D^\mu C)^a 
+ ig \frac{\alpha - \zeta}{\alpha} \epsilon^{ab} \bar{C}^a C^b \partial^\mu a_\mu 
+ ig \frac{\alpha - \zeta}{\alpha} (\bar{\omega} - \bar{\nu}) \epsilon^{ab} \bar{C}^a C^b a_\mu A^{ab} 
+ ig \frac{\alpha - \zeta}{\alpha} \epsilon^{ab} \bar{C}^a C^b \partial^\mu A^{ab} 
+ ig \epsilon^{ab} \partial^\mu \bar{C}^a C^b a_\mu A^{ab} 
+ i g \epsilon^{ab} \partial^\mu C^b A^{ab} \right\} \bar{C}^a D^\mu A^{ab} C^b 
+ g \int d^4x \left\{ \frac{(-\xi \epsilon^{ad} + \bar{\omega} \delta^{ad}) \bar{C}^a C^b + g \frac{\alpha}{2 \beta} (\eta \delta^{ab} - \zeta \delta^{ab}) \delta^{cd}}{2} \bar{C}^a C^b A^{cd}_{\mu} A^{\mu} 
+ g \frac{\alpha - \zeta}{\alpha} (\eta \delta^{ab} - \zeta \delta^{ab}) \bar{C}^a C^b C^c C^d 
+ g \frac{\alpha - \zeta}{\alpha} (\bar{\omega} \delta^{ab} - \bar{\nu} \delta^{ab}) \bar{C}^a C^b C^c C^d 
- g \frac{\alpha - \zeta}{\alpha} \eta \delta^{ad} + \bar{\omega} \delta^{ad} \right\} \partial^\mu \bar{C}^a C^b C^c C^d 
- \frac{1}{2 \alpha} \left( (D^\mu A^{a}_{\mu} + \bar{\nu} \epsilon^{ab} D^\mu A^{b}_{\mu} + \eta \delta^{ab}) A^{a}_{\mu} A^{b}_{\mu} \right)^2 
- \frac{1}{2 \beta} \left( \kappa \partial^\mu a_\mu + \frac{1}{2} g \partial^a A^{a}_{\mu} A^{a}_{\mu} + \frac{1}{2} g \chi a_\mu a^\mu \right)^2 \right\}. $$ (21)

The GF+FP term $S_{GF+FP}$ just obtained has eleven independent gauge fixing parameters $\xi$, $\alpha$, $\zeta$, $\eta$, $\kappa$, $\beta$, $\chi$, $\varrho$, $\varsigma$, $\vartheta$ and $\bar{\omega}$. If we adopt the most general gauge fixing term, therefore, we must treat the twelve dimensional parameter space in the Yang-Mills theory with a gauge coupling constant $g$. In order to simplify the theory, we try to find the fixed subspace in which the renormalization group flow is confined by being protected by some symmetries. In fact, there are some fixed subspaces in the parameter space protected by the following symmetries.¹

**Charge conjugation:** The exceptional part (19) breaks the “charge conjugation” symmetry[13] under the discrete transformation:

$$\Phi^1 \rightarrow \Phi^1, \quad \Phi^2 \rightarrow -\Phi^2, \quad \Phi^3 \rightarrow -\Phi^3.$$ (22)

¹Some of these symmetries were first pointed out by Hata and Niigata in Ref. [13].
where $\Phi^A$ denotes all fields. Any term belonging to the group (8) is not invariant under this charge conjugation, while any term belonging to the group (7) is invariant under the “charge conjugation”. Therefore, by setting the parameter $\chi = \varrho = \varsigma = \omega = \vartheta = \varpi = 0$, the charge conjugation symmetry is recovered. However, once we consider the non-perturbative effect, for instance ghost–antighost condensation proposed in the previous paper[9], the “charge conjugation” invariance\(^2\) is not expected to hold.

**Translational invariance for $\bar{C}^3$:** By setting the parameter to $\eta = 0$ and $\chi = \varrho = \varsigma = \omega = 0$, the GF+FP term respects a global symmetry under the translation of the diagonal antighost $\bar{C}^3(x)$ as $\bar{C}^3(x) \to \bar{C}^3(x) + \bar{\theta}^3$ where $\bar{\theta}^3$ is a constant Grassmann variable. This is because the diagonal antighost $\bar{C}^3$ appears only in the differentiated form $\partial_\mu \bar{C}^3$ for this choice of the parameters. Then the translational symmetry of $\bar{C}^3$ exists in the theory.

**Translational invariance for $C^3$:** By setting the parameter to $\alpha = \varsigma$, the action has a global symmetry under the translation of the diagonal ghost as $C^3(x) \to C^3(x) + \theta^3$ where $\theta^3$ is a constant Grassmann variable. In the similar manner to the previous case, we can confirm that the translational symmetry of $C^3$ exists in this case.

**Implicit residual $U(1)$ invariance:** By setting the parameter to $\xi = 1$, $\chi = \varrho = \varsigma = \omega = 0$ and $\varpi = \vartheta$, the action has the residual $U(1)$ gauge symmetry mentioned in the previous paper[10], although the gauge fixing for the residual $U(1)$ gauge symmetry has already been accomplished. As we have mentioned in the previous paper[10], there is the $U(1)$ gauge symmetry if the diagonal gluon does not appear in the action after replacing all the derivatives with the Abelian covariant derivative defined by (16) except for a quadratic term as $(\partial^\mu a^\mu)^2$. In the view of the background field method[18], there is a gauge symmetry with respect to the background diagonal field.

\[
S_{GF+FP} = \int d^4x \left\{ - \frac{1 + \varrho^2}{2\alpha} \left( D^\mu A^a_\mu \right)^2 - \frac{1}{2\beta} (\kappa \varrho^\mu a_\mu)^2 
+ i \bar{C}^a D^\mu D_\mu C^a + i \varrho \varepsilon^{ab} \bar{C}^a D^\mu D_\mu C^b + i \kappa \bar{C}^3 \partial^2 C^3 
+ i \frac{g}{\alpha} \left( \varepsilon^{ab} - \varrho \delta^{ab} \right) \left[ (\alpha - \varsigma) \bar{C}^a D^\mu A^b_\mu C^3 + \eta \bar{C}^3 C^a D^\mu A^b_\mu \right] 
+ i \frac{g\kappa}{\beta} \eta \varepsilon^{ab} \bar{C}^a C^b \partial^\mu a_\mu - i g\kappa \varepsilon^{ab} \partial^\mu \bar{C}^3 A^a_\mu C^b - i g^2 \varepsilon^{ad} \varepsilon^{bc} \bar{C}^a C^b A^c_\mu A^d_\mu 
+ \frac{g^2}{2} \left( \varsigma - \frac{\eta^2}{\beta} \right) \bar{C}^a \bar{C}^b C^a C^b - g^2 \frac{\alpha - \varsigma}{\alpha} \eta \bar{C}^3 C^a C^a C^3 \right\}. \tag{23}
\]

**Anti-BRST symmetry:** By setting the parameter to $\chi = \varrho = \varsigma = \omega = 0$, $\varpi = \varrho = 0$, $1 - \xi - \kappa = 0$ and $\alpha - \beta + \eta - \varsigma = 0$, the action has the anti-BRST invariance.

