On Brane Cosmology and Naked Singularities

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Abstract

Brane-world singularities are analysed, emphasizing the case of supergravity in singular spaces where the singularity puzzle is naturally resolved. These naked singularities are either time-like or null, corresponding to the finite or infinite amount of conformal time that massless particles take in order to reach them. Quantum mechanically we show that the brane-world naked singularities are inconsistent. Indeed we find that time-like singularities are not wave-regular, so the time-evolution of wave packets is not uniquely defined in their vicinity, while null singularities absorb incoming radiation. Finally we stress that for supergravity in singular spaces there is a topological obstruction, whereby naked singularities are necessarily screened off by the second boundary brane.

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1 Introduction

Brane cosmology[1] has recently appeared as a framework where thorny issues such as the cosmological constant problem can be tackled. In particular new mechanisms have been proposed whereby the vacuum energy of the brane-world curves the fifth dimension leaving a flat four dimensional brane-world intact[2]. Unfortunately this scenario seems to fail as the existence of a naked singularity in the bulk prevents one from obtaining a smooth five dimensional space-time. This singularity needs to be resolved leading to a fine-tuning between the tension on the brane-world and of the ghost brane sitting at the singularity[3].

Another proposal involves supergravity in singular spaces[4]. In that case $N = 2$ supergravity lives in the bulk and is broken on the brane-world. The breaking of supersymmetry ensures that a non-static configuration is generated[5]. The resulting brane-world metric is of the FRW type with an acceleration parameter $q_0 = -4/7$ and an equation of state $\omega = -5/7$, within the experimental ball-park[6]. Unfortunately a thorough analysis of the coupling of supergravity in singular spaces to ordinary matter on the brane-world shows that the amount of supersymmetry breaking needs to be fine-tuned to the level of the critical energy density of the universe[7]. Nevertheless this model is relevant as it realizes in an explicit way a quintessence scenario [8] in five dimensions. As in the self-tuned brane scenario there is a would-be singularity in the bulk. One of the aims of this paper is to provide a description of how a topological obstruction prevents the existence of such a naked singularity in the bulk.

In a first section we shall study the classical trajectories of massive and massless particles by studying the geodesics in warped geometries with naked singularities. For massless particles the singularity can be reached in either a finite or an infinite amount of conformal time corresponding to time-like or null singularities. The former corresponds to the self-tuned brane while the latter appears for supergravity in singular spaces. We then study the quantum mechanics of gravitons and show that the time-like naked singularities are repulsons[9], repelling all incoming radiation. We also find that they are not wave-regular[10, 11, 12], i.e. the time evolution of wave packets is not uniquely defined, prompting the necessity of imposing an appropriate boundary condition at the singularity, i.e. knowing enough about its possible resolutions. On the contrary, null singularities are wave-regular but absorb incoming radiation. In both case this signals a quantum inconsistency of the models. Fortunately, in the supergravity case, the would-be singularity is absent due to a topological obstruction associated with the presence of four-forms in the bulk. This obstruction is reminiscent of the tadpole cancellation mechanism in string theory[13, 14].

2 Geodesic Motion

In this section we are interested in the classical motion of massless and massive particles in the vicinity of the naked singularities arising in brane-world scenarios. More precisely we consider the bulk geometry to be described by the metric

$$ds^2 = a^2(u)(du^2 - dy^2 + dx_i dx^i)$$

(1)

in conformal coordinates. The behaviour of the scale factor $a(u)$ is given by a power law close to the singularity

$$a(u) = \left(\frac{u}{u_0}\right)\beta.$$

(2)
The curvature vanishes at the origin for $\beta > -1$ and at infinity for $\beta < -1$. We will investigate both situations in the following.

The classical motion is characterized by the point-particle Lagrangian \[ L \equiv \frac{ds^2}{d\tau^2} = a^2(u)(\dot{u}^2 - \dot{\eta}^2 + \dot{x}^i \dot{x}_i). \] \[ (3) \]

The time and space independence of the Lagrangian leads to the conservation of energy and momentum

$E = -2a^2(u)\dot{\eta}, \quad p^i = 2a^2(u)\dot{x}^i$ \[ (4) \]

thus giving the reduced Lagrangian

$L = a^2(u)(\dot{u}^2 + \frac{p^2 - E^2}{4a^4})$. \[ (5) \]

The trajectories are determined by the constraint

$L = \epsilon$ \[ (6) \]

where $\epsilon = 0$ for light-like paths and $\epsilon = -1$ for time-like paths. Let us consider a particle sitting initially on the brane-world with speed $\dot{u}|_0$. We can rewrite the Lagrangian constraint as

$\dot{u}^2 + \frac{\epsilon - \dot{u}^2|_0}{a^4} - \frac{\epsilon}{a^2} = 0$. \[ (7) \]

This is the classical motion of a massive particle with zero total energy in the potential

$V(u) = \frac{\epsilon - \dot{u}^2|_0}{a^4(u)} - \frac{\epsilon}{a^2(u)}$. \[ (8) \]

First of all notice that the particles with a vanishing initial velocity $\dot{u}_0 = 0$ remain on the brane.