\(^2\)There are two types ghost–antighost condensation, $C^a \bar{C}^a = C^1 \bar{C}^1 + C^2 \bar{C}^2$ and $\varepsilon^{ab} \bar{C}^a C^b = C^1 \bar{C}^2 - C^2 \bar{C}^1$. The “charge conjugation” invariance is broken by the latter one. (See Ref. [9] for more details.)
Then the action is given by

\[
S_{GF} \equiv -\int d^4x \left\{ i\delta_B \delta_B \left[ \frac{1}{2} A^a_{\mu} A^a_\mu - \frac{i}{2} (\eta + \zeta) C^a \bar{C}^a + \frac{\kappa}{2} a^\mu a_\mu - \eta i C^3 \bar{C}^3 \right] + \frac{1}{2} (\alpha - \eta - \zeta) B^A B^A \right\}. \tag{24}
\]

Here, the second term in the integrand of the right hand side of the Eq. (24) is not exact in the combined BRST and anti-BRST transformations, \( \delta_B \delta_B \), differently from the first term. However, the second term is both BRST and anti-BRST invariant since \( B^A B^A = -i \delta_B (\bar{C}^A B^A) = i \delta_B (\bar{C}^A B^A) \).

**Global SU(2) invariance:** After setting the parameter to \( \chi = \vartheta = \zeta = \omega = 0 \), \( \vartheta = \varpi = 0 \), \( \xi = 0 \), \( \kappa = 1 \), \( \alpha = \beta \) and \( \zeta = \eta \), the action has the global SU(2) invariance. Then the action is given by

\[
S_{GF} \equiv -\int d^4x i\delta_B \left\{ \bar{C}^A \left[ \partial^\mu A^A_\mu + \frac{\alpha}{2} B^A - \frac{\zeta}{2} e^{ABC} \bar{C}^B C^C \right] \right\}. \tag{25}
\]

It is easy to see that this choice of parameters is a special case of the anti-BRST symmetric case. As a result, the action (25) is recast into

\[
S_{GF} = \int d^4x \left[ i\delta_B \delta_B \left( \frac{1}{2} A^a_{\mu} A^a_\mu - \frac{\xi}{2} i C^A \bar{C}^A \right) + \frac{\xi'}{2} B^A B^A \right], \tag{26}
\]

by introducing \( \xi \) and \( \xi' \) as \( \alpha = \xi + \xi' \), \( \zeta = \xi \). This form agrees with the global SU(2) invariant action which is invariant under the BRST and anti-BRST transformation obtained by Baulieu and Thierry-Mieg[20].

**FP conjugation invariance:** After setting the parameter to \( \chi = \vartheta = \zeta = \omega = 0 \), \( \vartheta = \varpi = 0 \), \( \alpha = \zeta + \eta \), \( 1 - \xi - \kappa = 0 \) and \( \kappa(\beta - 2\eta) = 0 \) and integrating out \( B^3 \) and \( B^a \), the action has the invariance under the FP ghost conjugation:

\[
C^A \rightarrow \pm \bar{C}^A, \quad \bar{C}^A \rightarrow \mp C^A, \quad A^A_\mu \rightarrow \bar{A}^A_\mu. \tag{27}
\]

In the case of \( \kappa \neq 0 \), i.e., \( \beta = 2\eta \), the anti-BRST symmetry is also recovered and we obtain

\[
S_{GF} = \int d^4x \left\{ i\delta_B \delta_B \left[ \frac{1}{2} A^a_{\mu} A^a_\mu - \frac{i}{2} \alpha C^a \bar{C}^a + \frac{\kappa}{2} a^\mu a_\mu - \frac{i}{2} \beta C^3 \bar{C}^3 \right] \right\}. \tag{28}
\]

It is identical to the BRST–anti-BRST exact part of the FP+GF term (24) previously discussed. Another case, \( \kappa = 0 \), is very delicate since the gauge fixing term of the Abelian gluon is eliminated by the naive limit of \( \kappa \rightarrow 0 \).

**SL(2, R) symmetry:** After setting the parameter to \( \chi = \vartheta = \zeta = \omega = 0 \), \( \vartheta = \varpi = 0 \), \( \alpha = \zeta + \eta \), \( 1 - \xi - \kappa = 0 \) and \( \kappa(\beta - 2\eta) = 0 \) and integrating out \( B^3 \) and \( B^a \), just as in the FP conjugation invariance, the GF+FP term also become invariant under the two transformations:

\[
\delta_+ \bar{C}^A(x) = C^A(x), \quad \delta_+(\text{other fields}) = 0, \tag{29}
\]
\[ \delta_- C^A(x) = \bar{C}^A(x), \quad \delta_-(\text{other fields}) = 0. \]  

(30)

These symmetries are \( SL(2, R) \) symmetry for the multiplet of ghost and antighost \((C, \bar{C})\), see e.g. Ref. [9].

When we wish to perform thorough analyses for phenomena with unbroken global \( U(1) \) symmetry, it is desirable to employ the most general action (21). However, it is very tedious work so that we should require some additional symmetries and restrict the gauge parameter space properly.

Now we define the most general MA gauge. In the analysis of the conventional MA gauge, the residual \( U(1) \) symmetry is the most important symmetry since the MA gauge condition is originally defined as follows. The MA gauge is obtained by minimizing the functional \( R[A^U] \) with respect to the local gauge transformation \( U(x) \) of \( A^a_\mu(x) \). Here, \( R[A] \) is defined as the functional of off-diagonal gluon fields,

\[ R[A] := \int d^4x \frac{1}{2} A^a_\mu(x) A^{\mu a}(x). \]  

(31)

Then we obtain the differential form of the MA gauge condition,

\[ D_\mu A^{\mu a} \equiv \left( \partial_\mu \delta^{ab} - g \epsilon^{ab} a_\mu \right) A^{\mu b} = 0. \]  

(32)

To adopt this condition (32), we must set \( \xi = 1 \), \( \chi = \varrho = \varsigma = \omega = \varpi = \vartheta \) as we mentioned in Implicit residual \( U(1) \) invariance and (23).

A remarkable difference between our gauge fixing procedure and that of the previous works (Min, Lee and Pac [12] and Hata and Niigata [13]) is the existence of a new parameter \( \kappa \). If we do not require the recovery of global \( SU(2) \) gauge symmetry in our gauge fixing, then there is no need to set the parameter \( \kappa \) to 1 against Ref. [13]. At first sight, the parameter \( \kappa \) can be absorbed by rescaling the diagonal ghost \( C^3 \) and diagonal antighost \( \bar{C}^3 \). However, such a rescaling is not always valid. For instance, requiring the residual \( U(1) \) gauge symmetry to the gauge fixing term (28) which has invariance under \( SL(2, R) \) symmetry and FP conjugation, we must set \( \kappa = 0 \) since there is a relation \( 1 - \xi - \kappa = 0 \). Therefore, in this case, we cannot absorb the parameter \( \kappa \) by rescaling the field. On the contrary, setting \( \kappa = 1 \) in the gauge fixing term (28), the gauge fixing condition becomes \( \partial_\mu A^{\mu a} = 0 \) which is identical to the ordinary Lorentz gauge condition so that it is not the MA gauge in the ordinary sense in spite of the breaking of the global \( SU(2) \) symmetry for \( \alpha \neq \beta \).

Another advantage of introducing the parameter \( \kappa \) is that the symmetry of the renormalized theory under the FP conjugation (i.e., the symmetry of the renormalized theory under the exchange of the ghost and antighost) is easily examined for the renormalized theory, since the renormalized ghost and antighost fields are defined through the same renormalization factor.

3 The minimum choice of the gauge fixing terms

By requiring some of the symmetries listed in the previous section, that is, charge conjugation, translational invariance of the diagonal ghost \( C^3 \) or antighost \( \bar{C}^3 \) and
implicit $U(1)$ gauge symmetry, we obtain the minimum choice of the renormalizable MA gauge. Setting parameters as
\[
\alpha = \zeta, \quad \xi = 1, \quad \text{and} \quad \eta = \chi = \varrho = \varsigma = \omega = \vartheta = \varpi = 0,
\]
we arrive at the gauge fixing terms with three parameters $\alpha$, $\beta$ and $\kappa$.

\[
S_{GF} := i \int d^4 x \delta_B \tilde{\delta}_B \left[ \frac{1}{2} A^{\mu a} A_\mu^a - \frac{\alpha}{2} i C^a \tilde{C}^a \right] - i \int d^4 x \delta_B \left\{ \tilde{C}^a \left[ \kappa \partial^\mu a_\mu + \frac{\beta}{2} B^3 \right] \right\}
\]

\[
= \int d^4 x \left\{ B^a D^\mu A^b_\mu + \frac{\alpha}{2} B^a B^a + i \tilde{C}^a D^2 C^b - i g^2 \epsilon^{ad} \epsilon^{eb} \bar{C}^a C^b A^{uc} A^d_\mu \\
+ i g e^{ab} \tilde{C}^a (D^\mu A^b_\mu) C^3 - i \alpha g e^{ab} B^a \bar{C}^b C^3 + \frac{\alpha}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a C^b \tilde{C}^c C^d \right. \\
\left. + \kappa B^3 \partial^\mu a_\mu + \frac{\beta}{2} B^3 B^3 + i \kappa \tilde{C}^3 \partial^2 C^3 + i \kappa \tilde{C}^3 \partial^\mu (g e^{bc} A^b_\mu C^c) \right\}.
\]

By integrating out $B^3$ and $B^a$, we obtain

\[
S_{GF} = \int d^4 x \left\{ - \frac{1}{2\alpha} (D^\mu A^a_\mu)^2 - \frac{\kappa^2}{2\beta} (\partial^\mu a_\mu)^2 \\
+ i \tilde{C}^a D^2 C^b - i g^2 \epsilon^{ad} \epsilon^{eb} \bar{C}^a C^b A^{uc} A^d_\mu + \frac{\alpha}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a C^b \tilde{C}^c C^d \right. \\
\left. + i \kappa \tilde{C}^3 \partial^2 C^3 + i \kappa \tilde{C}^3 \partial^\mu (g e^{bc} A^b_\mu C^c) \right\}.
\]

In the following subsections, we consider the renormalizability of this action in the two different schemes. In the Scheme I, the parameter $\kappa$ is absorbed by rescaling the antighost field. In this case, we must distinguish the renormalization factors of the diagonal ghost and diagonal antighost fields. This approach is valid except for the case of $\kappa = 0$, in which the rescaling of the antighost field is ill-defined. In the Scheme II, the parameter $\kappa$ is left explicitly. In this case we can equate the renormalization factors of the diagonal ghost and diagonal antighost fields. This approach is valid even if $\kappa = 0$, and we demonstrate that the value of $\kappa = 0$ is meaningful from the viewpoint of renormalizability and symmetries.

### 3.1 Scheme I

In the case of $\kappa \neq 0$, the parameter $\kappa$ can be absorbed by rescaling the diagonal antighost field and parameter $\beta$ as

\[
\tilde{C}^3 \to \frac{\tilde{C}^3}{\kappa}, \quad \beta \to \kappa^2 \beta,
\]

and we obtain

\[
S_{GF} = \int d^4 x \left\{ - \frac{1}{2\alpha} (D^\mu A^a_\mu)^2 - \frac{1}{2\beta} (\partial^\mu a_\mu)^2 \\
+ i \tilde{C}^a D^2 C^b - i g^2 \epsilon^{ad} \epsilon^{eb} \bar{C}^a C^b A^{uc} A^d_\mu + \frac{\alpha}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \tilde{C}^b C^c C^d \right. \\
\left. + i \tilde{C}^3 \partial^2 C^3 + i \tilde{C}^3 \partial^\mu (g e^{bc} A^b_\mu C^c) \right\}.
\]
We notice that the diagonal ghost $C^3$ does not appear in the interaction terms in (37). Therefore we need not take account of the internal diagonal ghost in the calculation of perturbative loop expansions. The beta-function, the anomalous dimensions of the diagonal gluon $a_\mu$ and off-diagonal gluon $A_\mu^a$, and RG functions of two gauge fixing parameters $\alpha$ and $\beta$ have already been obtained in previous papers[10, 11]. In this paper, we determine the anomalous dimension of the ghost field $C$ and antighost field $\bar{C}$ by making use of the dimensional regularization at the one-loop level.