Let us now consider the geodesics obtained by launching the particles from the brane to the singularity. For $\beta > 0$ the potential goes to minus infinity at the singularity and vanishes at infinity. For $-1 < \beta < 0$ the potential vanishes at the singularity and goes to minus infinity at infinity. For $\beta < -1$ the potential goes to minus infinity at the singularity. There is a critical point at

$a^2_* = 2 + 4\dot{u}^2|_0$ \[ (9) \]

in the massive case. For $\beta > 0$ the critical point is beyond the brane-world located at $u_0$ while for $\beta < 0$ it is between the brane-world and the origin.

Let us consider massive particles first. In the case $\beta > 0$, as the total energy vanishes and the potential energy of the particle is always negative, we find that massive particles are attracted by the singularity. In the case $-1 < \beta < 0$ massive particles evolve in the bulk before reaching a turning point for

$a^2_{tp} = 1 + \dot{u}^2|_0$. \[ (10) \]

Notice that this ensures that massive particles never reach the singularity in the $-1 < \beta < 0$ case. For $\beta < -1$ the massive particles are attracted by the singularity.

Massless particles are attracted by the singularity. Indeed the geodesics are given by

$u = u_0 - \frac{2\dot{u}_0}{E\eta}$. \[ (11) \]
showing that the singularity is reached in a finite amount of conformal time for $\beta > -1$, corresponding to a time-like singularity whereas for $\beta < -1$ the amount of conformal time is infinite, i.e. a null singularity.

Two particularly relevant cases have been discussed lately. First of all the self-tuned brane scenario\cite{2} is such that $\beta = 1/3$. This corresponds to an attractive time-like singularity. All kinds of matter, whether massive or massless, are attracted and reach the singularity in a finite amount of conformal time. As such this singularity does not make sense classically. It has been argued that it needs to be resolved by putting an appropriate brane located at the singularity whose tension compensates for the tension of the original brane at $u_0$.

Another scenario invokes the presence of supergravity in the bulk with broken supersymmetry on the brane\cite{5}. There is a would-be singularity whose existence will be further discussed in section 4. It is characterized by $\beta = -3/2$. This corresponds to a null singularity.

In the next section we will study the quantum mechanical behaviour of massless particles in the vicinity of such naked singularities.

\section{Wave-Regularity of Naked Singularities}

We have seen that the brane-world singularities attract massless particles. It is then relevant to analyse their quantum mechanical behaviour in the neighbourhood of the naked singularity. We shall restrict our attention to massless particles in the s-wave channel. We assume that the only massless particle propagating in the bulk is the graviton. We are interested in gravitons polarized along the brane-world. The graviton wave function can be written as

$$h_{ij} = H(u, x) \epsilon_{ij}$$

where $H(u, x) = H(u) e^{ik \cdot x}$, $\epsilon_{ij}$ is the polarization tensor and $k$ is in the time direction. In the Einstein frame the graviton equation reduces to the Laplace equation

$$\Delta h_{ij} = 0.$$  \hfill (13)

The polarization tensor must be traceless $\eta^{ij} \epsilon_{ij} = 0$ and transverse to $k$ implying that $\epsilon_{0i} = 0$. Denoting by $\tilde{\epsilon}$ the spatial part of the polarization tensor we find a basis of these tensors given by off-diagonal symmetric matrices with $\tilde{\epsilon}_{ab} = \tilde{\epsilon}_{ba} = 1$ and zero otherwise along with diagonal matrices such that $\tilde{\epsilon}_{aa} = 1$, $\tilde{\epsilon}_{bb} = -1$, $a < b$. The latter are diagonal polarizations while the former are transverse polarizations. In the diagonal case put

$$H(u, x) = a \phi(u, x).$$  \hfill (14)

Then the scalar field $\phi$ satisfies the free wave equation

$$\nabla_\mu \nabla^\mu \phi = 0.$$  \hfill (15)

In the transverse case the function $H(u, x)$ satisfies the free scalar equation too.