### 3.1.1 Feynman rules

From the total action:

$$S := S_{YM} + S_{GF},$$

we obtain the following Feynman rules.

**Propagators**

(a) diagonal gluon propagator:

$$iD_{\mu\nu} = \frac{i}{p^2} \left[ g_{\mu\nu} - (1 - \beta) \frac{p_\mu p_\nu}{p^2} \right].$$
(b) off-diagonal gluon propagator:

\[ iD_{\mu\nu}^{ab} = -\frac{i}{p^2} \left[ g_{\mu\nu} - (1 - \alpha) \frac{p_{\mu}p_{\nu}}{p^2} \right] \delta^{ab}. \]  

(41)

(c) diagonal ghost propagator:

\[ i\Delta = -\frac{1}{p^2}. \]  

(42)

(d) off-diagonal ghost propagator:

\[ i\Delta^{ab} = -\frac{1}{p^2} \delta^{ab}. \]  

(43)

Three-point vertices

(e) One diagonal and two off-diagonal gluons:

\[ i \left< a_\mu (p) A_\rho^a (q) A_\sigma^b (r) \right>_{\text{bare}} = g \epsilon^{a b} \left[ (q - r)_\mu g_{\rho\sigma} + \left\{ r - p + \frac{2}{\alpha} \right\}_\rho g_{\sigma\mu} + \left\{ p - q - \frac{2}{\alpha} \right\}_\sigma g_{\mu\rho} \right]. \]  

(44)

(f) One diagonal gluon, one off-diagonal ghost and one off-diagonal antighost:

\[ i \left< C^a (p) C^b (q) a_\mu \right>_{\text{bare}} = -i(p + q)_\mu \epsilon^{a b}. \]  

(45)

(g) One off-diagonal gluon, one off-diagonal ghost and one diagonal antighost:

\[ i \left< \bar{C}^a (p) C^b (q) A_\mu^c \right>_{\text{bare}} = -i \epsilon^{c b} p_\mu. \]  

(46)

Four-point vertices

(h) Two diagonal gluons and two off-diagonal gluons:

\[ i \left< A_\mu^a A_\nu^a A_\rho^b A_\sigma^b \right>_{\text{bare}} = -ig^2 \delta^{ab} \left[ 2g_{\mu\nu} g_{\rho\sigma} - \left( 1 - \frac{1}{\alpha} \right) (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \right]. \]  

(47)

(i) Four off-diagonal gluons:

\[ i \left< A_\mu^a A_\nu^b A_\rho^c A_\sigma^d \right>_{\text{bare}} = -i2g^2 \epsilon^{a b} \epsilon^{c d} I_{\mu\nu,\rho\sigma} + \epsilon^{a c} \epsilon^{b d} I_{\mu\rho,\nu\sigma} + \epsilon^{a d} \epsilon^{b c} I_{\mu\sigma,\nu\rho}, \]  

where \( I_{\mu\nu,\rho\sigma} := (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})/2 \).  

(48)

(j) Two diagonal gluons, one off-diagonal ghost and one off-diagonal antighost:

\[ i \left< C^a C^b A_\mu^a A_\nu^b \right>_{\text{bare}} = 2g^2 \delta^{ab} g_{\mu\nu}. \]  

(49)

(k) Two off-diagonal gluons, one off-diagonal ghost and one off-diagonal antighost:

\[ i \left< \bar{C}^a C^b A_\mu^c A_\nu^d \right>_{\text{bare}} = g^2 \left[ \epsilon^{a d} \epsilon^{c b} + \epsilon^{a c} \epsilon^{d b} \right] g_{\mu\nu}. \]  

(50)

(l) Two off-diagonal ghosts and two off-diagonal antighosts:

\[ i \left< \bar{C}^a C^b C^c C^d \right>_{\text{bare}} = ig^2 \alpha \epsilon^{a b} \epsilon^{c d}. \]  

(51)
3.1.2 Counterterms

In order to construct the renormalized theory, we define the following renormalized fields$^3$ and parameters:

\[
\begin{align*}
  a_\mu &= Z_{a}^{1/2}a_{R\mu}, \quad C^3 = Z_{c}^{1/2}C_R^3, \quad \bar{C}^3 = Z_{\bar{c}}^{1/2}\bar{C}_R^3, \\
  A^a_\mu &= Z_{A}^{1/2}A^a_{R\mu}, \quad C^a = Z_{C}^{1/2}C_R^a, \quad \bar{C}^a = Z_{\bar{c}}^{1/2}\bar{C}_R^a, \\
  g &= Z_{g}g_R, \quad \alpha = Z_{\alpha}\alpha_R, \quad \beta = Z_{\beta}\beta_R.
\end{align*}
\]

(52)

By substituting the above renormalization relations (52) into the action (38), we obtain

\[
S = S_R + \Delta S_{\text{gauge}} + \Delta S_{\text{ghost}}.
\]

(53)

Here $S_R$ is the renormalized action obtained from the bare action (38) by replacing all the fields and parameters with the renormalized ones, while $\Delta S_{\text{ghost}}$ and $\Delta S_{\text{gauge}}$ are counterterms with and without ghost fields respectively. In this paper we focus on the renormalizability of the terms with ghost fields $\Delta S_{\text{ghost}}$. This is explicitly given by

\[
\Delta S_{\text{ghost}} = \int d^4x \left\{ i\delta_a \bar{C}^a \partial^\mu \Phi_a + i\delta_\bar{C}^\bar{a} \partial^\mu \Phi_{\bar{a}} \\
- i\delta \bar{c} g_R \epsilon^{ab} \bar{C}_R^a \Phi_a A_{R\mu}^b + \delta_{d}^\alpha \frac{g}{4} \epsilon^{ab} \epsilon^{cd} \bar{C}_R^a \Phi_a C_{R}^c \Phi_a C_{R}^d \\
+ i\delta \bar{c} \bar{C}_R^3 \partial^\mu (g_R \epsilon^{bc} A_{R\mu}^b \Phi_c) \right\},
\]

(54)

where we have defined the renormalized Abelian covariant derivative $D_R$ by

\[
D_{R\mu} \Phi^a := \left( \partial_{\mu} \delta^{ab} - g_R \epsilon^{ab} a_{R\mu} \right) \Phi^b.
\]

(55)

Abelian covariant derivative itself does not change under the renormalization. Indeed, substituting the renormalized relations (52) into the definition of the bare Abelian covariant derivative (16) and using the relation $Z_{g} = Z_{a}^{-1/2}$ due to the implicit residual $U(1)$ gauge symmetry pointed out in the previous paper[10], we obtain

\[
D_{R\mu} \Phi^a := \left( \partial_{\mu} \delta^{ab} - g_{R} \epsilon^{ab} a_{R\mu} \right) \Phi^b.
\]

(56)

$^3$Here, the renormalization factors for off-diagonal ghost $C^a$ and off-diagonal antighost $\bar{C}^a$ can be identical to each other. On the other hand, we must introduce different renormalization factors $Z_c$ and $Z_{\bar{c}}$ for diagonal ghost $C^3$ and diagonal antighost $\bar{C}^3$ respectively, since we absorbed the parameter $\kappa$ to the diagonal antighost field $\bar{C}^3$. See Appendix, for more details.
The coefficients \( \delta = (\delta_a, \delta_b, \delta_c, \delta_d, \delta_e) \) in the counter terms (54) are related to the renormalization factors \( Z_X = (Z_c, Z_\bar{c}, Z_C) \) as

\[
\begin{align*}
\delta_a &= Z_C - 1, \\
\delta_b &= Z_c^{1/2} Z_\bar{c}^{1/2} - 1, \\
\delta_c &= Z_a^2 Z_C Z_A - 1, \\
\delta_d &= Z_\alpha Z_a^2 Z_C^2 - 1, \\
\delta_e &= Z_\bar{c}^{1/2} Z_\bar{e} Z_A^{1/2} Z_C^{1/2} - 1.
\end{align*}
\]

(57)

Therefore we can determine the renormalization factors \( Zs \) by calculating \( \delta s \).

### 3.1.3 Anomalous dimensions and RG functions

In this subsection, we determine the renormalization factors and anomalous dimensions of the fields and parameters. The renormalization factor \( Z_X \) is expanded order by order of the loop expansion as

\[
Z_X = 1 + Z_X^{(1)} + Z_X^{(2)} + \cdots,
\]

(58)

where \( Z_X^{(n)} \) is the \( n \)th order contribution. The anomalous dimension of the respective field \( X = Z_X^{1/2} X_R \) is defined by

\[
\gamma_X := \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z_X := \frac{1}{2} \mu \frac{\partial}{\partial \mu} Z_X^{(1)} + \cdots,
\]

(59)

and the RG function of the respective parameter \( Y = Z_Y Y_R \) is defined by

\[
\gamma_Y := \mu \frac{\partial}{\partial \mu} Y_R := -Y_R \mu \frac{\partial}{\partial \mu} Z_Y^{(1)} + \cdots.
\]

(60)

The anomalous dimension of the diagonal gluon \( a_\mu \) can be determined by requiring the renormalizability for the transverse part of the diagonal gluon propagator. On the other hand, the RG function of the Abelian gauge fixing parameter \( \beta \) can be determined by requiring the renormalizability for the longitudinal part of the diagonal gluon propagator. Similarly, the anomalous dimensions of the off-diagonal gluons \( A_\mu^a \) and the RG function of the gauge fixing parameter \( \alpha \) can be respectively determined by considering the transverse and longitudinal part of the off-diagonal gluon propagators. Then the renormalization factors \( Z_a, Z_\beta, Z_A \) and \( Z_\alpha \) are obtained by calculating the counterterms \( \Delta S_{gauge} \). Moreover, from the counterterms \( \Delta S_{gauge} \) we can calculate also the RG function of the QCD coupling constant \( g \), that is, the \( \beta \)-function. These renormalization factors have already been calculated in Ref. [10, 11] by making use of the dimensional regularization. Consequently, the renormalization factors are given as

\[
\begin{align*}
Z_a^{(1)} &= Z_\beta^{(1)} = \frac{22}{3} \frac{(\mu^{-\epsilon} g_R)^2}{(4\pi)^2 \epsilon}, \\
Z_A^{(1)} &= \frac{(g_R \mu^{-\epsilon})^2}{(4\pi)^2 \epsilon} \left[ \frac{17}{6} - \frac{\alpha_R}{2} - \beta_R \right],
\end{align*}
\]

(61) (62)
where $\epsilon$ is defined as $\epsilon := (4 - d)/2$.

In this paper, we determine the remaining renormalization factors, $Z_c$, $Z_{\bar{c}}$ and $Z_C$, by making use of the dimensional regularization. In order to determine these three factors, we must calculate three independent coefficients $\delta s$ in Eqs. (57). For instance, $Z_C$ is obtained by calculating $\delta_a$ in Eq. (57). By calculating $\delta_b$, we obtain a relation of $Z_c$ and $Z_{\bar{c}}$. One more relation is obtained by calculating $\delta_e$. In the actual calculations, it is useful to remember the fact that the diagonal ghost does not appear in the internal line.

First, we consider the quantum correction to the diagonal ghost propagator. There is no divergent graph for the diagonal ghost propagator in the dimensional regularization, so that we immediately obtain a relation between $Z_c^{(1)}$ and $Z_{\bar{c}}^{(1)}$:

$$
\delta_b = \frac{1}{2} Z_c^{(1)} + \frac{1}{2} Z_{\bar{c}}^{(1)} = 0.
$$

Here, the identity $\delta_b = 0$ holds to all order of perturbative calculations, since there is no interaction term including the diagonal ghost in our action (37). In fact, we cannot write the diagram corresponding to the process which causes radiative corrections to the diagonal ghost propagator. In other words, the contribution to the diagonal ghost propagator comes from a tree level graph alone.

Next, we consider the quantum correction to the off-diagonal ghost propagators. The divergent graphs for the off-diagonal ghost propagators are enumerated in Fig. 2. The graph (a1) includes both the quadratic and logarithmic divergences. On the other hand, each graph (a2), (a3) and (a4) includes only quadratic divergence in the dimensional regularization. However, the quadratic divergence from four graphs are canceled so that non-trivial (logarithmic) contribution comes from only one graph (a1). Thus, by making use of the dimensional regularization, $\delta_a$ or $Z_C^{(1)}$ is obtained as

$$
\delta_a = Z_C^{(1)} = \frac{(g_R \mu^{-\epsilon})^2}{(4\pi)^2 \epsilon} (3 - \beta_R).
$$
Figure 3: The graphs corresponding to one-loop radiative corrections for the three point vertex of one diagonal antighost, one off-diagonal ghost and one off-diagonal gluon. The wavy line and the broken line labeled by 3 represent the diagonal gluon and diagonal ghost respectively, while the line without any label represents the off-diagonal gluon or off-diagonal ghost.