In the following we shall concentrate on the scalar wave equation in five dimensions. It is particularly useful to introduce

$$\phi = a^{-3/2} \psi$$  \hfill (16)

which satisfies the Schrodinger equation

$$\psi'' - V \psi = 0.$$  \hfill (17)

3
where
\[ V = -\omega^2 + \frac{(a^{3/2})''}{a^{3/2}}. \] (18)

It is possible to recast the Schrodinger equation into the form[12]
\[ (\bar{Q}Q - \omega^2)\psi = 0 \] (19)

where
\[ Q = -\frac{d}{du} + \frac{1}{2} \frac{d \ln a^3}{du}, \quad \bar{Q} = \frac{d}{du} + \frac{1}{2} \frac{d \ln a^3}{du}. \] (20)

The Hamiltonian \( \bar{Q}Q \) is a symmetric operator \( (f, \bar{Q}Qg) = (Q\bar{Q}f, g) \), where
\[ (f, g) = \int du f^*(u)g(u) + \int du D_u f^*(u)D_u g(u), \] (21)

for functions depending only on \( u \) if one restricts the domain of \( \bar{Q}Q \) to the infinitely differentiable functions with compact support. Following [11] we choose a Sobolev norm as it is related to the energy of the scalar field \( \phi \). In particular fields with finite norm have finite energy. With this choice the Hamiltonian is symmetric but is not guaranteed to be a self-adjoint operator[10, 11]. Notice too that the Hamiltonian is a positive operator with two zero modes
\[ \psi_1(u) = a^{3/2}, \quad \psi_2(u) = a^{3/2} \int^u \frac{dv}{a^3(v)}. \] (22)

The rest of the spectrum is positive preventing the existence of tachyons.

We can solve the Schrodinger equation corresponding to the brane-world singularities as
\[ V = -\omega^2 + \frac{3\beta}{2} \left( \frac{3\beta}{2} - 1 \right) \frac{1}{u^2}. \] (23)

The self-tuned brane scenario with \( \beta = 1/3 \) and the supergravity scenario with \( \beta = -3/2 \) lead to attractive singularities as the potential decreases at the origin in the former case, and at infinity in the latter case respectively. It is convenient to define \( z = \omega u \). The generalized eigenstates read
\[ \psi^1(z) = \sqrt{z} J_{(3\beta-1)/2}(z), \quad \psi^2(z) = \sqrt{z} J_{(1-3\beta)/2}(z) \] (24)

for \( \beta \neq 1/3 \). In the latter case, i.e. for the self-tuned brane scenario, the solutions are expressed in terms of zeroth order Bessel and Neumann functions
\[ \psi^1(z) = \sqrt{z} J_0(z), \quad \psi^2(z) = \sqrt{z} N_0(z). \] (25)

Close to the time-like singularity the constant term in the potential is negligible implying that all the solutions behave like the two zero modes
\[ \psi_1(z) = z^{3\beta/2}, \quad \psi_2(z) = z^{1-3\beta/2} \] (26)

for \( \beta \neq 1/3 \) and
\[ \psi_1(z) = \sqrt{z}, \quad \psi_2(z) = \sqrt{z} \ln z. \] (27)

for \( \beta = 1/3 \). As none of the eigenstates are oscillatory in the neighbourhood of the time-like singularity this implies that no flux reaches it. This is natural when the singularity is repulsive. For attractive singularities this is due to the extreme steepness of the potential. Such
singularities are repulsions\cite{9}. For $\beta < -1$ the singularity is at infinity where the eigenfunctions behave like plane waves. This implies that in a scattering experiment there will be some absorption by the null singularity.

We can now study whether the time evolution of wave packets is well defined in the vicinity of the naked singularities. To do that let us write the massless Klein-Gordon equation in the form

$$\frac{\partial^2 \phi}{\partial t^2} = -M \phi \quad (28)$$

where $M$ is a second order partial differential operator depending only on the spatial derivatives. After a change of variable, $M$ reduces to the Hamiltonian $\bar{Q}Q$. The Klein-Gordon equation defines a unique time evolution provided it can be written in the Schrodinger form

$$\frac{\partial \phi}{\partial t} = i M^{1/2} \phi \quad (29)$$

for a unique self-adjoint operator $M^{1/2}$. This is equivalent to finding a unique self-adjoint extension to the symmetric operator $\bar{Q}Q$, i.e. the Hamiltonian $\bar{Q}Q$ is essentially self-adjoint.