In order to determine $\delta_e$ we calculate the quantum correction to the three point vertex of one diagonal antighost, one off-diagonal ghost and one off-diagonal gluon. The divergent graphs for this vertex are collected in Fig. 3. Then we obtain

$$\delta_e = \frac{1}{2} Z_c^{(1)} + Z_g^{(1)} + \frac{1}{2} Z_A^{(1)} + \frac{1}{2} Z_C^{(1)}$$

or, by solving with respect to $Z_c$ and $Z_{\bar{c}}$, we also obtain

$$Z_c^{(1)} = -Z_{\bar{c}}^{(1)} = (3 + \alpha_R) \frac{(g_R \mu^{-\epsilon})^2}{(4\pi)^2\epsilon},$$

where we have made use of Eqs. (65), (62), (64) and (66).

Substituting one loop renormalization factors $Z_X^{(1)}$ and $Z_Y^{(1)}$ into the definitions of the anomalous dimension (59) or RG functions (60), we obtain the following anomalous dimensions and RG functions:

$$\gamma_a(g_R) = -\frac{22}{3} \frac{g_R^2}{(4\pi)^2},$$

$$\gamma_A(g_R) = -\frac{g_R^2}{(4\pi)^2} \left[ \frac{17}{6} - \frac{\alpha_R}{2} - \beta_R \right],$$

$$\gamma_\beta(g_R) = 44 \frac{\beta_R g_R^2}{(4\pi)^2},$$

$$\gamma_\alpha(g_R) = \frac{g_R^2}{(4\pi)^2} \left[ \frac{8}{3} \alpha_R - 2\alpha_R^2 - 6 \right],$$

$$\beta(g_R) = \gamma_\beta(g_R) = -\frac{22}{3} \frac{g_R^3}{(4\pi)^2},$$

where we have made use of Eqs. (65), (62), (64) and (66).
\[\gamma_c(g_R) = -(3 + \alpha_R)\frac{g_R^2}{(4\pi)^2} = -\gamma_c(g_R),\]  
(74)

\[\gamma_C(g_R) = -\frac{g_R^2}{(4\pi)^2}(3 - \beta_R).\]  
(75)

Thus we have obtained anomalous dimensions of all the fields and RG functions of all parameters at the one-loop level of perturbative expansion based on the dimensional regularization.

### 3.2 Scheme II

In the previous scheme, we absorbed \(\kappa\) by a rescaling (36). However, we can also leave this parameter explicitly. Especially, in the case of \(\kappa = 0\), we cannot perform such a rescaling (36). Here, some of Feynman rules enumerated in the previous subsection must be modified as follows.

**(a)'** Diagonal gluon propagator:

\[iD_{\mu\nu} = -\frac{i}{p^2} \left[ g_{\mu\nu} - \left(1 - \frac{\beta}{\kappa^2}\right) \frac{p_{\mu}p_{\nu}}{p^2} \right].\]  
(76)

**(c)'** Diagonal ghost propagator:

\[i\Delta = -\frac{1}{\kappa p^2}.\]  
(77)

**(g)'** One off-diagonal gluon, one off-diagonal ghost and one diagonal antighost:

\[i\left\langle \bar{C}^a(p)C^b(q)A_{\mu}^c \right\rangle_{\text{bare}} = -i\kappa g\epsilon^{ab}p_{\mu}.\]  
(78)

Due to the existence of \(\kappa\) and the renormalization of \(\kappa\), we can equate the renormalization factor of \(C^3\) and \(\bar{C}^3\), i.e., \(Z_c = Z_{\bar{c}}\). Therefore we can redefine the renormalized diagonal antighost field and a parameter \(\kappa\):

\[\bar{C}^3 = Z_{\bar{c}}^{1/2}\bar{c}_R^3, \quad \kappa = Z_{\kappa} \kappa_R.\]  
(79)

By straightforward calculations, we obtain the RG functions of \(\kappa\) and \(\hat{\beta} := \beta/\kappa^2\) as

\[\gamma_\kappa(g_R) = -2(3 + \alpha_R)\kappa_R\frac{g_R^2}{(4\pi)^2},\]  
(80)

\[\gamma_{\hat{\beta}}(g_R) = \frac{44}{3} \beta_R\frac{g_R^2}{(4\pi)^2}.\]  
(81)

The beta-function, the other anomalous dimensions and RG functions are identical to those calculated in the previous subsection. From (80), we notice that a fixed point of \(\kappa\) exists at \(\kappa = 0\). After setting \(\kappa = 0\), we can obtain a simpler action:

\[S_{GF} = \int d^4x \left\{ -\frac{1}{2\alpha} (D^\mu A^a_\mu)^2 - \frac{1}{2\hat{\beta}} (\partial^\mu a_\mu)^2 
+ i\bar{C}^a D^2 C^b - ig^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a C^b A^c_\mu A^d_\mu + \frac{\alpha}{4} \epsilon^{ab} \epsilon^{cd} \bar{C}^a C^b C^c C^d \right\}.\]  
(82)
The off-diagonal part is equal to the modified MA gauge term and the diagonal part is nothing but the Lorentz gauge for Abelian theory. It is remarkable that this gauge fixing term has FP conjugation invariance and \( SL(2, \mathbb{R}) \) symmetry for the multiplet of ghost and antighost fields.

## 4 Conclusion and discussion

In this paper, we have investigated how to construct the most general and renormalizable MA gauge for \( SU(2) \) Yang-Mills theory and performed perturbative calculations for a simplest version of the \( SU(2) \) Yang-Mills theory in the MA gauge.

First, we have constructed the most general gauge fixing term with the BRST symmetry and the global \( U(1) \) symmetry, but without the global \( SU(2) \) symmetry. By definition, the modified MA gauge partially fixes the gauge symmetry so that the residual \( U(1) \) gauge symmetry leaves intact. In order to make the gauge fixing term renormalizable in the exact sense, furthermore, we must fix the residual \( U(1) \) gauge symmetry and hence we need the gauge fixing term for fixing the residual \( U(1) \) symmetry. In the MA gauge, only the global \( U(1) \) symmetry remains unbroken.

Second, we have required several symmetries in order to restrict the parameter space. We expect that the renormalizability is not spoiled by restricting the parameter space to a subspace protected by the imposed symmetries. We have found that at least three independent parameters \( \alpha, \beta \) and \( \kappa \) are necessary and sufficient to maintain the renormalizability. The minimal choice coincides with the modified MA gauge proposed in the previous papers \([7, 9]\) from the viewpoint of renormalizability.

Third, we have calculated the beta function, anomalous dimensions of all fields and RG functions of all gauge parameters in the minimal action at one loop order.

In the construction of action, we should not impose so many restrictions. For instance, the ordinary Faddeev-Popov term \( i \partial_\mu \bar{C}^A D_\mu C^A \) is not invariant under the translation of the diagonal ghost, while it is invariant under the translation of the diagonal antighost. Thus, if we would like to compare the MA gauge with the ordinary Lorentz gauge, we should not require the translational symmetry for the diagonal ghost.

Similarly, in the modified MA gauge, we expect that the ghost–antighost composite operators \( \epsilon^{ab} C^a \bar{C}^b \) and \( C^a \bar{C}^a \) have non-trivial expectation values due to the condensation and hence the charge conjugation symmetry breaks down.\([9, 19]\) Therefore we must not require the charge conjugation symmetry from this viewpoint. The results of investigations to remedy these shortcomings will be reported elsewhere.

In this paper, we have used the full BRST transformation with nilpotency for constructing the MA gauge fixing. The GF+FP term obtained in this way contains the diagonal ghost and antighost for \( \kappa \neq 0 \). In a special case \( \kappa = 0 \), the diagonal ghost and antighost decouple from the Lagrangian, leaving only the simple conventional gauge fixing term for the diagonal gluon, see eq. (82). At \( \kappa = 0 \), the GF+FP term does not contain the diagonal ghost and antighost and reproduces the previous version of MA gauge \([9]\). If the diagonal ghost is not contained in the MAG Lagrangian from the beginning, the partition function vanishes, since the measure contains the
diagonal ghost, whereas the functional does not contain it. In this case, the correlation function must be obtained by differentiating \( \log Z[J]/Z[0] \) with respect to \( J \). Even if \( Z[J] \) contains such a zero, the normalization \( Z[0] \) carries the same zero. Therefore, the ratio is still well-defined and hence one can forget about zero. This is equivalent to removing the diagonal ghost from the measure. In fact, Schaden [19] has used a new equivariant BRST which contains no diagonal ghost from the beginning. In other words, he considered the BRST transformation on the functional subspace invariant under the residual gauge transformation. However, the equivariant BRST is not nilpotent. This may cause the difficulty to the renormalizability argument. Nevertheless, the final form of the Lagrangian he obtained is the same as ours set at \( \kappa = 0 \). Schaden’s consideration will be valid, as far as one does not perform the gauge fixing for the residual Abelian gauge. In this paper, the MA gauge without diagonal ghost and antighost used so far is obtained as a special case \( \kappa = 0 \) of our formulation which includes also the gauge fixing for the residual Abelian gauge without losing the nilpotency of the BRST transformation. This is an advantage of our formulation given in this paper.

The results obtained in this paper are the first step toward the non-perturbative study of the low energy physics, despite that they were calculated by means of perturbative way, since the high and low energy region of QCD are closely related to each other according to renormalization group equation and analyticity.[14]

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\section{A Rescaling of the fields preserving BRST transformation and its connection to the renormalization}

First, we consider the gauge fixing term with global \( SU(2) \) gauge symmetry given by

\[ \mathcal{L}_{GF} = -i \gamma \delta_{B} \left[ \bar{c}^{A} \left( \partial_{\mu} A^{A}_{\mu} + \frac{\alpha}{2} B^{A} - \zeta g f^{ABC} \bar{c}^{B} c^{C} \right) \right], \quad (83) \]

where \( \gamma, \alpha \) and \( \zeta \) are arbitrary parameters and \( \delta_{B} \) is the BRST transformation:

\[ \begin{align*}
\delta_{B} A^{A}_{\mu} &= D_{\mu} c^{A} - \frac{\alpha}{2} B^{A} - \zeta g f^{ABC} \bar{c}^{B} c^{C}, \\
\delta_{B} c^{A} &= -g f^{ABC} B^{B} c^{C}, \\
\delta_{B} \bar{c}^{A} &= i B^{A}, \\
\delta_{B} B^{A} &= 0.
\end{align*} \quad (84) \]

Because of the nilpotency of the BRST transformation, it is trivial that (83) is invariant under the transformation:

\[ \delta A^{A}_{\mu} = \epsilon \delta_{B} A^{A}_{\mu}, \quad \delta c^{A} = \epsilon \delta_{B} c^{A}, \quad \delta \bar{c}^{A} = \epsilon \delta_{B} \bar{c}^{A}, \quad \delta B^{A} = \epsilon \delta_{B} B^{A}, \quad (85) \]
for arbitrary Grassmann parameter $\epsilon$.

Now we show that the parameter $\gamma$ can be set equal to 1 without losing generality. It turns out that such a parameter can be absorbed by rescaling the fields $C^A$ and $\bar{C}^A$. Indeed, by rescaling the fields as

$$A'_\mu = x A_\mu, \quad B'_\mu = y B_\mu, \quad C' = u C, \quad \bar{C}' = v \bar{C},$$

(86)

the BRST transformation (84) is rewritten as

$$\delta_B A'^A_\mu = \left( \frac{x}{y} \right) \partial_\mu C'^A + \left( \frac{1}{x} \right) g f^{ABC} A'^B_\mu C'^C,$$

$$\delta_B C'^A = - \left( \frac{1}{x} \right) \frac{1}{2} g f^{ABC} C'^B C'^C,$$

$$\delta_B \bar{C}' = i \left( \frac{y}{x} \right) B'^A,$$

$$\delta_B B'^A = 0.$$  

(87)

If two conditions $x = 1$ and $y = uv$ are satisfied, the same form of the BRST transformation as the original BRST transformation (84) is obtained for the rescaled field (86) by defining a new BRST transformation $\delta'_B := u \delta_B$ as

$$\delta'_B A'^A_\mu = \partial_\mu C'^A + g f^{ABC} A'^B_\mu C'^C,$$

$$\delta'_B C'^A = - \frac{1}{2} g f^{ABC} C'^B C'^C,$$

$$\delta'_B \bar{C}' = i B'^A,$$

$$\delta'_B B'^A = 0.$$  

(88)

Then the Lagrangian (83) is rewritten as

$$L_{GF}' = -i \gamma' \delta'_B \left[ \bar{C}'^{\mu} \left( \partial_\mu A'^A_\mu + \alpha' B'^A - \zeta' g f^{ABC} \bar{C}'^{B} C'^C \right) \right],$$

(89)

where $\gamma' := \gamma / y$, $\alpha' := \alpha / y$ and $\zeta' := \zeta / y$. Therefore we can set $\gamma'$ to 1 by requiring a condition $y = \gamma$.