For null singularities at infinity, there is a single self-adjoint extension of the symmetric operator $\bar{Q}Q$ acting on functions decreasing fast enough at infinity\cite{10}. Hence the time-evolution of wave packets is well-defined. For the time-like case there is a useful criterion of essential self-adjointness\cite{10, 11}. Let us consider the eigenvalue problem

$$\bar{Q}Q \psi = \pm i \psi. \quad (30)$$

It reduces to a Schrodinger problem in a complex potential

$$\bar{V} = V \pm i. \quad (31)$$

Denote by $n_\pm$ the number of normalizable solutions to (30). As the operator $\bar{Q}Q$ is real one has $n_+ = n_-$, implying that there always exists self-adjoint extensions. Now the operator is essentially self-adjoint provided $n_\pm = 0$, i.e. the solutions are not normalizable. Due to the finiteness of the fifth dimension, the only possible source of divergence is at the singularity. Therefore one must check whether or not the solutions of (30) are normalizable close to the singularity.

In our case notice that in the vicinity of the singularity the extra complex term to $\bar{V}$ is negligible, implying that the solutions are expressed in terms of the two zero modes\footnote{The case $\beta = 2/3$ is special as the potential vanishes. It is easy to see that the eigenfunctions are normalizable.}. The issue of the quantum mechanical behaviour of the singularity is now dependent on the norm of these eigenfunctions. Using the fact that the covariant derivative of the metric vanishes, we find that the norm of $\psi_1$ is finite provided

$$\int du a^3 < \infty \quad (32)$$

which leads to

$$\beta > -\frac{1}{3}. \quad (33)$$

Similarly the norm of $\psi_2$ is finite provided

$$\int du \frac{1}{a^3} < \infty \quad (34)$$
leading to
\[ \beta < \frac{1}{3}. \tag{35} \]

Therefore we find that there is always one of the zero modes which is normalizable. This implies that the Hamiltonian is not essentially self-adjoint. Hence we cannot define a unique evolution operator in the neighbourhood of the singularity. Uniqueness of the evolution operator can be achieved if a physical choice of boundary condition at the singularity is imposed. This requires more knowledge about the physics of the singularity, i.e. its resolution.

We have thus shown that the quantum mechanical behaviour of brane-world singularities is pathological. Indeed the time-like singularities are repulsors while not wave-regular. This requires knowledge about the resolution of the singularity in order to define the time evolution of wave packets in their vicinity. On the contrary null singularities are wave regular allowing one to study the evolution of wave packets in their vicinity irrespective of the physical nature of the singularity. Unfortunately the null singularities have a non-vanishing absorption cross section which needs to be interpreted in order to make sense. In particular this absorption might signal the presence of fields at the null singularity to which the gravitons couple. In any case this requires a deep understanding of the nature of the singularity. The time-like case is exemplified by the self-tuned brane models while the null case occurs for supergravity in singular spaces. In the next section we will study the resolution of naked singularities in singular space supergravity.

4 Supergravity in Singular Spaces

We have seen that the brane-world naked singularity are either not wave-regular, so that the time evolution operator is not well-defined in their neighbourhood, or they absorb incoming radiation. This is an inconsistency of the models which needs to be cured. In the following we shall treat the case of supergravity in singular spaces [4] where such singularities may occur. Nevertheless we will show that there is a topological obstruction to the presence of naked singularities in the bulk.

Let us first discuss the on-shell bosonic part of the Lagrangian

\[ S_{\text{bulk}} = \frac{1}{2\kappa_5^2} \int \sqrt{-g_5} (R - \frac{3}{4} (g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + V)) \]  \hspace{1cm} \text{(36)}

for a particular sigma model metric \( g_{ij} \). The bulk potential is given by

\[ V = W_i W^i - W^2. \]  \hspace{1cm} \text{(37)}

as a function of the superpotential \( W \). On shell the bosonic Lagrangian (36) is supplemented with the boundary term

\[ S_{\text{bound}} = -\frac{3}{2\kappa_5^2} \int d^5x (\delta x_5 - \delta x_5 - R)(\sqrt{-g_4}W). \]  \hspace{1cm} \text{(38)}

In the case of a single scalar field in the bulk the superpotential reads

\[ W = \xi e^{\alpha \phi}. \]  \hspace{1cm} \text{(39)}
where $\xi$ is a scale. The corresponding solutions in conformal coordinates are given by

$$a = \left(\frac{u}{b_0}\right)^{1/(4\alpha^2-1)}. \quad (40)$$

Supergravity implies that $\alpha = 1/\sqrt{3}, \ -1/\sqrt{12}$. For the latter we find that $\beta = -3/2$.