Here, we have introduced four rescaling parameters ($x$, $y$, $u$, and $v$) and imposed three conditions ($x = 1$, $y = uv$, and $\gamma = y$). Then, we can require one more condition. We consider the renormalization of fields and parameter as

$$C = Z_C^{1/2} C_R, \quad \bar{C} = Z_{\bar{C}}^{1/2} \bar{C}_R, \quad u = Z_u u_R, \quad v = Z_v v_R,$$

(90)

while

$$C' = Z_C^{1/2} C'_R, \quad \bar{C}' = Z_{\bar{C}}^{1/2} \bar{C}'_R.$$  

(91)

In general, it is not necessary that the renormalization factors $Z_C$ and $Z_{\bar{C}}$ (similarly $Z_C'$ and $Z_{\bar{C}}'$) are equivalent. However, substituting (90) and (91) into (86), we have the relations:

$$Z_C^{1/2} Z_u C_R u_R = Z_C'^{1/2} C'_R \quad \text{and} \quad Z_{\bar{C}}^{1/2} Z_v \bar{C}_R v_R = Z_{\bar{C}}'^{1/2} \bar{C}'_R,$$

(92)
and we can require the relation $Z_{C'} = Z_{C'}$ by taking $u$ and $v$ appropriately. Therefore we adopt it as the last condition.

Next, we consider the case of the gauge fixing term with global U(1) gauge symmetry alone in $SU(2)$ Yang-Mills theory. Such a term is given by

$$\mathcal{L}_{GF} := i\gamma\partial^{\mu}\bar{C}^{a}\partial_{\mu}C^{a} + i\gamma\partial^{\mu}\bar{C}^{3}\partial_{\mu}C^{3} + \cdots,$$

where “…” denotes the interaction terms given in section 2. Decomposing the BRST transformation (84) into diagonal and off-diagonal components explicitly, we obtain

$$\delta_B a_\mu = \partial_\mu C^3 + g e^{ab} A^a_\mu C^b, \quad \delta_B A^a_\mu = \partial_\mu C^a + g e^{ab} A^b_\mu C^3 - g e^{ab} a_\mu C^b,$$

$$\delta_B C^3 = -\frac{i}{2} g e^{ab} C^a C^b, \quad \delta_B C^a = -g e^{ab} C^b C^3,$$

$$\delta_B \bar{C}^3 = i B^3, \quad \delta_B \bar{C}^a = i B^a,$$

and

$$\delta_B B^3 = 0, \quad \delta_B B^a = 0.$$

After rescaling the fields as

$$a'_\mu = k a_\mu, \quad B'^3 = l B^3, \quad C'^3 = m C^3, \quad \bar{C}'^3 = n \bar{C}^3,$$

$$A'^a_\mu = x A^a_\mu, \quad B'^a = y B^a, \quad C'^a = u C^a, \quad \bar{C}'^a = v \bar{C}^a,$$

the BRST transformation is rewritten as

$$\delta_B a'_\mu = \left(\frac{k}{m}\right) \partial_\mu C'^3 + \left(\frac{k}{x u}\right) g e^{ab} A'^a_\mu C'^b,$$

$$\delta_B A'^a_\mu = \left(\frac{x}{m}\right) \partial_\mu C'^a + \left(\frac{1}{m}\right) g e^{ab} A'^b_\mu C'^3 - \left(\frac{x}{k u}\right) g e^{ab} a'_\mu C'^b,$$

$$\delta_B C'^3 = - \left(\frac{m}{n^2}\right) \frac{i}{2} g e^{ab} C'^a C'^b, \quad \delta_B C'^a = - \left(\frac{1}{m}\right) g e^{ab} C'^b C'^3,$$

$$\delta_B \bar{C}'^3 = \left(\frac{u}{n}\right) i B'^3, \quad \delta_B \bar{C}'^a = \left(\frac{u}{n}\right) i B'^a,$$

$$\delta_B B'^3 = 0, \quad \delta_B B'^a = 0.$$

Similarly to the previous case, imposing the conditions:

$$\frac{k}{m} = \frac{k}{x u} = \frac{x}{m} = \frac{x}{k u} = \frac{m}{a^2} = \frac{n}{l} = \frac{v}{y},$$

or

$$k = 1, \quad x^2 = 1, \quad m^2 = u^2, \quad l = m n, \quad y = m v,$$

we can obtain the some BRST transformation as (94) for the rescaled fields:

$$\delta'_B a'_\mu = \partial_\mu C'^3 + g e^{ab} A'^a_\mu C'^b, \quad \delta'_B A'^a_\mu = \partial_\mu C'^a + g e^{ab} A'^b_\mu C'^3 - g e^{ab} a'_\mu C'^b,$$

$$\delta'_B C'^3 = - \frac{1}{2} g e^{ab} C'^a C'^b, \quad \delta'_B C'^a = -g e^{ab} C'^b C'^3,$$

$$\delta'_B \bar{C}'^3 = i B'^3, \quad \delta'_B \bar{C}'^a = i B'^a,$$

$$\delta'_B B'^3 = 0, \quad \delta'_B B'^a = 0,$$

where we have defined the new BRST transformation $\delta'_B$ as $\delta'_B := m \delta_B$ by using the rescaling factor $m$ of the diagonal ghost $C^3$.

Here, we have introduced five conditions (98) for eight parameters ($k$, $l$, $m$, $n$, $x$, $y$, $u$, $v$). Therefore, we can impose three more conditions. Note that there are four options:
(i) An absorption of a parameter $\gamma$,  
(ii) An absorption of a parameter $\kappa$,  
(iii) An equivalence of renormalization factor of $C^a$ and $\bar{C}^a$,  
(iv) An equivalence of renormalization factor of $\bar{C}^3$ and $C^3$.

Since we can impose only three conditions, one of the four options is never satisfied. It is possible to discard a condition (iii) or (iv) as done in Refs. [12] and [13]. However, in order to deal with the ghost and antighost on equal footing, for example, FP conjugation or BRST–anti-BRST field formalism, it is useful to retain the parameter $\gamma$ or $\kappa$ as we have adopted in this paper.

Thus we can restrict the gauge fixing terms with global $U(1)$ without losing generality to

$$L_{GF} := i\partial^\mu \bar{C}^a \partial_\mu C^a + i\kappa \partial^\mu \bar{C}^3 \partial_\mu C^3 + \cdots,$$

where the renormalization factors of $\bar{C}^a$ and $C^a$ are identical to each other and this is also the case for $\bar{C}^3$ and $C^3$. It is remarkable that the parameter $\kappa$ (or $\gamma$) cannot be absorbed.

References