Fortunately the bulk singularity is forbidden by the off-shell formulation of supergravity in singular spaces. The off-shell theory depends on two new fields. There is a supersymmetry singlet $G$ and a four form $A_{\mu\nu\rho\sigma}$. One also introduces a modification of the bulk action by replacing $g \rightarrow G$ and adding a direct coupling

$$S_A = \frac{1}{4!\kappa_5^2} \int d^5x \epsilon^{\mu\nu\rho\sigma} A_{\mu\nu\rho\sigma} \partial_5 G. \quad (41)$$

The boundary action is taken as

$$S_{\text{bound}} = -\frac{1}{\kappa_5^2} \int d^5x (\delta x_5 - \delta x_{5-R}) (\sqrt{-g_4} \frac{3}{2} W + g \frac{2}{4!} \epsilon^{\mu\nu\rho\sigma} A_{\mu\nu\rho\sigma}). \quad (42)$$

The supersymmetry singlet $G$ satisfies a first order constraint

$$\partial_5 G = 2g (\delta x_5 - \delta x_{5-R}) \quad (43)$$

where the left hand side is nothing but the $A_{\mu\nu\rho\sigma}$ charge associated with the boundary branes. The constraint (43) leads to the topological obstruction of singularities in the bulk. Indeed from

$$\int dx_5 \partial_5 G = 0 \quad (44)$$

due to the compactness of the fifth dimension we deduce that the total charge in the extra dimension must vanish. This is the equivalent to Gauss’ law, or the tadpole cancellation mechanism in M-theory and string theory. If one were to have a singularity in the bulk, the total charge would not vanish unless the singularity carries a charge, i.e. the extreme case where the singularity sits at the second brane. In all other cases the topological obstruction requires that the singularity be screened off by the second brane.

The same mechanism is at play when supersymmetry is broken on the brane by detuning one of the tensions. In that case the boundary Lagrangian becomes

$$S_{\text{bound}} = -\frac{1}{\kappa_5^2} \int d^5x (\delta x_5 - \delta x_{5-R}) (\sqrt{-g_4} \frac{3}{2} W + g \frac{2}{4!} \epsilon^{\mu\nu\rho\sigma} A_{\mu\nu\rho\sigma}) \quad (45)$$

on the non-supersymmetric brane. The supersymmetry breaking parameter is $T \neq 1$. The Lagrangian on the supersymmetric brane is not modified. As the brane charge is not modified, the first order constraint (43) remains leading, to the same topological obstruction as in the supersymmetric case. So supergravity in singular spaces leads to a natural resolution of the singularity puzzle.

However it may appear that the tension on the second brane (42) has been fine-tuned to the opposite value of the tension of the brane-world. As such this would be a phenomenon akin to the one presented in [3] where the ghost brane and the brane-world have the same tension. In the case of supergravity in singular spaces the mechanism is more subtle. Indeed Gauss’s law implies that the total charge vanishes. Hence, by supersymmetry, this leads to the vanishing of the total tension. Now when supersymmetry is broken on the brane-world by modifying the tension of the brane-world, we lose the exact cancellation between the brane tensions. Nevertheless the total charge still has to vanish, implying that the would-be singularity is screened off by the second brane.
5 Conclusions

In this paper we have analysed the naked singularities inherent in self-tuned branes or the supergravity in singular spaces. These theories have been put forward as possible five-dimensional explanations to the cosmological constant problem. We have shown that, for supergravity in singular spaces, the singularity problem resolves itself. This is due to the existence of a topological obstruction, requiring by Gauss’s law that the total charge vanishes. Thus in this theory the singularity must lie beyond the second brane, unless the singularity itself carries a charge, in which case it sits at the second brane. Thus there is a natural resolution to the singularity puzzle in this theory.

For the self-tuned brane there is no such natural resolution. We have analysed the behaviour of the singularity both classically and quantum mechanically. Classically the singularity attracts massless particles, in a finite amount of conformal time, this being due to the behaviour of the scale factor close to the singularity. Quantum mechanically we have shown that the singularity is a repulson, reflecting all incoming radiation. However it is not wave-regular, so that time evolution of wave packets is not uniquely defined in the vicinity of the singularity.

Our results suggest that the theory of supergravity in singular spaces is a well defined theory cosmologically. We have previously shown that this theory leads to a natural cosmological evolution of the universe, with a late stage of acceleration and cosmological constant consistent with experiment[6]. Thus this model deserves further investigation[16].

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References


